Aspects of the duality between supersymmetric Yang-Mills theory and string theory

Daniel Bundzik

Department of Theoretical Physics, Lund University
Technology and Society, Malmö University
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Thesis Advisor: Anna Tollstén

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Title and subtitle

Aspects of the duality between supersymmetric Yang-Mills theory and string theory

Abstract

The AdS/CFT-correspondence conjects how certain string theories are related to certain gauge theories.

This thesis covers three topics of the AdS/CFT-correspondence. Paper I presents solutions which describe the geometry of fractional D1-branes of Type IIB string theory. The running coupling constant is computed on the gauge theory side.

The AdS/CFT-correpondence predicts that the energy of the string has the dual interpretation of the anomalous dimension of a gauge theory operator. Paper II uses the idea that gauge theories can be interpreted as spin-chains. The eigenvalues of the spin-chain Hamiltonian is the anomalous dimension. The general Leigh-Strassler deformation is rewritten in terms of a spin-one spin chain and the integrability properties of the corresponding Hamiltonian is studied.

In the third paper the star product is defined which is a non-commutative multiplication law. From this star product the general Leigh-Strassler deformation is obtained. A star product defined theory is especially useful when computing amplitudes since the effects of the deformation results in a prefactor.

Sammanfattning

AdS/CFT-korrependensen förmodar hur särskilda strängteorier är relaterade till särskilda gaugeteorier.

Denna avhandling behandlar tre frågor i anknytning till AdS/CFT-korrepondensen. Papper I presenterar lösningar vilka beskriver geometrin för fraktionella D-brane av Type IIB strängteori. Kopplingkonstantens energiberoende beräknas på gaugeteori sidan.

AdS/CFT-korrependensen förutsäger att energin för en sträng har den duala tolkingen av att vara den anomaladimensionen för en gaugeteori operator.

Papper II använder sig av idén att gaugeteorier kan tolkas som spinnkedjor. Egenvärdena till spinnkedjans Hamiltonian är den anomaladimensionen. Den allmäna Leigh-Strassler deformationen är omskriven i termer av en spinn-ett spinnkedja och de integrerbara egenskaperna för den motsvarande Hamiltonian studeras.

I det tredje pappret är stjärnprodukten definierad vilken är en icke-kommutativ multiplikationslag. Från denna stjärnprodukt erhålls den allmänna Leigh-Strassler deformationen. En stjärnprodukten i en teori är speciellt användbar när amplituder beräknas eftersom effekten av deformationen resulterar i en förfaktor.

Key words: AdS/CFT-correspondence, gauge/gravity correspondence, Type IIB string theory, Fractional D-branes, Spin-chains, Integrability, R-matrix, Leigh-Strassler deformation, Star product.

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Contents

Introduction

A relation between field theory and string theory

The string as a physical object was first introduced to describe the strong force. The flux between two quarks was thought to be a string. Upon quantizing the string, a spin-two particle was found which was interpreted as the graviton, the quantum of the gravitational field. Instead of explaining the strong force, the idea that the string might be the key ingredient to unify all particles and all forces in Nature became appealing. Since it is believed in particle physics that all forces are described by quantum field theories, the string could be the missing link to the long lasting problem how the theory of gravitation is connected with the forces in quantum field theory.

A quantum field theory is usually only understandable when the coupling constant is small, since higher order interaction terms are proportional to powers of the coupling and can therefore be suppressed in calculations. A theory for which the coupling constant is small is called a perturbative theory. The perturbative theory of string theory is called supergravity.

To understand a theory in the non-perturbative regime is challenging and has in string theory resulted in a conjecture, which might explain not only how certain string theories, are related to field theories but also how the theory of the strong force, known as Quantum chromodynamics (QCD), at strong coupling can be interpreted in terms of a weakly coupled string theory. The most studied strong/weak-coupling duality which originates in string theory, is the so called AdS/CFT-correspondence [1]. This correspondence describes how a ten-dimensional string theory, which is effectively described as our four-dimensional space-time times a six-dimensional compact space which is too small to perceive, is related to a certain four-dimensional field theory.

To test this correspondence is very hard. The reason is that no one knows how to quantize the string on the curved background, the geometry on which the string theory is formulated. This makes it impossible to match the string states with states on the gauge theory side. Another reason is that when one side of the correspondence is weakly coupled, the other side is strongly coupled. In strongly coupled theories, perturbation theory is not valid. Thus, at strong coupling, the gauge theory is unknown and equally the supergravity description of the string is lost at strong coupling. However, the correspondence is powerful in the sense that even though the strongly coupled theory is not known it can be approached from the dual weakly coupled theory. Thus, the AdS/CFT-correspondence provides an indirect method of understanding strongly coupled theories.

Fortunately, there exists a “hidden” sector within the AdS/CFT-correspondence in which it is possible to find an exact match between string theory and gauge theory [2]. In this sector the string is described as a plane wave and the string states can be solved quantum mechanically. This makes it possible to compare with the gauge theory side finding an exact agreement.

To compute the properties of the gauge theory is in general very hard because of the operator mixing problem: The operators are not orthogonal and therefore the eigenvalues are hard to find. In [3], a method to diagonalize a large set of operators was achieved by translating the gauge theory into the description of the well-known Heisenberg spin chain. A spin chain can be thought of a number of interacting particles on a one-dimensional lattice. In this picture, each gauge theory operator is a state of the spin chain. The spin chain has a Hamiltonian which can be diagonalized, to find the eigenvalues, by the standard technique of the so called Bethe ansatz.

The original formulation of the AdS/CFT-correspondence involves a special field theory that is independent of a scale and is called a conformal field theory. It is possible to deform the conformal field theory by adding new coupling terms and coupling constants which preserve the conformal property. This procedure leads to new spin chains in which an enlarged set of eigenvalues of the spin-chain Hamiltonian can be computed exactly. In paper II the properties of the deformed conformal field theory of the so called general Leigh-Strassler deformation[4], which corresponds to a spin-one spin chain, was studied. Some new integrability points in parameter-space were found.

In Paper III the general Leigh-Strassler deformation was obtained by defining a star product of fields. This can be viewed as a generalized multiplication law of fields and enables us to study conformal properties of deformed conformal field theories with three deformation parameters. The procedure also simplifies computation of amplitudes, since the star product formulation has the effect of extracting the deformation in a prefactor.

A conformal theory which is scale-invariant is not a realistic theory, since all known theories which describe known particles and interactions are dependent on a scale and are therefore non-conformal theories. On the string theory side of the AdS/CFT-correspondence it is possible to construct theories which have the dual interpretation in terms of non-conformal field theories. This is obtained by formulating the string theory on other geometries. In Paper I, string theory is fomulated on a special geometry which makes the field theory on the gauge theory side of the AdS/CFT-correspondence scale-dependent.

Supersymmetry, super Yang-Mills theories and deformed theories

Supersymmetry

Supersymmetry is a symmetry between fermionic and bosonic particles. Each generator of supersymmetry is an operator with a half-integer spin which transforms a bosonic state to a fermionic state and vice versa. A supersymmetric gauge theory is called supersymmetric Yang-Mills theory or sometimes just super Yang-Mills theory. These theories are classified by their number of generators. There can be at the most four generators in four dimensions if we require the the supersymmetry to be a global symmetry. The theories are called simple $\mathcal{N} = 1$ and extended $\mathcal{N} = 2, 3, 4$ super Yang-Mills theory. These theories are consistent and as such renormalizable which means that divergences can be canceled by including counterterms. The highest spin of the particles is one. When more generators are included in a theory, higher spin particles are present which makes the theory non-renormalizable.

Super Yang-Mills theories

An $\mathcal{N} = 1$ super Yang-Mills theory contains two collections of fields where each is called a supermuliplet. The chiral supermultiplet contains a complex scalar field and a Majorana fermion. The vector supermultiplet contains a vector field and a Majorana fermion. The fields in the chiral supermultiplet are the matter fields and fields in the vector supermultiplet are the gauge fields.

Each supermultiplet can be assembled into a single field which is many times easier to handle mathematically than the individual fields separately. This single field is called a superfield and contains anti-commuting variables, which are called Grassman variables, in addition to the normal commuting space-time coordinates. Thus, the chiral supermultiplet can be arranged into a chiral superfield and the vector supermultiplet can be rewritten as a vector superfield.

It is possible to write $\mathcal{N} = 4$ super Yang-Mills theory in terms of $\mathcal{N} = 1$ superfields. This is obtained by a specific combination of one vector superfield and three chiral superfields which gives the whole $\mathcal{N} = 4$ super Yang-Mills theory. In addition, $\mathcal{N} = 4$ super Yang-Mills theory also contains a potential term which is not so surprisingly called the superpotential. In terms of $\mathcal{N} = 1$ superfields the superpotential is a product of the three chiral superfields.

The more symmetries a theory contains the more constrained it is. $\mathcal{N} = 4$ super Yang-Mills theory is special in this manner since it is a conformal field theory. A conformal field theory is a quantum field theory which is invariant under conformal symmetry which contains scale-invariance. In a scale-invariant theory, the gauge coupling renormalization constrain the coupling to be a constant. That is to say that the coupling constant is not running. The beta-function is defined as the change of the coupling with respect to the renormalization scale. So for a constant coupling constant, the beta-function is zero. This is true for $\mathcal{N} = 4$ super Yang-Mills theory.

Deformed Super Yang-Mills theories

By introducing additional coupling terms and coupling constants in a certain way, the conformal properties of $\mathcal{N} = 4$ super Yang-Mills can be preserved even though the theory only contains $\mathcal{N} = 1$ supersymmetry. The additional couplings can be viewed as deformations of the original superpotential of $\mathcal{N} = 4$ super Yang-Mills theory. An important deformation is the so called the general Leigh-Strassler deformation[4] which contains a one-parameter deformation of the original terms in the superpotential of $\mathcal{N} = 4$ super Yang-Mills, in addition to deformations related to new cubic terms.

In Paper III a generalized non-commutative multiplication law is introduced. It is called the star product and is used to simplify the study of deformed conformal theories. By replacing the ordinary multiplication of fields with the star product, the general Leigh-Strassler deformation of $\mathcal{N} = 4$ super Yang-Mills is obtained. One of the reasons why a star product defined theory is useful is that the effect of the deformation can be extracted into a prefactor when amplitudes are computed. This means that many of the properties of $\mathcal{N} = 4$ super Yang-Mills theory also hold for the deformed theories since the difference between the deformed and the undeformed theory lies in the prefactor.

In Paper II the general Leigh-Strassler deformation is also studied but now in terms of its integrable properties, which means exact solvability. Deformed conformal theories have shown to give important results when translated into the description of a spin chain. The Hamiltonian of a spin chain can then be investigated to see if it is integrable. This will be discussed further in Chapter "Integrability and spin chains".

String theory, supergravity and D-branes

String theory and supergravity

String theory is a theory of one-dimensional objects which are moving in a higher dimensional space-time. A string can either be open or closed. A propagating open string sweeps out a two-dimensional surface which is called the world sheet. The world sheet of a closed string is formed as a tube. An open string can interact with another open or closed string. Two closed strings can also interact. The bosonic formulation of string theory requires the number of dimensions to be twenty-six in order to be consistent. Thus, bosonic string theory describes strings moving in a twenty-six-dimensional space-time. Quantization of the bosonic string gives, at the massless level, gauge fields such as the photon and the graviton. The photon is an open string state and the graviton is a closed string state.

Superstring theory is a supersymmetric string theory which thus contains both fermions and bosons. The number of space-time dimensions is reduced to ten, again in order to have a consistent theory.

There are five consistent ten-dimensional superstring theories — Type I $SO(32)$, Type IIA, Type IIB, Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$. These theories are essentially distinguished by how the boundary conditions of the fermions are chosen. The only theory which contains open strings is Type I string theory.

These five theories are conjectured to be related and descendant from a unique 11-dimensional theory — M-theory. The exact form of M-theory is not known, only a low-energy approximation is known. The five ten-dimensional theories are also only known in the weak coupling limit. What is interesting is that the perturbative string theories and M-theory contain theories of supergravity which is the theory of local supersymmetry.

Since local supersymmetry is a theory of general coordinate transformations of space-time it is therefore a theory of gravity. All theories of supergravity contains the graviton in their particle spectrum. Supergravity plays an important part in string theory, and forms the basis for how quantum field theory and a quantum theory of gravitation may be related. Supergravity can either be viewed, as has been said, as a locally defined supersymmetric theory or as an effective field theory which describes low-mass degrees of freedom of a more fundamental theory which is believed to be string theory.

Type IIB supergravity is the most interesting theory in this context, since it is the theory on the supergravity side of the AdS/CFT-correspondence, which will be discussed in the next chapter. In the following, we will discuss the massless closed string spectrum of Type IIB supergravity.

The modes on the vibrating closed string can either be left-moving or right-moving. Even if they are propagating on the same string they are treated as independent. These modes can either be periodic or anti-periodic when going around the closed string. When the fermionic field modes are periodic they are said to belong to the Ramond (R) sector. When they are anti-periodic they are said to be in the Neveu-Schwarz (NS) sector. It can be shown that the ground state in the Ramond sector is fermionic and Neveu-Schwarz sector is bosonic.

There are four possibilities to construct a closed string state from the two sectors. If both the left and right movers are bosonic NS-states we obtain a closed bosonic string state, called NS-NS state. When both the left and right moving modes are fermionic, the closed string state is again a bosonic state and is called R-R states. The two remaining possibilities are both fermionic and corresponds to the closed string states NS-R and R-NS.

In Type IIB supergravity, the bosonic NS-NS fields are a scalar, called the dilaton, the symmetric metric and an anti-symmetric B-field. The metric and the B-field are two-tensors. The bosonic R-R fields are tensor potentials which are usually denoted C_0, C_2 and C_4 where the index counts the number of indices. The femionic NS-R and R-NS sectors contain two gravitinos, with the same chirality, and two dilatinos.

The number of supersymmetry generators is the same as the number of gravitinos. Since there are two gravitinos, Type IIB supergravity is a ten-dimensional $\mathcal{N} = 2$ supergravity. Note that there is no vector potential, which would be the normal gauge potential. Thus, there is no gauge symmetry in a normal sense in the theory, so other forces in Nature than gravity are absent. The particles that mediate the forces in this theory are described by other types of tensor fields.

The bosonic NS-NS B-field, with two indices, is the generalization of the electromagnetic potential, the photon, which has only one index. When the photon propagates as a point-particle it forms an one-dimensional line. Integrating over this world-line, represented by the electromagnetic potential, we obtain the action for the photon. The string is an one-dimensional object which forms a two-dimensional world-sheet when propagating. The action is obtained by integrating over the world-sheet of the two-index B-field. This is to say that the string is a source of the electric B-field.

String theory is a theory of strings living in a ten-dimensional space-time. The world we perceive is four-dimensional with one time and three spatial dimensions. For string theory to be a description of our world, there are six dimensions to many. One way of solving this apparent contradiction is by making these extra dimensions small enough not to be seen. The ten-dimensional string is thus moving in our extended four-dimensional world and at the same time wrapping, bending and curling up in the six small and compact dimensions.

D-branes

As we have discussed, the string is the source for the B-field. In a famous paper by Joseph Polchinski[5], it was shown that D-branes are the sources for the bosonic R-R fields. In Type IIB supergravity, the R-R fields C_0, C_2 and C_4 are charges of the D-branes. D-branes are as fundamental as the string.

The D-brane is a dynamic object which can wrap and bend in the compact dimensions, just like the string. A D0-brane is a point and a D1-brane looks like an infinitely long string. A Dp-brane is a $p + 1$-dimensional object with one time dimension and p spatial dimensions. A Dp-brane is the source for the C_{p-1} R-R field. Type IIB supergravity therefore contains the D1-, D3- and the D5-brane.

In the low energy description of the D-brane, the brane is a rigid object and can be viewed as a flat hyperplane in space-time. The open string ends on the D-brane. The D3-brane has the world-volume of a four-dimensional infinite space-time. A ten-dimensional string which ends on a D3-brane therefore has four space-time coordinates parallel to the D-brane, in which the string can freely move. In the six transverse coordinates, the string is constrained to the D-brane. Mathematically, this means that the Neumann boundary conditions are replaced in the compact directions by Dirichlet boundary conditions. D-brane is short for Dirichlet-brane.

The understanding of how gauge theories are related to string theory came first after D-branes were introduced. When an open string ends on a D-brane the coordinates parallel to the brane can be regarded as a vector field and since there is a $U(1)$ symmetry, this vector field can be interpreted as a gauge vector field. The transverse coordinates are scalar fields. Thus, the open string sector introduces in a simple way gauge fields and scalar fields in the context of D-branes.

D-branes have many more special features. One is that a D-brane has mass and charge equal. This means that two D-branes can be pushed together without any external force. In other words, there is no force between two D-branes since the attractive force of gravity exactly cancels the repulsive force from R-R charge. When N D-branes are pulled together the $U(1)$ symmetry becomes an enhanced $U(N)$ gauge symmetry. Thus, the open string sector with N D3-branes on top of each other contains non-abelian $U(N)$ gauge fields living within the D-brane and six scalar fields in the adjoint representation. The low-energy theory of the D-brane is a non-abelian Yang-Mills gauge theory with $U(N)$ gauge symmmetry.

In Type IIB string theory there are no open strings. The closed strings can be viewed as excitations of the D-brane. The graviton can be emitted from the boundary into the directions transverse to the D-brane, propagate for a while and then disappear into the vacuum. Two D-branes interact by exchanging closed strings.

D-branes are also localized solutions to supergravity. Localized static solutions to classical fields equations with finite energy are a type of solitions. For example, solving Type IIB supergravity gives, among other solutions, that the D3-brane is represented as the R-R C_2 field and the ten-dimensional space-time metric. The more D3-branes the solution contains the more the space-time is curved. In short, the presence of D3-branes dictates the geometry of the four-dimensional space-time.

In Paper I the geometry of a special kind of D1-branes, called fractional

D1-branes, is found.

The AdS/CFT-correspondence

The open string sector on the D-brane describes super Yang-Mills gauge theory. The closed string sector contains supergravity solutions in terms of D-branes which dictates the geometry of space-time. The D-brane therefore has a two-folded interpretation — gauge theories and theories of gravitation. This duality can be understood from the properties of the string. The string is modular invariant. Modular invariance interchanges the two world-sheet parameters, one time and space, of the string so that the one-loop open string amplitude can also be viewed as a tree diagram of a closed string. From this symmetry we can understand that there should also be a duality between gauge theories (open strings) and gravity (closed strings). The first exact gauge/gravity-correspondence was presented by Maldacena in 1995. His statement is that $\mathcal{N} = 4$ super Yang-Mills in four dimensions is dual to Type IIB supergravity compactified on the ten-dimensional space-time $AdS_5 \times S^5$ [1]. This duality is called Maldacena's conjecture or the AdS/CFT-correspondence.

The $\mathcal{N} = 4$ super Yang-Mills theory is a conformal field theory (CFT) with gauge group $SU(N)$. The AdS/CFT-correspondence is only valid when N is very large. In the so called large-N limit the Feynman diagrams are very simple. Only planar diagrams, that is diagrams which can be drawn on a paper without any crossing lines, survive.

Type IIB supergravity is compactified on a very special geometry. Five of the ten coordinates have the geometry of a sphere. The remaining five coordinates have the geometry of a hyperboloid which is called an anti-de Sitter space (AdS_5). The boundary of the AdS_5 space is four-dimensional and without gravity. Loosely speaking, the correspondence says that the conformal super Yang-Mills theory is the same theory as string theory on the boundary of the anti-de Sitter space.

It is possible to extend the AdS/CFT-correspondence to more realistic field theories which are non-conformal and with less or no supersymmetry. See reference [6] for details and references within. This is achived by considering other D-brane configuration than the ordinary D-branes.

If there exists a small compact circle in the compactification space one can take a $D(p+2)$-brane and wrap it around the circle to a obtain a Dp-brane stuck at the circle. This new D-brane, in the limit of the vanishing of the radius of the circle, is called a fractional D-brane. By including fractional D-branes into the Type IIB supergravity it is possible to obtain solutions which corresponds to gauge theories with $\mathcal{N} = 2$, $\mathcal{N} = 1$ or no supersymmetry. These theories are all non-conformal and the fractional D-brane at the circle is responsible for the running of the coupling constant.

In Paper I, Type IIB supergravity in terms of fractional D1-branes is studied and perturbative features such as the running coupling constant on the gauge theory side are computed.

Integrability and spin chains

Since the AdS/CFT-correspondence is a strong/weak coupling duality it is very hard to test this conjecture. At strong coupling, the perturbation theory breaks down and no predictions can be made. However, there is a "hidden" subsector within the AdS/CFT-correnspondence where it is possible to find exact solutions on both sides of the correspondence. This subsector describes, on the supergravity side, a string which moves very fast around the equator of the compact five-sphere with large angular momentum J. This special double-scaling limit of large angular momentum J and large number of D-branes N is called the BMN-limit[2]. In this limit, the string becomes a plane wave which can be solved exactly in terms of a quantum mechanical system.

The AdS/CFT-correnspondence predicts that the energy of the string states on the supergravity side has the dual interpretation as the anomalous dimension of gauge theory operators on the gauge theory side. The anomalous dimension of an operator is usually hard to compute because of the operator mixing problem, that is that the operators are not orthogonal and therefore the eigenstates and eigenvalues are hard to find.

In the important paper [3], a technique was developed to compute the anomalous dimension for a large set of operators. The key idea was to write the fields in $\mathcal{N} = 4$ super Yang-Mills in terms of the Hamiltonian of an $SO(6)$ Heisenberg spin chain. This was achieved by considering the scalar fields of the chiral sector of $\mathcal{N} = 4$ super Yang Mills. The scalar fields transform under an internal $SO(6)$ symmetry. The main idea in [3] is to regard the operator as a one-dimensional lattice with J sites where each site host a six-dimensional real vector. This lattice forms an $SO(6)$ spin chain, or perhaps more correct an $SO(6)$ vector chain. It was shown that the Hamiltonian of the spin chain could be identified as the matrix of the anomalous dimension for a gauge theory operator.

The eigenstates and eigenvalues are usually not diagonal. To find these, the Hamiltonian has to be diagonalized. This can be done by using the algebraic Bethe ansatz. Obtaining the equations of the Bethe ansatz is a standard procedure in finding the eigenstates and eigenvalues. The strategy is rather technical and the details can be found in [7], but the chain of thoughts is the following. Once the Hamiltonian of a spin chain is found it can be rewritten as an R-matrix. The derivative of the R-matrix, with respect of a spectral parameter, is essentially the Hamiltonian. If this R-matrix satisfies the so called Yang-Baxter equation the Hamiltonian is known to be integrable, which means it is exactly solvable. Like the Hamiltonian, the R-matrix is acting on two adjacent vector sites of the spin chain. The Yang-Baxter equation contains a product of three matrices where each matrix is defined over three vector sites and is obtained from the R-matrix. This equation can be thought of as describing a three particle scattering and dictates the conditions of the specific momentum and scattering angles which are necessary for integrability.

The idea of translating a gauge theory into a spin chain has turned out to be very fruitful and has been extended to deformations of $\mathcal{N}=4$ SYM. In Paper II the integrability properties of the deformed $\mathcal{N}=4$ SYM of the general Leigh-Strassler deformation is studied. The corresponding Hamiltonian describes a spin-one spin chain. The integrability properties of this spin-chain is studied in terms of an R-matrix and some new integrability points in parameter-space is found.

Outline of the thesis

This thesis is based on three articles which cover three different topics related to the AdS/CFT-correspondence.

Paper I

The first article "The geometry of fractional D1-branes" provides solutions to Type IIB supergravity in terms of fractional D1-branes. Perturbative features such as the running coupling constant on the gauge theory side are computed.

Paper II

The second article "The general Leigh-Strassler deformation and integrability" translates marginal deformations of $\mathcal{N}=4$ super Yang-Mills theory in terms of $\mathcal{N}=1$ superfields including new cubic potential terms into a spin-chain. Properties of integrability of the corresponding dilation operator is studied. Some integrable points are found.

Paper III

The third article "Star product and the general Leigh-Strassler deformation" shows that the general Leigh-Strassler deformation is obtainable from a generalized version of the Lunin-Maldacena star product including three-parameter deformations. The conformal properties are discussed.

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The Geometry of Fractional D1-branes

Paper I

LU TP 04-13
March 2004

The Geometry of Fractional D1-branes

Daniel Bundzik[a,b] **and Anna Tollstén**[a]

[a] *School of Technology and Society*
Malmö University, Citadellsvägen 7, S 205 06 Malmö, Sweden
Daniel.Bundzik@ts.mah.se, Anna.Tollsten@ts.mah.se

[b] *Department of Theoretical Physics 2*
Lund University, Sölvegatan 14A, S 223 62 Lund, Sweden

abstract

We find explicit solutions of Type IIB string theory on $\mathbb{R}^4/\mathbb{Z}_2$ corresponding to the classical geometry of fractional D1-branes. From the supergravity solution obtained, we capture perturbative information about the running of the coupling constant and the metric on the moduli space of $\mathcal{N} = 4$, $D = 2$ Super Yang Mills.

1.1 Introduction

The success of gauge theories to describe interactions in QED and QCD indicate the possibility that all fundamental interactions in Nature are of gauge type. Despite many challenging results of non-perturbative field theories, calculations are stuck at the perturbative level. Progress in string theory has opened up new perspectives. As a consequence, new important perturbative and non-perturbative results have been obtained. Supersymmetric gauge theories can be seen as embedded in a higher dimensional string theory containing D-branes. On the one hand, the lightest open string excitations can be viewed as gauge fields living in the world volume of the D-brane. On the other hand, the lightest closed string modes correspond to D-brane solutions of supergravity. From the duality between the open string loop channel and the closed string tree channel one hence expects a possible relation between gauge theory and supergravity in general. The first exact gauge/gravity correspondence was conjectured by Maldacena, proposing that $\mathcal{N} = 4, D = 4$ Super Yang-Mills theory (SYM) is dual to Type IIB string theory compactified on $AdS_5 \times S^5$[1].

To extend the AdS/CFT correspondence to non-conformal theories with less supersymmetry, one can study string theories with wrapped D-brane configurations in the vicinity of singularities on orbifold or conifold backgrounds. The number of supercharges which are preserved, and hence the possible SYM theory, is decided by the details of the particular background. The way conformal invariance is broken depends on how the D-branes are wrapped around the singularity.

In order to study the wrapped D-branes alone, we should decouple all other states. Since the mass of a static, wrapped D-brane is proportional to the volume it encircles times the mass of the "normal" string states, we should make this volume very small. In the limit of vanishing volume these light, wrapped brane states become massless and correspond to particles in the uncompactified space-time. One only expects perturbative features of the gauge dual from this singular geometry. When keeping the volume finite non-perturbative effects, such as gaugino condensate and instanton effects, occur.

A general feature of fractional D3-branes on orbifold fixed points[2, 3, 4] or at conical singularities[5, 6], is the presence of naked singularities at small radial distance. The fractional brane acts as a source for a twisted field which represents the flux of an NS-NS two-form through the two-cycle. This twisted scalar field gets radial dependence and is interpreted, in the gauge dual, as the running coupling constant in the IR.

In some cases, the IR singularity can be avoided by considering wrapped D5-branes on non-vanishing Calabi-Yau two-spheres[7, 8, 9, 10], or deformed conifolds[11]. In both these situations, the gauge theory interpretation of chiral symmetry breaking and gaugino condensate is controlled by a single function in the gravitational counterpart. Moreover, it was shown in ref.[10] that the occurrence of non-perturbative instanton corrections in $\mathcal{N}=1$ SYM smooth out the running of the coupling constant in the IR and the theory is thus without divergences. For a more detailed discussion and complete reference list see for instance the review articles [12] and [13].

In this article we will use the gauge/gravity correspondence to study $\mathcal{N}=4$ SYM in $D=2$. In Section 2 we consider the action of Type IIB string theory on $\mathbb{R}^{1,5}\times\mathbb{R}^4/\mathbb{Z}_2$ using the wrapping ansatz for the fractional D1-brane. In Section 3 we find solutions to the equations of motion. These solutions can be expressed as a warp factor for the untwisted fields and a radially dependent function for the twisted fields. In Section 4 the singular fractional D1-brane geometry is probed. Before reaching the singularity the enhançon radius is revealed and the breakdown of supergravity is discussed. From the probe analysis the running Yang-Mills coupling constant is extracted. In Section 5 we show that the one-loop running gauge coupling for the two-dimensional gauge theory, using the background field method, is in exact agreement with the running coupling constant obtained from probe analysis. The explicit equations of motion can be found in the Appendix.

1.2 Action on the Orbifold

The action of Type IIB supergravity in ten dimensions can be written (in the Einstein frame) as[1]

$$\begin{aligned} S_{IIB} = \frac{1}{2\kappa_{10}^2}\Bigg\{\int d^{10}x\sqrt{-\det G}R - \frac{1}{2}\int \Big[& d\phi\wedge{}^*d\phi + e^{-\phi}H_{(3)}\wedge{}^*H_{(3)} \\ & + e^{2\phi}F_{(1)}\wedge{}^*F_{(1)} + e^{\phi}\widetilde{F}_{(3)}\wedge{}^*\widetilde{F}_{(3)} + \frac{1}{2}\widetilde{F}_{(5)}\wedge{}^*\widetilde{F}_{(5)} \\ & - C_{(4)}\wedge H_{(3)}\wedge F_{(3)}\Big]\Bigg\}, \end{aligned} \tag{1.1}$$

where the field strengths

$$H_{(3)} = dB_{(2)}\ , \qquad F_{(1)} = dC_{(0)}\ , \qquad F_{(3)} = dC_{(2)}\ , \qquad F_{(5)} = dC_{(4)}\ , \tag{1.2}$$

correspond to the NS-NS 2-form potential and the R-R 0-form, 2-form, and 4-form with

$$\widetilde{F}_{(3)} = F_{(3)} + C_{(0)}\wedge H_{(3)}, \qquad \widetilde{F}_{(5)} = F_{(5)} + C_{(2)}\wedge H_{(3)}. \tag{1.3}$$

[1]The conventions in this paper for curved indices and forms are: $\varepsilon_{0\ldots9}=+1$, signature $(-,+^9)$,
$\omega_{(n)} = \frac{1}{n!}\omega_{\mu_1\ldots\mu_n}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n}$ and ${}^*\omega_{(n)} = \frac{\sqrt{-detG}}{n!(10-n!)}\omega_{\mu_1\ldots\mu_n}\varepsilon^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_{10-n}}dx^{\nu_1}\wedge\ldots\wedge dx^{\nu_{10-n}}$

The field $\widetilde{F}_{(5)}$ is self-dual, i.e $\widetilde{F}_{(5)} = {}^*\widetilde{F}_{(5)}$, which condition can only be imposed on the equations of motion. The overall factor $\kappa_{10} = 8\pi^{7/2} g_s \alpha'^2$ is written in terms of the string coupling constant g_s and $\alpha' = l_s^2$ where l_s is the string scale.

Type IIB supergravity is now studied on the orbifold, $\mathbb{R}^{1,5} \times \mathbb{R}^4/\mathbb{Z}_2$. A fractional D1-brane arises when a D3-brane is wrapped on a compact 2-cycle of an ALE-manifold, wherupon we take the orbifold limit [14]. Although the cycles shrink to zero size in the orbifold limit the fractional brane can exist because the non-vanishing $B_{(2)}$-flux persists and is therefore keeping the brane tensionful [15, 16]. Since the four-form $C_{(4)}$ couples to the D3-brane, the "wrapping ansatz" for the fractional D1-brane is

$$B_{(2)} = b\omega_2, \qquad C_{(4)} = \widehat{C}_{(2)} \wedge \omega_2, \tag{1.4}$$

where ω_2 is the anti-self dual 2-form related to the vanishing 2-cycle at the orbifold fixed point. The scalar field b and the 2-form $\widehat{C}_{(2)}$ will be refered to as "twisted" fields since they correspond to the massless states of the twisted sector of Type IIB string theory on the orbifold.

The fractional branes are free to move in the flat transverse directions but are forced to stay on the fixed orbifold hyperplanes $x^{6,7,8,9} = 0$, and the corresponding twisted fields are functions of directions transverse to the orbifold, $\rho \equiv \sqrt{(x^2)^2 + \ldots + (x^5)^2}$. The bulk branes can move freely in the orbifold directions, and the untwisted fields are instead functions of directions transverse to the fractional D1-brane world-volume, i.e. $r \equiv \sqrt{(x^2)^2 + \ldots + (x^9)^2}$.

It is here appropriate to list the notation of indices used throughout this paper. The indices for the world-volume are denoted by $\alpha, \beta = 0, 1$. The transverse space $i, j = 2, \ldots, 9$ consists of four flat directions $a, b = 2, \ldots, 5$ plus four orbifold directions $\mu, \nu = 6, \ldots, 9$.

The fractional branes couple to closed string states. Using the boundary state formalism[2] one can determine which fields couple to the branes. In ref.[15] the authors study how fractional branes in general couple to boundary states and, in particular, it was found that, in the the fractional D1-brane case, the boundary state emits the NS-NS graviton G_{ij} and the dilaton ϕ and the R-R 2-form $C_{(2)}$ in the untwisted sector. In the twisted sector, the two-form $\widehat{C}_{(2)}$ and the scalar $\tilde{b}$ couple to the boundary. $\tilde{b}$ is the fluctuation part of b around the background value characteristic of the $\mathbb{Z}_2$ orbifold [18, 19], $b = 2\pi^2\alpha' + \tilde{b}$.

Inserting the "wrapping ansatz" (1.4) into the action of Type IIB string theory we obtain the action

[2]For a good review of the boundary state formalism see for an example ref.[17].

$$S_{orbifold} = \frac{1}{2\kappa_{orb}^2}\left\{\int d^{10}x\sqrt{-\det G}R - \frac{1}{2}\int_{10}\left[d\phi\wedge {}^*d\phi + e^{\phi}dC_{(2)}\wedge {}^*dC_{(2)}\right]\right.$$
$$\left. -\frac{1}{2}\int_6\left[e^{-\phi}d\widetilde{b}\wedge {}^{*6}d\widetilde{b} + \frac{1}{2}G_3\wedge {}^{*6}G_3 + \widehat{C}_{(2)}\wedge d\widetilde{b}\wedge dC_{(2)}\right]\right\} \quad (1.5)$$

on the orbifold. Here we have introduced $\kappa_{orb} = \sqrt{2}\kappa_{10}$ and an anti-self dual 3-form defined as $G_3 = d\widehat{C}_{(2)} + C_{(2)}\wedge db$. The anti-self dual orbifold 2-cycles are normalized to

$$\int \omega_2\wedge {}^*\omega_2 = -\int \omega_2\wedge\omega_2 = 1. \quad (1.6)$$

It is straightforward to show that the action (1.5) is consistent with the equations of motion of the full Type IIB supergravity.

The boundary action is

$$S_{boundary} = \frac{1}{2\kappa_{orb}^2}\left\{-\frac{2T_1\kappa_{orb}}{\sqrt{2}}\int dx^2 e^{-\phi/2}\sqrt{-\det G_{\alpha\beta}}\left(1+\frac{1}{2\pi^2\alpha'}\widetilde{b}\right)\right.$$
$$\left. +\frac{2T_1\kappa_{orb}}{\sqrt{2}}\int_{\mathcal{M}_2}\left[C_{(2)}\left(1+\frac{1}{2\pi^2\alpha'}\widetilde{b}\right)+\frac{1}{2\pi^2\alpha'}\widehat{C}_{(2)}\right]\right\} \quad (1.7)$$

where $T_p = \sqrt{\pi}(2\pi\sqrt{\alpha'})^{(3-p)}$ is the normalization of the boundary state related to the brane tension in units of the gravitational coupling constant and $\mathcal{M}_2$ is the world volume of the fractional brane.

1.3 The Ansatz and the Classical Solutions

To find the classical solution of the low-energy string effective action (1.5) with boundary term (1.7), we make the ansatz that the geometry of the fractional D1-brane is described with the extremal metric in the Einstein frame:

$$(ds)^2 = H^{-3/4}\eta_{\alpha\beta}dx^{\alpha}dx^{\beta} + H^{1/4}\delta_{ij}dx^i dx^j. \quad (1.8)$$

The harmonic function H is a function of the transverse world volume directions. The ansatz for the untwisted 2-form and the dilaton are

$$C_{(2)} = \left(\frac{1}{H}-1\right)dx^0\wedge dx^1, \qquad e^{\phi} = H^{1/2}, \quad (1.9)$$

while the twisted fields $\widehat{C}_{(2)}$ and scalar field b are assumed to have the form

$$\widehat{C}_{(2)} = f dx^0\wedge dx^1 + \widehat{C}_{ab}dx^a\wedge dx^b, \qquad b = f + \text{constant}. \quad (1.10)$$

The function f depends on the directions tranverse to the orbifold. The axion field $C_{(0)}$ is assumed to be zero in agreement with the "wrapping ansatz" (1.4) and the requirement $C_{\mu\nu} = 0$. This can be concluded from the equation of motion for the axion field.

The above ansatz implies that the solution is restricted to a subspace of the complete moduli space of solutions. To relax the self-consistent condition $\widehat{C}_{\alpha a} = C_{\alpha a} = 0$ might give a more general set of solutions. Note, however, that the components $\widehat{C}_{ab}$ differ from zero. This is a necessary requirement to sustain the anti-self duality of $G_{(3)}$ which leads to the condition

$$\partial_a \widehat{C}_{bc} + \partial_b \widehat{C}_{ca} + \partial_c \widehat{C}_{ab} = -\varepsilon_{abc}{}^d \partial_d f. \tag{1.11}$$

Solutions to this relation can be interpreted as a solitonic brane and look like generalized Dirac monopoles.

In the Appendix the equations of motion and more details on their solution are presented. The equations for the twisted fields $\tilde{b}$ and $\widehat{C}_{01}$ both give, after lengthy calculations,

$$\partial_a \partial^a f - K\delta^4(x^{2,\dots,5}) = 0. \tag{1.12}$$

The constant is $K = T_1 \kappa_{orb}/\sqrt{2}\pi^2\alpha'$. In a similar fashion, the equations for the untwisted fields; the metric G_{ij}, the dilaton ϕ and the the R-R 2-form C_{01} are all equivalent to

$$\partial_i \partial^i H + \partial_a f \partial^a f \delta^4(x^{6,\dots,9}) + \Delta\delta^8(x^{2,\dots,9}) = 0, \tag{1.13}$$

where $\Delta = \sqrt{2} T_1 \kappa_{orb}$. The singular terms of both equations are source terms coming from the boundary action truncated to the first order in the fluctuations around the background values of the fields.

To solve the equations (1.12) and (1.13) standard Fourier transform techniques are used with the resulting solutions

$$f(\rho) = -\frac{K}{(2\pi)^2}\frac{1}{\rho^2} \tag{1.14}$$

for the twisted fields and

$$\begin{aligned} H \;=\; & 1 + \frac{\Delta}{2\pi^4}\frac{1}{r^6} + \frac{K^2}{2\pi^6}\frac{1}{r^6}\left[\frac{1}{\epsilon^2} + 3\frac{r^2 - 2\rho^2}{r^4}\ln\frac{(r^2-\rho^2)\epsilon^2}{r^4}\right. \\ & \left. + \frac{2}{r^2} - \frac{10\rho^2}{r^4} + \frac{1}{2(r^2-\rho^2)}\right] \end{aligned} \tag{1.15}$$

for the untwisted fields. The presence of the cutoff ϵ reflects the unknown short-distance physics in directions transverse to the orbifold. Another indication of this unknown physics is the presence of the enhançon radius which is discussed in the following section.

It is interesting to note that although the warp factor, H, appears to differ very much from the expression in the case of fractional D3-branes[2], they are actually very similar. They both contain one term which is just the spherical solution to the Laplacian in $9-p$ dimensions with a point source, and the remaining terms originate from the same expression in terms of a $5-p$-dimensional integral

$$\frac{\Gamma\left(\frac{7-p}{2}\right)\Gamma^2\left(\frac{5-p}{2}\right)}{16\pi^{\frac{19-3p}{2}}}\int\frac{d^{5-p}u}{u^{8-2p}((u-x)^2+r^2-\rho^2)^{\frac{7-p}{2}}} \tag{1.16}$$

where $(u-x)^2=\delta_{ab}(u^a-x^a)(u^b-x^b)$. It would be interesting to find out if this solution is valid for fractional Dp-branes in general.

1.4 Probe analysis of the fractional brane solution

In this section we wish to relate our result to the non-conformal extension of the gauge/gravity-correspondence and to compare the supergravity solution with the low-energy dynamics of the gauge theory living on the fractional brane. The previously found background, consisting of N fractional D1-branes, is approached by a slowly moving fractional D1-brane probe. To find the gauge/gravity-relations we identify the gauge theory Higgs field Φ^a with the transverse directions x^a on the supergravity side through the relation $x^a=2\pi\alpha'\Phi^a$. The probe action is defined as the boundary action (1.7) expanded to second order in the Higgs field. Expressed in static gauge, $x^0=\xi^0$, $x^1=\xi^1$ and $x^a=\xi^a(x^0)$, the probe action becomes

$$S_{probe}=-\frac{T_1}{4\kappa_{10}}V_2-(2\pi\alpha')^2\frac{T_1}{4\kappa_{10}}\left(1+\frac{\tilde{b}N}{2\pi^2\alpha'}\right)\int d^2\xi\frac{1}{2}\partial_\alpha\Phi^a\partial^\alpha\Phi^b\delta_{ab}. \tag{1.17}$$

The first term is a constant, which shows that all position dependent terms have cancelled. This is related to the fact that fractional branes are BPS states and hence there is no force between the probe and the source. The second order term survives which enables us to define a non-trivial four-dimensional metric on the moduli space

$$g_{ab}=(2\pi\alpha')^2\frac{T_1}{4\kappa}\left(1+\frac{\tilde{b}N}{2\pi^2\alpha'}\right)\delta_{ab}=\frac{\pi\alpha'}{2g_s}\left(1-\frac{4g_s\alpha'N}{\rho^2}\right)\delta_{ab}. \tag{1.18}$$

In the last step we have inserted our explicit solution (1.13). From the second term in the probe action (1.17), which is interpreted as the gauge field kinetic term, the running of the Yang-Mills coupling constant can be extracted. It equals

$$\frac{1}{g_{YM}^2(\rho)} = \frac{1}{g_{YM}^2(\infty)} \left(1 - g_{YM}^2(\infty)\frac{2\pi\alpha'^{\,2}N}{\rho^2}\right), \tag{1.19}$$

where the bare coupling constant is defined as $g_{YM}^2(\infty) = 2g_s/\pi\alpha'$. If we now change the scale parameter to $\rho = 2\pi\alpha'\mu$, we obtain the running coupling constant

$$\frac{1}{g_{YM}^2(\mu)} = \frac{1}{g_{YM}^2(\infty)} \left(1 - g_{YM}^2(\infty)\frac{N}{2\pi\mu^2}\right). \tag{1.20}$$

for our fractional D1-branes. In the following section this result will be compared to the running coupling constant of $\mathcal{N} = 4$, $D = 1 + 1$ super Yang-Mills theory.

To end this section we note that when the probe approaches the radius $\rho = \rho_e$ where

$$\rho_e = \sqrt{4g_s\alpha' N}, \tag{1.21}$$

the metric (1.18) on the moduli space vanishes which means that the fractional brane becomes tensionless. This is the enhançon radius [20]. For values $\rho < \rho_e$ the tension becomes negative and hence undefined. The supergravity solutions can not be trusted inside the enhançon radius. If we insert the value for ρ_e into the solution (1.14) for the $\widetilde{b}$ field, we find it equal to the background value for the b field with opposite sign. This means that at the enhançon radius the b field vanishes. If we express the Yang-Mills coupling constant in terms of the b-field

$$\frac{1}{g_{YM}^2(\rho)} = \frac{1}{4\pi g_s}\int_{\Sigma_2} B_{(2)} = \frac{b}{4\pi g_s}, \tag{1.22}$$

we see that at the enhançon radius the coupling constant g_{YM} goes to infinity. To overcome this artifact one should remember that the supergravity action is truncated to first order. This suggests that when the probe approaches the enhançon radius new physical degrees of freedom, which extrapolate the reliability of supergravity to smooth geometries, become important. The lack of a trustworthy solution inside the enhançon radius means that we cannot predict the infrared behavior of the dual non-conformal gauge theory within this framework.

1.5 The running coupling constant of $\mathcal{N} = 4$, $D = 2$ SYM

The background field method is an efficient approach to calculate the Yang-Mills one-loop running gauge coupling for a D-dimensional field theory. The standard procedure is to write down a Lagrangian, gauge invariant even after gauge-fixing, with a background external gauge field which is a solution to the classical field equations. From the effective action[21]

$$S_{eff} = \frac{1}{4}\int d^D x F^2 \left(\frac{1}{g_{YM}^2} + I\right), \tag{1.23}$$

the quadratic terms in the gauge fields can then be extracted with

$$I = \frac{1}{(4\pi)^{D/2}}\int_0^\infty ds \frac{e^{-\mu^2 s}}{s^{D/2-1}} R. \tag{1.24}$$

Here μ is the mass of the fields and

$$R = 2\left[\frac{N_s}{12}c_s + \frac{D-26}{12}c_v + \frac{2^{[D/2]}N_f}{6}c_f\right]. \tag{1.25}$$

The bracket means the integer part, that is $[D/2] = D/2$ if D is even and $[D/2] = (D-1)/2$ if D is odd. The constants c are the normalization of the generators of the gauge group with $Tr(\lambda^a\lambda^b) = c\delta^{ab}$ and depend on the representation under which the scalars, vectors and fermions transform. N_s and N_f are the numbers of scalars and Dirac fermions in the theory.

For the specific case of fractional D1-branes, there are $N_s = 4$ scalars and $N_f = 2$ Dirac fermions in a $\mathcal{N}=4$, $D=1+1$ super Yang-Mills theory. If we choose the gauge group to be $SU(N)$ the scalars and Dirac fermions are in the adjoint representation which implies that $c_s = c_v = c_f = N$. With all this in hand, we find for $D = 1+1$

$$I = \frac{1}{4\pi}\int_0^\infty ds\, e^{-\mu^2 s} R = \frac{1}{4\pi\mu^2} R \tag{1.26}$$

with $R = -2N$. This means that $I = -N/2\pi\mu^2$. The running gauge coupling constant is then

$$\frac{1}{g_{YM}^2(\mu)} = \frac{1}{g_{YM}^2(\infty)}\left(1 - g_{YM}^2(\infty)\frac{N}{2\pi\mu^2}\right) \tag{1.27}$$

which is in exact agreement with what we previously found from the fractional D1-brane solution.

We can also calculate the β-function,

$$\beta(g_{YM}(\mu)) \equiv \mu\frac{\partial g_{YM}(\mu)}{\partial\mu} = g_{YM}(\infty)\left(\frac{g_{YM}(\mu)}{g_{YM}(\infty)} - \left(\frac{g_{YM}(\mu)}{g_{YM}(\infty)}\right)^3\right) \tag{1.28}$$

which has a UV-stable fixed point at $g_{YM}(\infty)$.

1.6 Discussion

We have shown that perturbative features of $\mathcal{N} = 4$ super Yang-Mills in two-dimensions are qualitatively inherent in the obtained supergravity solutions for the fractional D1-brane. The running of the coupling constant is governed by the twisted b-field which represents the flux of the NS-NS two-form through the compactification two-cycle. When the geometry is studied at sub-stringy length scales, the probe becomes tensionless before reaching the singularity. At the enhançon radius the b-field vanishes and supergravity is no longer a trustworthy description. It would be interesting to study this short-distance physics further, in context of wrapped D3-branes where the singular orbifold is replaced by a non-vanishing two-sphere. One expects, in a similar manner as in ref.[7], that identifying the spin connection with an external gauge field would give a (4,4) SYM theory in D=1+1 with a corresponding gravity dual free of singularities. The running of the coupling constant is now dependent on the volume of the two-sphere rather than the b-field. The abelian topological twist should be performed in the UV regime but becomes, presumably, non-abelian in the IR which smooths out the geometry of the supergravity solutions. This enhanced gauge symmetry have been studied for wrapped D5-branes[8, 9, 10] and it would be interesting to see if wrapped D3-branes share the same behaviour and account for non-perturbative results such as gaugino condensate and instanton effects.

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1.7 Appendix

In this appendix more details of the calculations are presented. the equations of motion obtained from the action (1.5) with boundary terms (1.7) are presented. Inserting the ansatz (1.8)-(1.10) in these equations yield the equations (1.13) and (1.12).

The equation of motion for the field $\widehat{C}_{(2)}$ is

$$d^{*_6}G_3 - db \wedge dC_{(2)} + 2K\Omega_4 = 0, \tag{1.29}$$

where

$$\begin{aligned} G_{(3)} &= d\widehat{C}_{(2)} + C_{(2)} \wedge db \\ K &= T_1 \kappa_{orb}/(\sqrt{2}\pi^2\alpha') \\ \Omega_4 &= \delta(x^2)\ldots\delta(x^5)dx^2 \wedge \ldots \wedge dx^5. \end{aligned} \tag{1.30}$$

The equation of motion for the field b is

$$d(e^{-\phi}\, {}^{*_6}db) + G_{(3)} \wedge dC_{(2)} - KV_2\Omega_4 = 0 \tag{1.31}$$

where V_2 spans the world-volume of the fractional D1-brane.

The equation of motion for the field $C_{(2)}$ is

$$d(e^{\phi *_{10}} dC_{(2)}) + db \wedge d\widehat{C}_{(2)} \wedge \widetilde{\Omega}_4 + \Delta\Omega_8 = 0 \tag{1.32}$$

where

$$\begin{aligned} \Delta &= \sqrt{2}T_1\kappa_{orb} \\ \widetilde{\Omega}_4 &= \delta(x^6)\ldots\delta(x^9)dx^6 \wedge \ldots \wedge dx^9 \\ \Omega_8 &= \delta(x^2)\ldots\delta(x^9)dx^2 \wedge \ldots \wedge dx^9. \end{aligned} \tag{1.33}$$

The equation of motion for the dilaton ϕ is

$$d^{*_{10}}d\phi - \frac{1}{2}e^{\phi}dC_{(2)} \wedge {}^{*_{10}}dC_{(2)} + \frac{1}{2}e^{-\phi}db \wedge {}^{*_6}db \wedge \widetilde{\Omega}_4 + \frac{1}{2}\Delta V_2\Omega_8 = 0. \tag{1.34}$$

To obtain the equation of motion for the metric is not quite so simple. It is more convenient to use the language of components instead of forms. The equation can symbolically be expressed as $R_{MN} = L_{MN}$ where the left-hand side is the Ricci tensor with ten-dimensional indices. There are three cases to consider; when the indices are $\alpha, \beta = 0, 1,\ a, b = 2, \ldots, 5$ and $\mu, \nu = 6, \ldots, 9$ (remember that $i, j = 2, \ldots, 9$). The result is

$$R_{\alpha\beta} = \frac{3}{8}H^{-2}\left(\partial_k\partial^k H - H^{-1}\partial_k H\partial^k H\right)\eta_{\alpha\beta}, \tag{1.35}$$

$$R_{ij} = -\frac{3}{8}H^{-2}\partial_i H\partial_j H - \frac{1}{8}\left(H^{-1}\partial_k\partial^k H - H^{-2}\partial_k H\partial^k H\right)\delta_{ij}, \tag{1.36}$$

and

$$\begin{aligned} L_{\alpha\beta} = & \Bigg(-\frac{3}{8}H\partial_k C_{01}\partial^k C_{01} - \frac{1}{8}H^{-2}\partial_c b\partial^c b\ \delta^4(x) \\ & -\frac{1}{4}G_{c01}G^c{}_{01}\delta^4(x) - \frac{3}{8}\Delta\delta^8(x)\Bigg)\eta_{\alpha\beta}, \end{aligned} \tag{1.37}$$

$$
\begin{aligned}
L_{ab} &= \frac{1}{2}\partial_a\phi\partial_b\phi \\
&\quad -\frac{1}{2}H^2\partial_a C_{01}\partial_b C_{01} + \frac{1}{8}H^2\partial_k C_{01}\partial^k C_{01}\delta_{ab} \\
&\quad +\frac{1}{2}H^{-1}\partial_a b\,\partial_b b\delta^4(x) - \frac{1}{8}H^{-1}\partial_c b\partial^c b\,\delta^4(x)\,\delta_{ab} \\
&\quad -\frac{1}{2}HG_{a01}G_{b01}\delta^4(x) + \frac{1}{4}HG_{c01}G^c{}_{01}\delta^4(x)\,\delta_{ab} \\
&\quad +\frac{1}{8}\Delta\delta^8(x)\delta_{ab},
\end{aligned} \tag{1.38}
$$

$$
\begin{aligned}
L_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}H^2\partial_\mu C_{01}\partial_\nu C_{01} + \frac{1}{8}H^2\partial_k C_{01}\partial^k C_{01}\delta_{\mu\nu} \\
&\quad +\frac{1}{8}H^{-1}\partial_c b\partial^c b\,\delta^4(x)\,\delta_{\mu\nu} + \frac{1}{8}\Delta\delta^8(x)\delta_{\mu\nu}.
\end{aligned} \tag{1.39}
$$

Combining these relations in an appropriate manner gives the same equation (1.13) just like the equations for $C_{(2)}$ and ϕ do.

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The general Leigh-Strassler deformation and Integrability

Paper II

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The general Leigh-Strassler deformation and Integrability

Daniel Bundzik
School of Technology and Society, Malmö University,
Östra Varvsgatan 11A, S-205 06 Malmö, Sweden
Department of Theoretical Physics, Lund University,
Sölvegatan 14A, S-223 62, Sweden
E-mail: Daniel.Bundzik@ts.mah.se
Teresia Månsson
NORDITA, Blegdamsvej 17, DK-2100 Copenhagen, Denmark
E-mail: teresia@nordita.dk

abstract

The success of the identification of the planar dilatation operator of $\mathcal{N} = 4$ SYM with an integrable spin chain Hamiltonian has raised the question if this also is valid for a deformed theory. Several deformations of SYM have recently been under investigation in this context. In this work we consider the general Leigh-Strassler deformation. For the generic case the S-matrix techniques cannot be used to prove integrability. Instead we use R-matrix techniques to study integrability. Some new integrable points in the parameter space are found.

KEYWORDS: AdS-CFT correspondence, Integrable field theories, bethe ansatz

2.1 Introduction

In the last few years, several new discoveries have shed light on the AdS/CFT correspondence [1, 2, 3]. This correspondence maps strings moving in an $AdS_5 \times S^5$ background to an $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. The eigenvalues of the dilatation operator are mapped to the energies of closed string states [4]. A step in understanding this duality better was the discovery that the dilatation operator of the $\mathcal{N} = 4$ SYM is proportional to the Hamiltonian of an integrable spin chain [5, 6, 7].

Recently, progress has been made to extend the gauge/gravity-correspondence, in context of spin chains, towards more realistic models with less supersymmetry [8, 9, 10, 11, 12]. For instance, if the background geometry for the string is $AdS_5 \times W$, where W is some compact manifold, the dual gauge theory should still be conformal. Other geometries, mainly orbifolds of $AdS_5 \times S^5$, corresponding to non-conformal theories have also been investigated [13, 14].

The success in using spin chains to study the duality beyond the BMN limit motivates studies of integrability of deformed correlators. One question that naturally arises in this context is whether integrability is related to supersymmetry, conformal invariance or have more geometrical reasons.

The Leigh-Strassler deformations [15] preserve $\mathcal{N} = 1$ supersymmetry and conformal invariance, at least up to one loop. It is hence of great interest to investigate if there exist points in the parameter space where the dilatation operator is mapped to an integrable spin-chain Hamiltonian. This question has been under investigation in [16, 17, 18, 19]. In [17] this deformation was studied in a special case corresponding to a q-deformed (often called β-deformed) commutator. It was found that for the sector with three chiral fields the dilatation operator is integrable for q equals root of unity.

In reference [20], a way of generating supergravity duals to the β-deformed field theory was introduced, and in [8, 9, 21] agreement between the supergravity sigma model and the coherent state action coming from the spin chain describing the β-deformed dilatation operator was demonstrated. This way of creating supergravity duals was used in [22] to construct a three-parameter generalization of the β-deformed theory. The gauge theory dual to this supergravity solution was found in [22, 18] for $q = e^{i\gamma_j}$ with γ_j real, corresponding to certain phase deformations in the Lagrangian. This gauge theory is referred to as twisted SYM, from which the β-deformed theory is obtained when all the $\gamma_j = \beta$. The result is that the theory is integrable for any $q = e^{i\gamma_j}$ with γ_j real [18]. The general case with complex γ_j is not integrable [17, 19].

In the present work, the q-deformed analysis is extended to the more general Leigh-Strassler deformations with an extra complex parameter h, in order to find new integrable theories. A site dependent transformation is found which relates the γ_j-deformed case to a site dependent spin-chain Hamiltonian with nearest-neighbour interactions. In particular when all γ_j are equal, the transformation relates the q-deformed theory to the h-deformed theory, *i.e.* the theory only involving the parameter h. In particular, we find a new R-matrix, at least in the context of $\mathcal{N} = 4$ SYM, for $q = 0$ and $h = e^{i\theta}$ with θ real. We also find all R-matrices with a linear dependence on the spectral parameter which give the dilatation operator. A general ansatz for the R-matrix is given. Unfortunately, the most general solution is not found. However, we find a new hyperbolic R-matrix which corresponds to a basis-transformed Hamiltonian with only diagonal entries [19]. A reformulation of the general R-matrix shows that the structure of the equations obtained from the Yang-Baxter equations resemble the equations obtained in the eight vertex model. This gives a clear hint how to proceed.

In the dual supergravity theory, some attempts to construct backgrounds for non-zero h have been done [23, 24]. Apart from the five-flux there is also a three-flux. A step going beyond supergravity was taken in [25] where the BMN limit was considered. We hope our results will make it easier to find the supergravity dual of the general Leigh-Strassler deformed theory.

2.2 Marginal deformations of $\mathcal{N}$=4 supersymmetric Yang-Mills

To study marginal deformations of $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group, it is convenient to use $\mathcal{N} = 1$ SYM superfields. The six real scalar fields of the $\mathcal{N} = 4$ vector multiplet are combined into the lowest order terms of three complex $\mathcal{N} = 1$ chiral superfields Φ_0, Φ_1 and Φ_2. It is well known that the $\mathcal{N} = 1$ superpotential

$$W_{\mathcal{N}=1} = \frac{1}{3!} C^{IJK}_{abc} \Phi^a_I \Phi^b_J \Phi^c_K \,, \tag{2.1}$$

where C^{IJK}_{abc} is the coupling constant, describes a finite theory at one-loop if the following two conditions are fulfilled [26, 27]

$$3C_2(G) = \sum_I T(A_I), \qquad \text{and} \qquad C^{IKL}_{acd} \bar{C}^{bcd}_{JKL} = 2g^2 T(A_I) \delta_a{}^b \delta^I{}_J. \tag{2.2}$$

The constant $C_2(G)$ is the quadratic Casimir operator defined here[1] as $C_2(G) \cdot \mathbf{1} = \delta_{ab} T^a_A T^b_A$ where A is the adjoint representation of the group G which in the present context is the symmetry group $SU(N)$. The constant $T(M)$ is defined through $T(M)\delta^{ab} = \mathrm{Tr}(T^a_M T^b_M)$ for the representation M. The first condition

[1]Our conventions are: T^a are the $SU(N)$ group generators, satisfying $T^a = T^{a\dagger}$. The normalization of T^a is given by $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$ from where it follows that $\mathrm{Tr}(T^a_A T^b_A) = N\delta^{ab}$ in the adjoint representation A.

of (2.2) implies that the β-function is zero. For an $SU(N)$ group with the superpotential (2.1) this is automatically fulfilled. The choice $C^{IJK}_{abc} = g\varepsilon^{IJK} f_{abc}$ therefore gives a superconformal $\mathcal{N} = 1$ theory at one-loop. However, there are more general superpotentials satisfying the one-loop finiteness conditions. To explore marginal deformations of $\mathcal{N} = 4$ SYM we consider the Leigh-Strassler superpotential [15]

$$W = \frac{1}{3!}\lambda\varepsilon^{IJK}\mathrm{Tr}\left[\left[\Phi_I, \Phi_J\right]\Phi_K\right] + \frac{1}{3!}h^{IJK}\mathrm{Tr}\left[\left\{\Phi_I, \Phi_J\right\}\Phi_K\right], \qquad (2.3)$$

where h^{IJK} is totaly symmetric. The coupling constants can now be written as $C^{IJK}_{abc} = \lambda\varepsilon^{IJK} f_{abc} + h^{IJK}\mathrm{Tr}\left[\{T_a, T_b\}\, T_c\right]$. The non-zero couplings are chosen to be $h^{012} = \lambda(1-q)/(1+q)$ and $h^{III} = 2\lambda h/(1+q)$. In terms of the deformation parameters q and h the superpotential (2.3) becomes

$$W = \frac{2\lambda}{1+q}\mathrm{Tr}\left[\Phi_0\Phi_1\Phi_2 - q\Phi_1\Phi_0\Phi_2 + \frac{h}{3}\left(\Phi_0^3 + \Phi_1^3 + \Phi_2^3\right)\right]. \qquad (2.4)$$

This deformed superpotential will be our main focus.

The presence of q and h in the superpotential (2.4) breaks the $SU(3)$ symmetry in the chiral sector. What is left of the symmetry is a $Z_3 \times Z_3$ symmetry. The first Z_3 permutes the Φ's and the second takes $\Phi_0 \to \omega\Phi_0$, $\Phi_1 \to \omega^2\Phi_1$ and $\Phi_2 \to \Phi_2$, where ω is a third root of unity.

The one-loop finiteness condition (2.2) is satisfied if

$$g^2 = \frac{\lambda^2}{(1+q)^2}\left[(1+q)^2 + \left((1-q)^2 + 2h^2\right)\left(\frac{N^2-4}{N^2}\right)\right]. \qquad (2.5)$$

In the large-N limit, which we consider, the relation (2.5) becomes even more simple. The one-loop finiteness condition (2.2) also implies that the scalar field self-energy contribution from the fermion loop is the same as in the $\mathcal{N} = 4$ scenario, due to the fact that the fermion loop has the contraction $C^{IKL}_{acd}\bar{C}^{bcd}_{JKL}$. The parameters in (2.5) span a space within which there exists a manifold, or perhaps just isolated points, $\beta(g, \lambda, q, h) = 0$ of superconformal theories to all loops [15]. In the limit $q \to 1$ and $h \to 0$ the $\mathcal{N} = 4$ SYM is restored. Marginal deformations away from this fixed point will be explored in the following sections by means of integrable spin chains.

2.3 Dilatation operator

From the Leigh-Strassler deformation (2.4) of the $\mathcal{N} = 4$ SYM theory it is possible to obtain the dilatation operator in the chiral sector. In this sector, the only contribution is coming from the F-term in the Lagrangian, under the assumption that the one-loop finiteness condition (2.2) is fulfilled. The scalar field part of the F-term can be expressed in terms of the superpotential as

$$\mathcal{L}_F = \left|\frac{\partial W}{\partial \Phi_0}\right|^2 + \left|\frac{\partial W}{\partial \Phi_1}\right|^2 + \left|\frac{\partial W}{\partial \Phi_2}\right|^2 . \tag{2.6}$$

Using ϕ_0, ϕ_1 and ϕ_2 to denote the complex component fields, the Lagrangian becomes (omitting the overall factor $2\lambda/(1+q)$ in (2.4))

$$\begin{aligned} \mathcal{L}_F &= \mathrm{Tr}\left[\phi_i\phi_{i+1}\bar{\phi}_{i+1}\bar{\phi}_i - q\phi_{i+1}\phi_i\bar{\phi}_{i+1}\bar{\phi}_i - q^*\phi_i\phi_{i+1}\bar{\phi}_i\bar{\phi}_{i+1}\right] \\ &+ \mathrm{Tr}\left[qq^*\phi_{i+1}\phi_i\bar{\phi}_i\bar{\phi}_{i+1} - qh^*\phi_{i+1}\phi_i\bar{\phi}_{i+2}\bar{\phi}_{i+2} - q^*h\phi_{i+2}\phi_{i+2}\bar{\phi}_i\bar{\phi}_{i+1}\right] \\ &+ \mathrm{Tr}\left[h\phi_{i+2}\phi_{i+2}\bar{\phi}_{i+1}\bar{\phi}_i + h^*\phi_i\phi_{i+1}\bar{\phi}_{i+2}\bar{\phi}_{i+2} + hh^*\phi_i\phi_i\bar{\phi}_i\bar{\phi}_i\right], \end{aligned} \tag{2.7}$$

where a summation over $i = 0, 1, 2$ is implicitly understood and the indices of the fields ϕ_i are identified modulo three. In order to see how the dilatation operator acts on a general operator $O = \psi^{i_1\cdots i_L}\mathrm{Tr}\phi_{i_1}\ldots\phi_{i_L}$ to first loop order in the planar limit we calculate the Feynman graphs and regularize in accordance with [16, 17]. The vector space, spanning these operators, can be mapped to the vector space of a spin-1 chain (see [5] for details). We define the basis states $|0\rangle$, $|1\rangle$ and $|2\rangle$ for the spin chain which correspond to the fields ϕ_0, ϕ_1 and ϕ_2. By introducing the operators E_{ij}, which act on the basis states as $E_{ij}|k\rangle = \delta_{jk}|i\rangle$, the dilatation operator can be written as a spin-chain Hamiltonian with nearest-neighbour interactions, *i.e.* $\Delta = \sum_l H^{l,l+1}$ where

$$\begin{aligned} H^{l,l+1} &= E^l_{i,i}E^{l+1}_{i+1,i+1} - qE^l_{i+1,i}E^{l+1}_{i,i+1} - q^*E^l_{i,i+1}E^{l+1}_{i+1,i} \\ &+ qq^*E^l_{i+1,i+1}E^{l+1}_{i,i} - qh^*E^l_{i+1,i+2}E^{l+1}_{i,i+2} - q^*hE^l_{i+2,i+1}E^{l+1}_{i+2,i} \\ &+ hE^l_{i+2,i}E^{l+1}_{i+2,i+1} + h^*E^l_{i,i+2}E^{l+1}_{i+1,i+2} + hh^*E^l_{i,i}E^{l+1}_{i,i}. \end{aligned} \tag{2.8}$$

The direct product between the operators E_{ij} is suppressed. If we use the convention

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.9}$$

the Hamiltonian can be expressed as the matrix

$$H^{l,l+1} = \begin{pmatrix} h^*h & 0 & 0 & 0 & 0 & h & 0 & -q^*h & 0 \\ 0 & 1 & 0 & -q^* & 0 & 0 & 0 & 0 & h^* \\ 0 & 0 & q^*q & 0 & -qh^* & 0 & -q & 0 & 0 \\ 0 & -q & 0 & q^*q & 0 & 0 & 0 & 0 & -qh^* \\ 0 & 0 & -q^*h & 0 & h^*h & 0 & h & 0 & 0 \\ h^* & 0 & 0 & 0 & 0 & 1 & 0 & -q^* & 0 \\ 0 & 0 & -q^* & 0 & h^* & 0 & 1 & 0 & 0 \\ -qh^* & 0 & 0 & 0 & 0 & -q & 0 & q^*q & 0 \\ 0 & h & 0 & -q^*h & 0 & 0 & 0 & 0 & h^*h \end{pmatrix} . \tag{2.10}$$

We will now search for special values of the parameters h and q for which the spin-chain Hamiltonian (2.8) is integrable. When h is absent, the analysis simplifies considerably, because the usual S-matrix techniques can be used [17, 19, 28]. The existence of a homogeneous eigenstate, an eigenstate of the form $|\,a\rangle\otimes\,|\,a\rangle\ldots\otimes\,|\,a\rangle$, is crucial for the S-matrix techniques to work. From this reference state, excitations can be defined. In this context, the state $|a\rangle$ is a pure state, that is, one of the states $|0\rangle$, $|1\rangle$ or $|2\rangle$.

When h is non-zero, the analysis become significantly harder. The only values for the parameters, for which it is possible to define a homogeneous eigenstate are $q = 1 + e^{i2\pi n/3}h$ or $q = -1$ and $h = e^{i2\pi n/3}$, where n is an arbitrary integer. In these cases the homogeneous eigenstates are

$$|\,a\rangle =|\,0\rangle + e^{\frac{i2\pi m}{3}}\,|\,1\rangle + e^{-\frac{i2\pi m}{3}}\,|\,2\rangle, \qquad m \in Z\,. \tag{2.11}$$

Clearly, the two Z_3 symmetries are manifest. For $q = 1+he^{i2\pi n/3}$, the eigenvalues are zero, thus the corresponding states are protected. This case is related to the q-deformed Hamiltonian by a simple change of variables. We introduce a new basis

$$\begin{aligned} |0\rangle &= \frac{e^{\frac{i2\pi n}{3}}}{\sqrt{3}}(|\tilde{0}\rangle + |\tilde{1}\rangle + \tilde{2}\rangle)\,, \\ |1\rangle &= \frac{1}{\sqrt{3}}(|\widetilde{0}\rangle + e^{\frac{i2\pi}{3}}|\widetilde{1}\rangle + e^{-\frac{i2\pi}{3}}|\widetilde{2}\rangle)\,, \\ |2\rangle &= \frac{1}{\sqrt{3}}(|\widetilde{0}\rangle + e^{-\frac{i2\pi}{3}}|\widetilde{1}\rangle + e^{\frac{i2\pi}{3}}|\widetilde{2}\rangle), \end{aligned} \tag{2.12}$$

where n is an integer. It will shortly be shown that the phase shift in $|0\rangle$ will imply that a phase $e^{\pm i2\pi/3}$ can be transformed away from h. The Hamiltonian expressed in the new basis (2.12) takes the same form as (2.8), but with new parameters $\tilde{q}$ and $\tilde{h}$ and an overall proportionality factor

$$e^{\frac{i2\pi}{3}} - qe^{-\frac{i2\pi}{3}} + he^{-\frac{i2\pi n}{3}}\,. \tag{2.13}$$

The new parameters $\tilde{q}$ and $\tilde{h}$ can then be expressed in terms of the old parameters as

$$\tilde{q} = \frac{qe^{\frac{i2\pi}{3}} - e^{\frac{-i2\pi}{3}} - he^{-\frac{i2\pi n}{3}}}{e^{\frac{i2\pi}{3}} - qe^{\frac{-i2\pi}{3}} + he^{-\frac{i2\pi n}{3}}} \tag{2.14}$$

$$\tilde{h} = \frac{1 - q + he^{-\frac{i2\pi n}{3}}}{e^{\frac{i2\pi}{3}} - qe^{\frac{-i2\pi}{3}} + he^{-\frac{i2\pi n}{3}}}. \tag{2.15}$$

The case $q = he^{-i2\pi n/3} + 1$ corresponds to the q-deformed case, and if h's imaginary part comes from the phase $e^{i2\pi n/3}$, the remaining part is phase independent. This is in agreement with reference [29]. The integrable case q equals a phase will correspond to the case $q = he^{-i2\pi n/3} + 1$ with $h = \rho e^{i2\pi n/3}$ with ρ and q being real. It is also clear that the case $q = -1$ and $h = e^{i2\pi n/3}$ is related by the change of basis to a Hamiltonian of the form

$$H = \sum_i 3\left[E^i_{22}E^{i+1}_{22} + E^i_{00}E^{i+1}_{00} + E^i_{11}E^{i+1}_{11}\right] . \tag{2.16}$$

This case looks perhaps trivial, but it is not. The different eigenvalues equal $3n$ with $n = 0, 1, 2, \ldots, L-2, L$. Note that the value $L-1$ is absent for this periodic spin chain[2]. The states have a large degree of degeneration.

For other values of q and h, a reference state does not have a precise meaning. Hence, we cannot adapt the S-matrix formalism. Instead, we will try to find an R-matrix, from which the Hamiltonian (2.8) is obtainable. The existence of an R-matrix $R(u)$, depending on the spectral parameter u, is sufficient to ensure integrability. All R-matrices necessarily have to satisfy the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{2.17}$$

The Hamiltonian can be obtained from the R-matrix through the following relation

$$\mathcal{P}\frac{d}{du}R(u)|_{u=u_0} = H\,, \tag{2.18}$$

where $\mathcal{P}$ is the permutation operator, with the additional requirement $R(u_0) = \mathcal{P}$ for some point $u = u_0$.

2.4 A first look for integrability

In this section, we will show how the transformation of basis (2.12) combined with a position dependent phase shift, sometimes called a twist, gives rise to

[2]Excitations can be created if two states of the same number are next to each other. For example the state $|112012\rangle$ has energy three and the next highest energy state is $|111112\rangle$ with energy 4×3. The state with the highest energy, equals to 6×3, is $|111111\rangle$.

new interesting cases of integrability. In [17], the q-deformed case was studied. It was shown that for q equals a root of unity, the phases can be transformed away into the boundary conditions. Furthermore, it was shown in [18] that the integrability properties do not get affected for any $q = e^{i\beta}$, where β is real. It was also established that a generalised SYM Lagrangian deformed with three phases γ_i, instead of just one variable, is integrable. The deformed theory is referred to as the twisted (or γ-deformed) SYM and the corresponding one-loop dilatation operator in the three scalar sector is

$$\begin{aligned} H^{l,l+1} &= \left[E^l_{00}E^{l+1}_{11} + E^l_{11}E^{l+1}_{22} + E^l_{22}E^{l+1}_{00}\right] \\ &- \left[e^{i\gamma_1}E^l_{10}E^{l+1}_{01} + e^{i\gamma_2}E^l_{21}E^{l+1}_{12} + e^{i\gamma_3}E^l_{02}E^{l+1}_{20}\right] \\ &- \left[e^{-i\gamma_1}E^l_{01}E^{l+1}_{10} + e^{-i\gamma_2}E^l_{12}E^{l+1}_{21} + e^{-i\gamma_3}E^l_{20}E^{l+1}_{02}\right] \\ &+ \left[E^l_{11}E^{l+1}_{00} + E^l_{22}E^{l+1}_{11} + E^l_{00}E^{l+1}_{22}\right] . \end{aligned} \tag{2.19}$$

A natural question to ask is if the phases can also be transformed away in a generic Hamiltonian of the form (2.8). If both q and h are present we can not, at least in any simple way, transform away the phase of the complex variables. However, when $q = re^{\pm 2\pi i/3}$ it is possible to do a position dependent coordinate transformation

$$|\tilde{0}\rangle_k = e^{i2\pi/3}|0\rangle_k\,, \qquad |\tilde{1}\rangle_k = e^{i2k\pi/3}|1\rangle_k\,, \qquad \text{and} \qquad |\tilde{2}\rangle_k = e^{-i2k\pi/3}|2\rangle_k\,, \tag{2.20}$$

as in [17][3] so that the phase of q is transformed away. Here, k refers to the site of the spin-chain state. This transformation changes the generators in the Hamiltonian as

$$\tilde{E}^l_{n,n+m} = e^{\frac{i2\pi ml}{3}}E^l_{n,n+m}\,. \tag{2.21}$$

This kind of transformation of basis generally results in twisted boundary conditions. Thus, the periodic boundary condition $|a\rangle_0 = |a\rangle_L$ for the original basis becomes in the new basis

$$|\tilde{0}\rangle_0 = |\tilde{0}\rangle_L\,, \qquad |\tilde{1}\rangle_0 = e^{\frac{i2\pi L}{3}}|\tilde{1}\rangle_L\,, \qquad \text{and} \qquad |\tilde{2}\rangle_0 = e^{\frac{-i2\pi L}{3}}|\tilde{2}\rangle_L\,, \tag{2.22}$$

where L is the length of the spin chain. A consequence is that the system is invariant under a rotation of q by introducing appropriate twisted boundary conditions (2.22). As an example, the q-deformed Hamiltonian with periodic boundary conditions with $q = he^{i2\pi n/3}+1$ (see text above (2.16)), is equivalent to $qe^{i2\pi m/3} = he^{i2\pi n/3} + 1$ with twisted boundary conditions. Hence, the following cases are integrable

$$h = \rho e^{\frac{i2\pi n}{3}}\,, \quad q = (1+\rho)e^{\frac{i2\pi m}{3}} \qquad \text{and} \qquad q = -e^{\frac{i2\pi m}{3}}\,, \quad h = e^{\frac{i2\pi n}{3}}\,, \tag{2.23}$$

[3]Note that the phase factor in $|0\rangle$ is not position dependent, it was only added in order to cancel the extra phase which would have appeared in front of the terms having h in them.

where ρ is real and can take both negative and positive values and n and m are arbitrary independent integers.

One can actually combine the twist transformation above with the shift of basis (2.12) in a non-trivial way. This combination will turn out to give a relation which maps the Hamiltonian with arbitrary q and vanishing h into the Hamiltonian with vanishing q and arbitrary h. The periodic boundary condition will, however, change for spin chains where the length is not a multiple of three.

In terms of matrices the transformation can be represented as follows. Let us represent the shift of basis (2.12) by the matrix T (with n set to zero)

$$T = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i2\pi/3} & e^{-i2\pi/3} \\ 1 & e^{-i2\pi/3} & e^{i2\pi/3} \end{pmatrix}, \tag{2.24}$$

and the transformation matrix related to the phase shift (2.20) by (but without the phase-shift in the zero state $|0\rangle$)

$$U_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i2\pi k/3} & 0 \\ 0 & 0 & e^{-i2\pi k/3} \end{pmatrix}. \tag{2.25}$$

The transformation that takes the q-deformed to the h-deformed Hamiltonian is then

$$\widetilde{H} = T_1 H T_1^{-1}, \tag{2.26}$$

where

$$T_1 = (T \otimes T)(U_k \otimes U_{k+1})(T^{-1} \otimes T^{-1}). \tag{2.27}$$

Acting with this transformation on the Hamiltonian (2.8) we get the new Hamiltonian

$$\begin{aligned} \widetilde{H}^{l,l+1} &= q^* q E^l_{i,i} E^{l+1}_{i+1,i+1} - h q^* E^l_{i+1,i} E^{l+1}_{i,i+1} - h^* q E^l_{i,i+1} E^{l+1}_{i+1,i} \\ &+ h h^* E^l_{i+1,i+1} E^{l+1}_{i,i} + h E^l_{i+1,i+2} E^{l+1}_{i,i+2} + h^* E^l_{i+2,i+1} E^{l+1}_{i+2,i} \\ &- q^* E^l_{i+2,i} E^{l+1}_{i+2,i+1} - q E^l_{i,i+2} E^{l+1}_{i+1,i+2} + E^l_{i,i} E^{l+1}_{i,i}, \end{aligned} \tag{2.28}$$

Up to an overall factor, the transformation (2.26) change the couplings as

$$q \neq 0 \quad \text{and} \quad h = 0 \quad \Longleftrightarrow \quad \tilde{q} = 0 \quad \text{and} \quad \tilde{h} = -1/q \tag{2.29}$$

In terms of states, the map (2.26) generates the following change

$$|a\rangle_{1+3k} \to |a-1\rangle_{1+3k}, \quad |a\rangle_{2+3k} \to |a+1\rangle_{2+3k}, \quad |a\rangle_{3k} \to |a\rangle_{3k}, \tag{2.30}$$

where a takes the values 0, 1 or 2. Let us investigate how the transformation (2.28) affect the boundary conditions. From equation (2.30) we see that the original periodic boundary conditions $|a\rangle_0 = |a\rangle_L$ translate into

$$|0^{new}\rangle_0 = |2^{new}\rangle_L, \qquad |1^{new}\rangle_0 = |0^{new}\rangle_L \qquad \text{and} \qquad |2^{new}\rangle_0 = |1^{new}\rangle_L \,, \tag{2.31}$$

if the length L of the spin chain is one modulo three and the opposite, $|0^{new}\rangle_0 = |1^{new}\rangle_L$ *etc*, for the two modulo three case. If the length is a multiple of three the boundary conditions remain the same.

If we start from the Hamiltonian of the γ-deformed SYM (2.19), the transformation (2.26) leads to the Hamiltonian

$$\begin{aligned} H^{l,l+1} &= \left[E^l_{00}E^{l+1}_{11} + E^i_{11}E^{l+1}_{22} + E^i_{22}E^{l+1}_{00}\right] \\ &- \left[e^{-i\gamma_{3-l}}E^l_{20}E^{l+1}_{21} + e^{-i\gamma_{1-l}}E^l_{01}E^{l+1}_{02} + e^{-i\gamma_{2-l}}E^l_{12}E^{l+1}_{10}\right] \\ &- \left[e^{i\gamma_{3-l}}E^l_{02}E^{l+1}_{12} + e^{i\gamma_{1-l}}E^l_{10}E^{l+1}_{20} + e^{i\gamma_{2-l}}E^l_{21}E^{l+1}_{01}\right] \\ &+ \left[E^l_{00}E^{l+1}_{00} + E^l_{11}E^{l+1}_{11} + E^l_{22}E^{l+1}_{22}\right] . \end{aligned} \tag{2.32}$$

This Hamiltonian describes interactions which differ from systems we have previously encountered, since here the interactions are site dependent. This behavior shows up naturally in a non-commutative theory. In [18], it was discussed that the γ-deformed SYM corresponds to a form of non-commutative deformation of $\mathcal{N} = 4$ SYM.

If all the γ_i are equal, the Hamiltonian above will corresponds to our original Hamiltonian (2.8) with $q = 0$ and $h = e^{i\theta}$. The associated R-matrix is

$$\begin{aligned} R(u) &= \left[E^i_{01}E^{i+1}_{10} + E^i_{12}E^{i+1}_{21} + E^i_{20}E^{i+1}_{02}\right] \\ &- ue^{-i\theta}\left[E^i_{20}E^{i+1}_{21} + E^i_{01}E^{i+1}_{02} + E^i_{12}E^{i+1}_{10}\right] \\ &- ue^{i\theta}\left[E^i_{12}E^{i+1}_{02} + E^i_{20}E^{i+1}_{10} + E^i_{01}E^{i+1}_{21}\right] \\ &+ \left[E^i_{00}E^{i+1}_{00} + E^i_{11}E^{i+1}_{11} + E^i_{22}E^{i+1}_{22}\right] \\ &+ (1-u)\left[E^i_{10}E^{i+1}_{01} + E^i_{21}E^{i+1}_{12} + E^i_{02}E^{i+1}_{20}\right] . \end{aligned} \tag{2.33}$$

We have checked explicit that (2.33) satisfies the Yang-Baxter equation. This means that the theory is integrable!

In the rest of this section we will discuss the spectrum when the spin-chain Hamiltonian (2.8) is either q-deformed or h-deformed. Figure 2.1 shows the spectrum for a three-site spin-chain Hamiltonian. The left graph shows how the energy depends on the phase ϕ, with $q = e^{i\phi}$ and $h = 0$. The right graph shows instead how the eigenvalues vary as a function $\tilde{\theta}$, when $\tilde{h} = e^{i\tilde{\theta}}$ and $\tilde{q} = 0$.

Figure 2.2 shows the same spectra for a four-site spin chain. All graphs

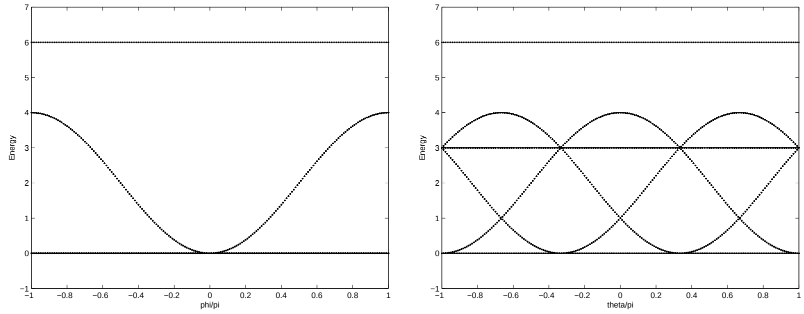

Figure 2.1: Spin chain with three sites. The left graph shows the energy spectrum as a function of the phase ϕ, when $q = e^{i\pi\phi}$ and $h = 0$. The right graph shows the spectrum as a function of the phase $\tilde{\theta}$, when $\tilde{h} = e^{i\tilde{\theta}}$ and $\tilde{q} = 0$.

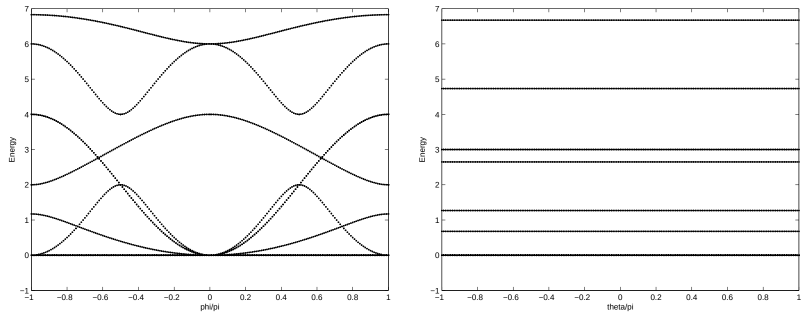

Figure 2.2: Spin chain with four sites. The left graph shows the energy spectrum as a function of the phase ϕ, when $q = e^{i\pi\phi}$ and $h = 0$. The right graph shows the spectrum as a function of the phase $\tilde{\theta}$, when $\tilde{h} = e^{i\tilde{\theta}}$ and $\tilde{q} = 0$.

contain energies which are the eigenvalues of several states. Highly degenerate states are generally a sign of integrability because they reflect a large number of symmetries in the theory.

Let us start by explaining the spectra in Figure 2.1. When h is zero there is only one sinus curve while when q is zero there are three sinus curves. The reason is the transformation (2.29), since it maps $q = e^{i\phi}$ and $h = 0$ into $\tilde{h} = e^{i\tilde{\theta}}$ and $\tilde{q} = 0$ with the relation of the phases $\tilde{\theta} = \pi - \phi + 2\pi n/3$. Therefore, for each value of q there exist several values of $\tilde{h}$ which differ by a phase $2\pi/3$. For $q = 0$, there is a state, independent of the phase, with energy three. This state is absent for $h = 0$. One example of such a state is $|000\rangle - |111\rangle$. The "inverse" transformation, see (2.30), of this state is $|120\rangle - |201\rangle$, which is zero due to periodicity.

The four-site spin chain (see Figure 2.2) differs substantially from the spin chain with three sites. The case $q = 0$ is completely phase-independent. The reason is the boundary conditions. Actually, spin chains with the number of sites differing from multiples of three will have spectra which do not depend on the phase. It will just coincide with the spectra for the case $q = e^{-i2\pi/3}$ and $h = 0$. Starting with the case q equal to a root of unity it is possible to make a transformation, changing the boundary conditions, such that the phase of q is removed [17]. The change in the boundary conditions is then

$$|0^o\rangle_0 = |0^o\rangle_L\,, \qquad |1^o\rangle_0 = e^{i\Phi}|1^o\rangle_L \qquad \text{and} \qquad |2^o\rangle_0 = e^{-i\Phi}|2^o\rangle_L\,, \tag{2.34}$$

where Φ is a phase factor, the exact form of which is not important for our purposes. The effect (2.34) has on the boundary conditions (2.31) is, when L is one modulo three,

$$|0^{new}\rangle_0 = |2^{new}\rangle_L, \;\; |1^{new}\rangle_0 = e^{i\Phi}|0^{new}\rangle_L \;\; \text{and} \;\; |2^{new}\rangle_0 = e^{-i\Phi}|1^{new}\rangle_L. \tag{2.35}$$

If we make the shift $|1^{new}\rangle \to e^{i\Phi}|1^{new}\rangle$ we see that this corresponds to the boundary conditions (2.31). The same procedure can be made when L is two modulo three. This means that any q equal to root of unity[4] can be mapped to any $\tilde{h}$ with the phase $\tilde{\theta} = \pi + 2\pi p/n + 2\pi m/3$. All values of h will then give the same energy spectrum due to the fact that p,n and m are arbitrary integer numbers, so the possible values of $\tilde{\theta}$ will in principle fill up the real axis. This implies that the energy must be the same for all values of $\tilde{\theta}$. For $q = e^{-i2\pi/3}$ and $h = 0$ there is a direct map (see (2.14)) to the case $q = 0$ and $h = -e^{2\pi m/3}$ which does not change the boundary conditions. The energy spectra for these two cases must be the same. Consequently, the spectra for "all" points coincide with the spectrum of $q = e^{-i2\pi/3}$ and $h = 0$. The fact that the shape of the eigenvalue distribution changes drastically depending on how many sites there are suggests that a well-defined large L-limit does not exist. However, it might still be possible to find a well-defined large L-limit if only L-multiples of three is considered.

2.5 R-matrix

We will now try to make a general ansatz for an R-matrix which has the possibility to give rise to our Hamiltonian (2.8). A linear ansatz will turn out to lead to the cases we found in the previous section. To find a new solution the ansatz need to be more complicated, for instance consisting of hyper-elliptic

[4] $q = e^{i\phi}$ is a root of unity iff $n\phi = 0 \bmod 2\pi$ for n an integer. The phase is then $\phi = 2\pi p/n$ where p is an integer.

functions. We are interested in an R-Matrix of the following form

$$\begin{aligned} R(u) &= aE_{i,i} \otimes E_{i,i} + bE_{i,i} \otimes E_{i+1,i+1} + \bar{b}E_{i+1,i+1} \otimes E_{i,i} \\ &+ cE_{i,i+1} \otimes E_{i+1,i} + \bar{c}E_{i+1,i} \otimes E_{i,i+1} \\ &+ dE_{i+1,i+2} \otimes E_{i,i+2} + \bar{d}E_{i+2,i+1} \otimes E_{i+2,i} \\ &+ eE_{i+2,i} \otimes E_{i+2,i+1} + \bar{e}E_{i,i+2} \otimes E_{i+1,i+2}, \end{aligned} \tag{2.36}$$

where the coefficients are functions of a spectral parameter u.

Written on matrix form the R-matrix is

$$R = \begin{pmatrix} a & 0 & 0 & 0 & 0 & e & 0 & \bar{d} & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 & e \\ 0 & 0 & \bar{b} & 0 & d & 0 & \bar{c} & 0 & 0 \\ 0 & \bar{c} & 0 & \bar{b} & 0 & 0 & 0 & 0 & d \\ 0 & 0 & \bar{d} & 0 & a & 0 & e & 0 & 0 \\ \bar{e} & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ 0 & 0 & c & 0 & \bar{e} & 0 & b & 0 & 0 \\ d & 0 & 0 & 0 & 0 & \bar{c} & 0 & \bar{b} & 0 \\ 0 & e & 0 & \bar{d} & 0 & 0 & 0 & 0 & a \end{pmatrix}. \tag{2.37}$$

A natural first step to look for a R-matrix solution is to make a linear ansatz which will give the Hamiltonian (2.8) as in (2.18).

The Hamiltonian can also be defined through the permuted $\mathfrak{R}$-matrix

$$\mathfrak{R} \equiv \mathcal{P}R, \tag{2.38}$$

where $\mathcal{P}$ is the 9×9 permutation matrix.

If $R(u)|_{u=u_0} = \mathcal{P}$ or $\mathfrak{R}(u)|_{u=u_0} = \mathcal{P}$, the Hamiltonian is obtained as

$$H = \mathcal{P}\frac{d}{du}R(u)|_{u=u_0} \quad \text{or} \quad H = \mathcal{P}\frac{d}{du}\mathfrak{R}(u)|_{u=u_0}. \tag{2.39}$$

The linear ansatz below has the property that it gives the Hamiltonian (2.8) in accordance with the first formula in (2.39)

$$\begin{aligned} a(u) &= (h^*h - k)u + \alpha\,, \qquad b(u) = -qu\,, \qquad \bar{d}(u) = -q^*hu\,, \\ c(u) &= (q^*q - k)u + \alpha\,, \qquad \bar{b}(u) = -q^*u\,, \qquad e(u) = hu\,, \\ \bar{c}(u) &= (1 - k)u + \alpha\,, \qquad d(u) = h^*u\,, \qquad \bar{e}(u) = -qh^*u\,, \end{aligned} \tag{2.40}$$

with k and α being free parameters, the Yang-Baxter equations turn out to be independent of α while they demand k to be $k = \frac{1}{2}(1 + h^*h + q^*q)$. Inserting the linear ansatz in the Yang-Baxter equation we find that the equation is satisfied either if

$$q = e^{i\phi} \text{ and } h = 0 \quad \text{or} \quad q = 0 \text{ and } h = e^{i\theta}\,, \tag{2.41}$$

where ϕ and θ can be any phase, or if the following equations holds

$$e^{i3\phi}r = \left(1+\rho e^{i3\theta}\right) , \tag{2.42}$$

$$e^{-i3\phi}r = \left(1+\rho e^{-i3\theta}\right) , \tag{2.43}$$

$$r = \pm(1\pm\rho) , \tag{2.44}$$

where we used the notation that $q = re^{i\phi}$ and $h = \rho e^{i\theta}$ and let r and ρ be any real numbers. Here we immediately see that the relations between the real parts of q and h are given by the last equation, hence we only need to consider which angles are not in contradiction to that. The solution is

$$q = re^{i2\pi n/3} , \qquad h = (1+r)e^{i2\pi m/3} , \tag{2.45}$$

where we once again let r take any real number. Now we would like to see whether there exist solutions if an ansatz is made with the permuted version of the R-matrix ansatz (2.40). We obtain

$$\begin{aligned} a(u) &= (h^*h-k)u+\alpha , & c(\mu) &= -q^*u , & e(u) &= hu , \\ b(u) &= (1-k)u+\alpha , & \bar{c}(u) &= -qu , & \bar{d}(u) &= -q^*hu , \\ \bar{b}(u) &= (q^*q-k)u+\alpha , & d(u) &= -qh^*u , & \bar{e}(u) &= h^*u . \end{aligned} \tag{2.46}$$

The conditions from the Yang-Baxter equation read

$$q^* = -q^2 , \qquad h^* = h^2 , \tag{2.47}$$

with no restriction on k and α. The only solution to this is

$$q = -e^{2\pi n/3} , \qquad h = e^{2\pi m/3} , \tag{2.48}$$

(or $q = 0$ and $h = 0$). This is the other type of solution we expected from the last section. The one corresponding to $q = -1$ and $h = e^{i2\pi m/3}$ and that one but with twisted boundary conditions. Hence a R-matrix with a linear dependence on the spectral parameter u can not give us more integrable cases than already found. We need a more general R-matrix solution to find new interesting cases.

2.5.1 Symmetries revealed

In order to address the problem of finding the most general solution for the R-matrix (2.37) it is an advantage to make use of the symmetries. We choose the representation

$$R = \sum_{i=1}^{3}(\omega_i T_i \otimes S_i + \bar{\omega}_i S_i \otimes T_i + \gamma_i E_i \otimes E_{2i}) . \tag{2.49}$$

All indices in this section are modulo three if not otherwise stated. The generators S_i, T_i and E_i are

$$S_1 = \begin{pmatrix} 0 & 0 & 1 \\ e^{-\frac{i2\pi}{3}} & 0 & 0 \\ 0 & e^{\frac{i2\pi}{3}} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ e^{\frac{i2\pi}{3}} & 0 & 0 \\ 0 & e^{-\frac{i2\pi}{3}} & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$T_1 = \begin{pmatrix} 0 & e^{\frac{i2\pi}{3}} & 0 \\ 0 & 0 & e^{-\frac{i2\pi}{3}} \\ 1 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & e^{-\frac{i2\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{i2\pi}{3}} \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{i2\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{i2\pi}{3}} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{i2\pi}{3}} & 0 \\ 0 & 0 & e^{-\frac{i2\pi}{3}} \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

How the functions in the R-Matrix (2.37) are expressed in terms of the functions ω_i, $\bar{\omega}_i$ and γ_i can be found in Appendix (2.66). The generators are related by

$$\begin{array}{lll} S_kS_l = e^{-i\frac{2\pi(l-k)}{3}}T_{2k-l} & S_kT_l = E_{k-l} & S_kE_l = e^{i\frac{2\pi l}{3}}S_{k+l} \\ T_kS_l = e^{-\frac{i2\pi(l-k)}{3}}E_{l-k} & T_kT_l = e^{-i\frac{2\pi(l-k)}{3}}S_{2k-l} & T_kE_l = T_{k-l} \\ E_kS_l = S_{k+l} & E_kT_l = e^{i\frac{2\pi}{3}k}T_{l-k} & E_kE_l = E_{k+l} \end{array} \tag{2.50}$$

Using these relations it is straightforward to obtain the Yang-Baxter equations which can be found in Appendix (2.67). A nice feature of these equations is that all of them, except the fourth, the fifth and the sixth, can be generated from the first equation through the cyclic permutations $\omega_{n+1} \to \bar{\omega}_{n+1} \to \gamma_3$ and $\gamma_2 \to \omega_n \to \bar{\omega}_n \to \gamma_1 \to \omega_{n+2} \to \bar{\omega}_{n+2}$. The remaining three equations are related to each other by the same cyclic permutation. The structure of the equations (2.67) is similar to the Yang-Baxter equations in the eight vertex model [30, 31]

$$\omega_n\omega_l'\omega_j'' - \omega_l\omega_n'\omega_k'' + \omega_j\omega_k'\omega_n'' - \omega_k\omega_j'\omega_l'' = 0\ , \tag{2.51}$$

for all cyclic permutations (j, k, l, n) of (1,2,3,4). These equations can neatly be represented by writing the elements in rectangular objects

$\omega_{\mathbf{n}}$	ω_l	$\omega_{\mathbf{j}}$	ω_k
ω_l	$\omega_{\mathbf{n}}$	ω_k	$\omega_{\mathbf{j}}$
$\omega_{\mathbf{j}}$	ω_k	$\omega_{\mathbf{n}}$	ω_l
ω_k	$\omega_{\mathbf{j}}$	ω_l	$\omega_{\mathbf{n}}$
$\omega_{\mathbf{n}}$	ω_l	$\omega_{\mathbf{j}}$	ω_k

. (2.52)

Note the beautiful toroidal pattern. The object above should be interpreted as follows. The first three rows represent the equation (2.51) with the first column representing the first term in (2.51)

$$\begin{array}{|c|}\hline \omega_{\mathbf{n}} \\ \hline \omega_l \\ \hline \omega_{\mathbf{j}} \\ \hline \end{array} = \omega_n \omega'_l \omega''_j \,, \tag{2.53}$$

and the next column is equal to the second term in (2.51)

$$\begin{array}{|c|}\hline \omega_l \\ \hline \omega_{\mathbf{n}} \\ \hline \omega_k \\ \hline \end{array} = -\omega_l \omega'_n \omega''_k \,. \tag{2.54}$$

The next three rows represent another equation of eight vertex model

$$\begin{array}{|c|c|c|c|}\hline \omega_l & \omega_{\mathbf{n}} & \omega_k & \omega_{\mathbf{j}} \\ \hline \omega_{\mathbf{j}} & \omega_k & \omega_{\mathbf{n}} & \omega_l \\ \hline \omega_k & \omega_{\mathbf{j}} & \omega_l & \omega_{\mathbf{n}} \\ \hline \end{array} = \omega_l \omega'_j \omega''_k - \omega_n \omega'_k \omega''_j + \omega_k \omega'_n \omega''_l - \omega_j \omega'_l \omega''_n = 0 \,. \tag{2.55}$$

Our equations can also be represented in terms of similar rectangular objects, with the same toroidal pattern

$$\begin{array}{|c|c|c|c|c|c|}\hline \omega_{\mathbf{2}} & \omega_1 & \bar{\omega}_{\mathbf{2}} & \gamma_1 & \gamma_{\mathbf{3}} & \bar{\omega}_3 \\ \hline \omega_1 & \omega_{\mathbf{2}} & \gamma_1 & \bar{\omega}_{\mathbf{2}} & \bar{\omega}_3 & \gamma_{\mathbf{3}} \\ \hline \gamma_{\mathbf{3}} & \gamma_1 & \omega_{\mathbf{2}} & \bar{\omega}_3 & \bar{\omega}_{\mathbf{2}} & \omega_1 \\ \hline \gamma_1 & \gamma_{\mathbf{3}} & \bar{\omega}_3 & \omega_{\mathbf{2}} & \omega_1 & \bar{\omega}_{\mathbf{2}} \\ \hline \bar{\omega}_{\mathbf{2}} & \bar{\omega}_3 & \gamma_{\mathbf{3}} & \omega_1 & \omega_{\mathbf{2}} & \gamma_1 \\ \hline \end{array} \,. \tag{2.56}$$

The first three rows give the second equation in (2.67) with $n = 3$. The next three rows are the seventh equation in (2.67) with $n = 1$. This suggests that the system of equations (2.67) should have a nice solution, just like the eight vertex model. The first row determines the rest of the entries, thus all equations can be represented with just the upper row. Hence, all the 36 equations can be represented by the following rows

$$\begin{array}{c}
\begin{array}{|c|c|c|c|c|c|}\hline \omega_{n+1} & \omega_n & \bar{\omega}_{n+1} & \gamma_1 & \gamma_3 & \bar{\omega}_{n+2} \\ \hline \end{array} \\[4pt]
\begin{array}{|c|c|c|c|c|c|}\hline \omega_n & \omega_{n+1} & \bar{\omega}_n & \gamma_2 & \gamma_3 & \bar{\omega}_{n+2} \\ \hline \end{array} \\[4pt]
\begin{array}{|c|c|c|c|c|c|}\hline \bar{\omega}_2 & \omega_2 & \bar{\omega}_1 & \omega_1 & \bar{\omega}_3 & \omega_3 \\ \hline \end{array} \\[4pt]
\begin{array}{|c|c|c|c|c|c|}\hline \omega_{n+1} & \omega_n & \bar{\omega}_n & \gamma_2 & \gamma_1 & \bar{\omega}_{n+1} \\ \hline \end{array} \\[4pt]
\begin{array}{|c|c|c|c|c|c|}\hline \omega_2 & \gamma_2 & \omega_1 & \gamma_1 & \omega_3 & \gamma_3 \\ \hline \end{array} \\[4pt]
\begin{array}{|c|c|c|c|c|c|}\hline \gamma_1 & \bar{\omega}_2 & \gamma_2 & \bar{\omega}_1 & \gamma_3 & \bar{\omega}_3 \\ \hline \end{array}
\end{array} \tag{2.57}$$

The solution to the eight vertex model is a product of theta functions. The cyclicity and periodicity properties of the eight vertex model is mirrored into the rectangular object. Due to the combination of addition theorems for theta functions and the intrinsic properties of the equations, the rectangular objects make it easy to see if an ansatz solves all the equations. We believe that the addition theorems for theta functions generating the solution of the eight vertex model should be possible to generalize to any even sized rectangular object. It would then be interesting to see if those equations are related to an R-matrix of arbitrary dimension.

2.5.2 A hyperbolic solution

If the following ansatz $\omega_i = e^{uQ_i}$, $\bar{\omega}_i = e^{u\bar{Q}_i}$ and $\gamma_i = e^{uK_i}$, where we let Q_i, $\bar{Q}_i$ and K_i be arbitrary constants is made, it leads us to the following solution

$$\begin{aligned}
\omega_1 &= e^{Q_1 u}\,, & \bar{\omega}_1 &= e^{Q_2 u}\,, & \gamma_1 &= e^{Q_2 u}\,,\\
\omega_2 &= e^{Q_2 u}\,, & \bar{\omega}_2 &= e^{Q_1 u}\,, & \gamma_2 &= e^{Q_1 u}\,,\\
\omega_3 &= e^{Q_3 u}\,, & \bar{\omega}_3 &= e^{Q_3 u}\,, & \gamma_3 &= e^{Q_3 u}\,.\,,
\end{aligned} \tag{2.58}$$

The following Hamiltonian is obtained from the above R-matrix solution

$$\begin{aligned}
H^{l,l+1} = {} & E^l_{i,i} \otimes E^{l+1}_{i+1,i+1} + e^{i\phi} E^l_{i+1,i} \otimes E^{l+1}_{i,i+1} + e^{-i\phi} E^l_{i,i+1} \otimes E^{l+1}_{i+1,i}\\
& + E^l_{i+1,i+1} \otimes E^{l+1}_{i,i} + e^{-i\phi} E^l_{i+1,i+2} \otimes E^{l+1}_{i,i+2} + e^{i\phi} E^l_{i+2,i+1} \otimes E^{l+1}_{i+2,i}\\
& + e^{-i\phi} E^l_{i+2,i} \otimes E^{l+1}_{i+2,i+1} + e^{i\phi} E^l_{i,i+2} \otimes E^{l+1}_{i+1,i+2} + E^l_{i,i} \otimes E^{l+1}_{i,i}\,.
\end{aligned} \tag{2.59}$$

where $e^{i\phi} = (Q_2 e^{i2\pi/3} + Q_1 e^{-i2\pi/3})/(Q_1^2 + Q_2^2 - Q_1 Q_2)$ (we put Q_3 to zero because it does not give us any more information). Here we also made use of the fact that the Hamiltonian obtained from the procedure (2.18) can be rescaled plus that something proportional to the identity matrix can be added.

Actually this Hamiltonian can be related with the transformation (2.14) to a completely diagonal Hamiltonian, such that it is included in the integrable models mentioned in [19]. In figure 2.3 the graph to the left shows how the energy eigenvalues of the Hamiltonian (2.59) depends on the phase ϕ. The graph to the right shows the eigenvalues, of the Hamiltonian if we change the sign in front of the second and third term in (2.59), depending on the phase ϕ. The graph to the right looks very amusing. It looks very similar to the graph to the left if that is turned upside down and deformed in a considerable symmetrical way.

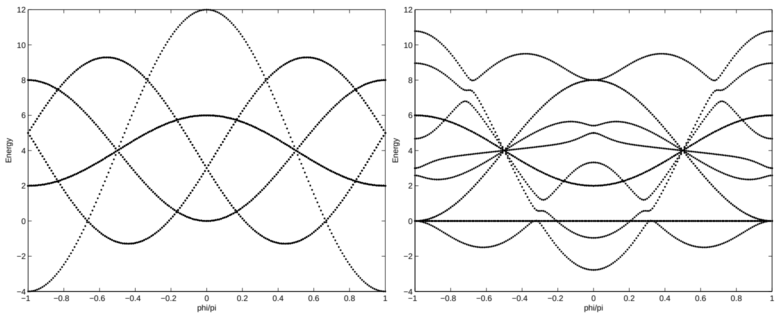

Figure 2.3: The eigenvalue dependence on the phase for the the Hamiltonian (2.59).

2.6 Broken $\mathbf{Z_3 \times Z_3}$ symmetry

Relaxing the one-loop finiteness condition (2.2), by choosing $h^{000} = h^{222} = 0$ and $h^{111} \propto h$ in the superpotential (2.3) breaks the $Z_3 \times Z_3$ symmetry. The superpotential is

$$W \propto \mathrm{Tr}\left[\Phi_0\Phi_1\Phi_2 - q\Phi_1\Phi_0\Phi_2 + \frac{h}{3}\Phi_1^3\right] , \tag{2.60}$$

where an overall factor is excluded. This superpotential is actually easier to study since the dilatation operator has homogeneous vacua $|0\rangle|0\rangle \dots |0\rangle$ and $|2\rangle|2\rangle \dots |2\rangle$. The mixing-matrix for the anomalous dimensions has the form of a spin-chain Hamiltonian arising from R-matrices found by Fateev-Zamolodchikov (or XXZ) [32] and the Izergin-Korepin [33]. This type of models were considered in [34] even though the authors never completely classified them. They have a $U(1)$-symmetry which can be used to get rid of the phase in the complex variable h.

In this setting, there is no longer a cancelation between the fermion loop and the scalar self-energy. The additional contribution to the Hamiltonian is of the form (2.72) (see Appendix 2.B for details). The spin chain obtained from the superpotential 2.60 is, with $q = -1$,

$$H = \left(\begin{array}{ccc|ccc|ccc}
0 & & & & & & & & \\
 & 1-\frac{h^*h}{2} & & 1 & & & & & \\
 & & 1 & & h^* & & 1 & & \\
\hline
 & 1 & & 1-\frac{h^*h}{2} & & & & & \\
 & & h & & 0 & & h & & \\
 & & & & & 1-\frac{h^*h}{2} & & 1 & \\
\hline
 & & 1 & & h^* & & 1 & & \\
 & & & & & 1 & & 1-\frac{h^*h}{2} & \\
 & & & & & & & & 0
\end{array}\right). \tag{2.61}$$

The term $h^*h/2$, is the fermion loop contribution from the self-energy. We will show that for the special values $q = -1$ and $h = e^{i\phi}\sqrt{2}$, this Hamiltonian can be obtained from the spin-1 XXZ R-matrix. The phase of h is redundant, the energy does not depend on it, and can be phased away through the transformation $|\tilde{1}\rangle = e^{-i\phi/2}|1\rangle$. The R-matrix for the XXZ-model is [32]

$$R(u) = \left(\begin{array}{ccc|ccc|ccc}
s & & & & & & & & \\
 & t & & r & & & & & \\
 & & T & & a^* & & R & & \\
\hline
 & r & & t & & & & & \\
 & & a & & \mathbf{s} & & a & & \\
 & & & & & t & & r & \\
\hline
 & & R & & a^* & & T & & \\
 & & & & & r & & t & \\
 & & & & & & & & s
\end{array}\right) \qquad
\begin{array}{l}
s = 1 \\
t = \epsilon J \sinh(u) \\
r = J \sinh 2\eta \\
a = e^{i\phi} J \frac{\sinh u \sinh 2\eta}{\sinh(u+\eta)} \\
R = J \frac{\sinh \eta \sinh 2\eta}{\sinh(u+\eta)} \\
T = J \frac{\sinh u \sinh(u-\eta)}{\sinh(u+\eta)} \\
\sigma = \epsilon t + R \\
J = \frac{1}{\sinh(u+2\eta)}
\end{array} \tag{2.62}$$

where $\epsilon = \pm 1$. The ϵ in t in (2.62) is added after checking that the R-matrix still satisfies Yang-Baxter equation. If we put $u = 0$, the R-matrix becomes the permutation matrix. Thus, a Hamiltonian can be obtained from the R-matrix by the usual procedure $H = PR'|_{u=0}$. Performing the derivatives at the point $u = 0$ gives

$$\begin{aligned}
s' &= 0\,, \qquad t' = \epsilon \frac{1}{\sinh 2\eta}\,, \qquad r' = -\frac{\cosh 2\eta}{\sinh 2\eta}\,, \qquad a' = e^{i\phi}\frac{1}{\sinh \eta}\,, \\
R' &= -\frac{\cosh \eta}{\sinh \eta} - \frac{\cosh 2\eta}{\sinh 2\eta}\,, \qquad T' = -\frac{1}{\sinh 2\eta}\,, \qquad \sigma' = \epsilon t' + R'\,.
\end{aligned} \tag{2.63}$$

Multiplying all parameters with $\sinh 2\eta$, the new variables, evaluated at $\eta = \pi/4$, leads to

$$\begin{aligned}
&\tilde{s}' = 0\,, \quad \tilde{t}' = -1\,, \quad \tilde{r}' = 0\,, \quad \tilde{a}' = e^{i\phi}\sqrt{2}\,, \\
&\tilde{R}' = -1\,, \quad \tilde{T}' = -1\,, \quad \tilde{\sigma}' = 0\,.
\end{aligned} \tag{2.64}$$

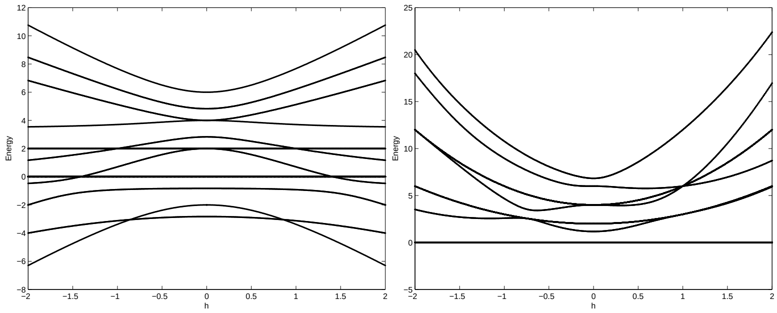

Figure 2.4: To the left is the spectra for the case $h^{000} = h^{222} = 0$ and $h^{111} = h$ depending on h when $q = -1$, and to the right is the spectra for the case h^{III} all equal to h (up to a constant factor) depending on h when $q = -1$

with the corresponding Hamiltonian

$$H = \left(\begin{array}{ccc|ccc|ccc} 0 & & & & & & & & \\ & 0 & & \pm 1 & & & & & \\ & & -1 & & e^{-i\phi}\sqrt{2} & & -1 & & \\ \hline & \pm 1 & & 0 & & & & & \\ & & e^{i\phi}\sqrt{2} & & 0 & & e^{i\phi}\sqrt{2} & & \\ & & & & & 0 & & \pm 1 & \\ \hline & & -1 & & e^{-i\phi}\sqrt{2} & & -1 & & \\ & & & & & \pm 1 & & 0 & \\ & & & & & & & & 0 \end{array}\right) \tag{2.65}$$

If we make the choice $\epsilon = -1$, this is the spin chain Hamiltonian with deformation $h = e^{i\phi}\sqrt{2}$ and $q = -1$! Looking at the left graph of Figure 2.4 of a four-site spin chain we see that two lines cross at this point.

This might, however, just be a coincidence. A special feature with $q = -1$ is that there is a Z_2-symmetry due to the invariance under exchange of the fields Φ_0 and Φ_2.

The right graph shows the same spectrum, but with all couplings h^{III} equal to h, up to a constant factor. The point $h = 1 - \sqrt{3}$ is special, since at this point the transformation (2.14) is “self-dual” , which means here that $\tilde{q} = q$ and $\tilde{h} = h$.

2.7 Conclusions

We have studied the dilation operator, corresponding to the general Leigh-Strassler deformation with h non-zero of $\mathcal{N} = 4$ SYM, in order to find new integrable points in the parameter-space of couplings. In particular we have found a relationship between the γ-deformed SYM and a site dependent spin-chain Hamiltonian. When all parameters γ_i are equal, this relates an entirely q-deformed to an entirely h-deformed superpotential. For $q = 0$ and the $h = e^{i\theta}$, where θ is real, we have found a new R-matrix (see 2.33).

We found a way of representing a general ansatz for the R-matrix, with the right form to give the dilatation operator, which makes the structure of the Yang-Baxter equations clear. The equations can be represented in terms of rectangular objects, which reveals that the underlying structure is a generalized version of the structure of the eight-vertex model. We presented all values of the parameters q and h for which the spin-chain Hamiltonian can be obtained from R-matrices with a linear dependence on the spectral parameter. Most of them were related to the q-deformed case through a simple shift of basis with a real phase β, or a shift with a twist with the phase $\pm 2\pi/3$, which reflects the Z_3-symmetry.

We also found a new hyperbolic R-matrix (2.59) which, through a simple change of basis, gives a Hamiltonian with only diagonal terms which was included in the cases studied in [19]. We had a brief look at a case with broken $Z_3 \times Z_3$ symmetry and found that the matrix of anomalous dimensions can for some special values of the parameters be obtained from the Fateev-Zamolodchikov R-matrix.

We conjecture that the Yang-Baxter equations found for the general R-matrix have a solution which is a generalized version of the solution to the eight-vertex model. If this solution exists, it is plausible that there will exist more points in the parameter space for which the dilatation operator is integrable. To find a general solution to these equations would be of interest in its own right. From a mathematical point of view, it is then interesting to generalize the solution to an R-matrix of arbitrary dimension.

The found relationship between the q- and the h-deformed superpotential should be visible in the dual string theory, and should also give a clue of what that string theory looks like. Another way to approach the problem, as mentioned in [9], is to first find a coherent state spin chain and from that reconstruct the dual geometry. The coherent state spin chain [9] is valid for small β, *i.e.* $q \approx 1$. We believe that making use of the basis transformation (2.14) makes it possible to create a coherent state spin chain for $q \approx 1$ and small h acting with the transformation (2.14) on a q close to one gives a new q close to one and a new small h. We also hope that due to the relation between vanishing h and vanishing q it is possible to write a coherent sigma model for both q and h close to one. It would then be very interesting to find the dual geometry, which corresponds to a further away deformation of the $\mathcal{N} = 4$ SYM.

One other thing of interest is to extend the analysis to other sectors of the theory and to higher loop order. In the β-deformed case it is possible to argue that the integrability holds to higher loop order [8], because the dilatation operator is related with a unitary transformation to the case of the usual $\mathcal{N} = 4$ SYM. In the same way can we argue about the h-deformed case, even though we have to consider the induced effects of the spin chain periodicity.

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2.A Yang-Baxter equations for the general case

The functions in the R-Matrix (2.37) are expressed in terms of the functions ω_i, $\bar{\omega}_i$ and γ_i as

$$\begin{aligned}
a(u) &= \gamma_1(u) + \gamma_2(u) + \gamma_3(u)\,, \\
b(u) &= \gamma_1(u)e^{i2\pi/3} + \gamma_2(u)e^{-i2\pi/3} + \gamma_3(u)\,, \\
\bar{b}(u) &= \gamma_2(u)e^{i2\pi/3} + \gamma_1(u)e^{-i2\pi/3} + \gamma_3(u) \\
c(u) &= \omega_1(u) + \omega_2(u) + \omega_3(u)\,, \\
c(u) &= \bar{\omega}_1(u) + \bar{\omega}_2(u) + \bar{\omega}_3(u)\,, \\
d(u) &= \omega_2(u)e^{i2\pi/3} + \omega_1(u)e^{-i2\pi/3} + \omega_3(u)\,, \\
\bar{d}(u) &= \bar{\omega}_1(u)e^{i2\pi/3} + \bar{\omega}_2(u)e^{-i2\pi/3} + \bar{\omega}_3(u)\,, \\
e(u) &= \omega_1(u)e^{i2\pi/3} + \omega_2(u)e^{-i2\pi/3} + \omega_3(u)\,, \\
\bar{e}(u) &= \bar{\omega}_2(u)e^{i2\pi/3} + \bar{\omega}_1(u)e^{-i2\pi/3} + \bar{\omega}_3(u)\,,\,.
\end{aligned} \tag{2.66}$$

Yang-Baxter equations from the R-matrix ansatz (2.49) read

$$\begin{aligned}
\omega_{n+1}\omega'_{n+2}\gamma''_3 - \omega_{n+2}\omega'_{n+1}\gamma''_2 + \gamma_3\bar{\omega}'_n\bar{\omega}''_{n+1} - \gamma_2\bar{\omega}'_{n+1}\bar{\omega}''_n & \\
+\bar{\omega}_{n+1}\gamma'_2\omega''_{n+1} - \bar{\omega}_n\gamma'_3\omega''_{n+2} &= 0\,, \\
\omega_{n+1}\omega'_{n+2}\gamma''_1 - \omega_{n+2}\omega'_{n+1}\gamma''_3 + \gamma_1\bar{\omega}'_{n+2}\bar{\omega}''_n - \gamma_3\bar{\omega}_n\bar{\omega}''_{n+2} & \\
+\bar{\omega}_n\gamma'_3\omega''_{n+1} - \bar{\omega}_{n+2}\gamma'_1\omega''_{n+2} &= 0\,, \\
\omega_{n+1}\omega'_{n+2}\gamma''_2 - \omega_{n+2}\omega'_{n+1}\gamma''_1 + \gamma_2\bar{\omega}'_{n+1}\bar{\omega}''_{n+2} - \gamma_1\bar{\omega}_{n+2}\bar{\omega}''_{n+1} & \\
+\bar{\omega}_{n+2}\gamma'_1\omega''_{n+1} - \bar{\omega}_{n+1}\gamma'_2\omega''_{n+2} &= 0\,, \\
\omega_1\bar{\omega}'_{n+1}\omega''_2 - \bar{\omega}_1\omega'_{2n+1}\bar{\omega}''_3 + \omega_2\bar{\omega}'_{n+2}\omega''_0 - \bar{\omega}_2\omega'_{2n-1}\bar{\omega}''_1 & \\
+\omega_0\bar{\omega}'_n\omega''_1 - \bar{\omega}_0\omega'_{2n}\bar{\omega}''_2 &= 0\,, \\
\gamma_2\omega'_{n+1}\gamma''_1 + \gamma_3\omega'_{n-1}\gamma''_2 + \gamma_1\omega'_n\gamma''_3 - \omega_1\gamma'_n\omega''_2 & \\
-\omega_2\gamma'_{n+1}\omega''_3 - \omega_3\gamma'_{n-1}\omega''_1 &= 0\,, \\
\gamma_1\bar{\omega}'_{n-1}\gamma''_2 + \gamma_2\bar{\omega}'_{n+1}\gamma''_3 + \gamma_3\bar{\omega}'_n\gamma''_1 - \bar{\omega}_1\gamma'_{n-1}\bar{\omega}''_2 & \\
-\bar{\omega}_2\gamma'_{n+1}\bar{\omega}''_3 - \bar{\omega}_3\gamma'_n\bar{\omega}''_1 &= 0\,, \\
\bar{\omega}_{n+1}\bar{\omega}'_{n+2}\omega''_{n+1} - \bar{\omega}_{n+2}\bar{\omega}'_{n+1}\omega''_n - \omega_n\gamma'_3\gamma''_1 + \omega_{n+1}\gamma'_1\gamma''_3 & \\
-\gamma_1\omega'_{n+1}\bar{\omega}''_{n+2} + \gamma_3\omega'_n\bar{\omega}''_{n+1} &= 0\,,
\end{aligned}$$

$$\begin{aligned}
\bar{\omega}_{n+2}\bar{\omega}'_{n+1}\omega''_{n+2} - \bar{\omega}_{n+1}\bar{\omega}'_{n+2}\omega''_{n} - \omega_n\gamma'_3\gamma''_2 + \omega_{n+2}\gamma'_2\gamma''_3 & \\
-\gamma_2\omega'_{n+2}\bar{\omega}''_{n+1} + \gamma_3\omega'_n\bar{\omega}''_{n+2} &= 0\,, \\
\bar{\omega}_{n+1}\bar{\omega}'_{n+2}\omega''_{n+2} - \bar{\omega}_{n+2}\bar{\omega}'_{n+1}\omega''_{n+1} - \omega_{n+1}\gamma'_1\gamma''_2 + \omega_{n+2}\gamma'_2\gamma''_1 & \\
+\gamma_1\omega'_{n+1}\bar{\omega}''_{n+1} - \gamma_2\omega'_{n+2}\bar{\omega}''_{n+2} &= 0\,, \\
\omega_{n+2}\omega'_n\bar{\omega}''_{n+1} - \omega_n\omega'_{n+2}\bar{\omega}''_n + \bar{\omega}_{n+1}\gamma'_3\gamma''_1 - \bar{\omega}_n\gamma'_1\gamma''_3 & \\
+\gamma_1\bar{\omega}'_n\omega''_{n+2} - \gamma_3\bar{\omega}'_{n+1}\omega''_n &= 0\,, \\
\omega_n\omega'_{n+1}\bar{\omega}''_n - \omega_{n+1}\omega'_n\bar{\omega}''_{n+2} - \bar{\omega}_{n+2}\gamma'_3\gamma''_2 + \bar{\omega}_n\gamma'_2\gamma''_3 & \\
-\gamma_2\bar{\omega}'_n\omega''_{n+1} + \gamma_3\bar{\omega}'_{n+2}\omega''_n &= 0\,, \\
\omega_{n+2}\omega'_{n+1}\bar{\omega}''_{n+1} - \omega_{n+1}\omega'_{n+2}\bar{\omega}''_{n+2} + \bar{\omega}_{n+1}\gamma'_2\gamma''_1 - \bar{\omega}_{n+2}\gamma'_1\gamma''_2 & \\
+\gamma_1\bar{\omega}'_{n+2}\omega''_{n+2} - \gamma_2\bar{\omega}'_{n+1}\omega''_{n+1} &= 0\,,
\end{aligned} \tag{2.67}$$

Here, we have defined $\omega = \omega(u-v)$, $\omega' = \omega(u)$ and $\omega'' = \omega(v)$.

2.B Self-energy with broken $\mathbf{Z_3} \times \mathbf{Z_3}$ symmetry

We will follow the prescription of [35] to compute the contribution to the Hamiltonian from the superpotential (2.60), when conformal invariance is broken. The additional terms are coming from the self-energy fermion loop.

The scalar self-energy of the vertices is, in $\mathcal{N}=4$ SYM,

$$\frac{g_{YM}^2(L+1)}{8\pi^2} N : \mathrm{Tr}\left(\bar{\phi}_i\phi_i\right) :, \tag{2.68}$$

where $L = \log x^{-2} - (1/\epsilon + \gamma + \log\pi + 2)$. The scalar-vector contribution to this is $-\frac{g_{YM}^2(L+1)}{8\pi^2}$, and the fermion loop contribution is $\frac{g_{YM}^2(L+1)}{4\pi^2}$. Half of the fermion contribution comes from the superpotential; this is the part which will be altered by the extra h-dependent part of the superpotential. Hence, the additional term to the new spin chain, besides the F-term scalar part, is

$$\frac{h^*h}{1+q^*q}\frac{g^2(L+1)}{8\pi^2} N : \mathrm{Tr}\left(\bar{\phi}_1\phi_1\right) : . \tag{2.69}$$

Then, we will have an effective scalar interaction which just comes from the F-term (since we have the same cancelation as in the $\mathcal{N}=4$ SYM)[35]

$$\pm\sqrt{\frac{2}{(1+q^*q)}}\frac{g_{YM}^2 L}{16\pi^2} : V_F :, \tag{2.70}$$

where

$$
\begin{aligned}
V_F &= \big(\mathrm{Tr}\left[\phi_i\phi_{i+1}\bar{\phi}_{i+1}\bar{\phi}_i - q\phi_{i+1}\phi_i\bar{\phi}_{i+1}\bar{\phi}_i - q^*\phi_i\phi_{i+1}\bar{\phi}_i\bar{\phi}_{i+1}\right] \\
&+ \mathrm{Tr}\left[qq^*\phi_{i+1}\phi_i\bar{\phi}_i\bar{\phi}_{i+1} - qh^*\phi_0\phi_2\bar{\phi}_1\bar{\phi}_1 - q^*h\phi_1\phi_1\bar{\phi}_2\bar{\phi}_0\right] \\
&+ \mathrm{Tr}\left[h\phi_1\phi_1\bar{\phi}_0\bar{\phi}_2 + h^*\phi_2\phi_0\bar{\phi}_1\bar{\phi}_1 + hh^*\phi_1\phi_1\bar{\phi}_1\bar{\phi}_1\right]\big)\ . \qquad (2.71)
\end{aligned}
$$

The plus-minus sign in (2.70) depends on which sign we choose for the superpotential. Since all terms are multiplied by the same divergent factor we can set $L = \log x^{-2}$, just as in the case of $\mathcal{N} = 4$. The contribution from the self-energy to the dilatation operator is

$$
\frac{h^*h}{1+q^*q}(E_{11}\otimes I + I\otimes E_{11})\,, \qquad (2.72)
$$

and the F-term scalar interaction contribute with

$$
\begin{aligned}
&\pm\sqrt{\frac{2}{(1+q^*q)}}\,\big(E^l_{i,i}E^{l+1}_{i+1,i+1} - qE^l_{i+1,i}E^{l+1}_{i,i+1} - q^*E^l_{i,i+1}E^{l+1}_{i+1,i} \\
&+ \; qq^*E^l_{i+1,i+1}E^{l+1}_{i,i} - qh^*E^l_{i+1,i+2}E^{l+1}_{i,i+2} - q^*hE^l_{1,0}E^{l+1}_{1,2} \\
&+ \; hE^l_{1,2}E^{l+1}_{1,0} + h^*E^l_{2,1}E^{l+1}_{0,1} + hh^*E^l_{1,1}E^{l+1}_{1,1}\big)\ . \qquad (2.73)
\end{aligned}
$$

We will now consider the case when $q = -1$. The total dilatation operator simplifies to

$$
H = \left(\begin{array}{ccc|ccc|ccc}
0 & & & & & & & & \\
 & 1-\frac{h^*h}{2} & & 1 & & & & & \\
 & & 1 & & -qh^* & & 1 & & \\
\hline
 & 1 & & 1-\frac{h^*h}{2} & & & & & \\
 & & h & & 0 & & h & & \\
 & & & & & 1-\frac{h^*h}{2} & & 1 & \\
\hline
 & & 1 & & h^* & & 1 & & \\
 & & & & & 1 & & 1-\frac{h^*h}{2} & \\
 & & & & & & & & 0
\end{array}\right). \qquad (2.74)
$$

Here we have chosen a relative minus sign between the contribution from the fermion loop and the scalar interaction term.

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Star product and the general Leigh-Strassler deformation

Paper III

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Star product and the general Leigh-Strassler deformation

Daniel Bundzik

School of Technology and Society, Malmö University,
Östra Varvsgatan 11A, S-205 06 Malmö, Sweden
Department of Theoretical Physics, Lund University,
Sölvegatan 14A, S-223 62, Sweden
E-mail: Daniel.Bundzik@ts.mah.se

abstract

We extend the definition of the star product introduced by Lunin and Maldacena to study marginal deformations of $\mathcal{N} = 4$ SYM. The essential difference from the latter is that instead of considering $U(1)\times U(1)$ non-R-symmetry, with charges in a corresponding diagonal matrix, we consider two $\mathbb{Z}_3$-symmetries followed by an $SU(3)$ transformation, with resulting off-diagonal elements. From this procedure we obtain a more general Leigh-Strassler deformation, including cubic terms with the same index, for specific values of the coupling constants. We argue that the conformal property of $\mathcal{N} = 4$ SYM is preserved, in both β-(one-parameter) and γ_i-deformed (three-parameters) theories, since the deformation for each amplitude can be extracted in a prefactor. We also conclude that the obtained amplitudes should follow the iterative structure of MHV amplitudes found by Bern, Dixon and Smirnov.

KEYWORDS: marginal deformations, β-deformation, γ_i-deformation, three-parameter deformation

3.1 Introduction

The exactly marginal deformations of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) preserving $\mathcal{N} = 1$ supersymmetry, systematically investigated by Leigh and Strassler in [1], have been studied extensively since the finding, by Lunin and Maldacena in [2], of the supergravity dual of the so-called β-deformed[1] $\mathcal{N} = 4$ SYM theory. Marginal deformations provide an interesting opportunity to study the AdS/CFT-correspondence [3] in new supergravity backgrounds.

The perturbative behaviour of the β-deformed theory shares many features of the undeformed theory [4, 5, 6, 7]. In [8] it was found that maximally helicity violating (MHV) planar amplitudes in $\mathcal{N} = 4$ SYM have an iterative structure for all n-point amplitudes. These results were then transferred to the β-deformed theory in [7] by placing the deformation into the so-called star product. The use of the star product, which was first introduced in this context in [2], to study marginal deformations is especially convenient when calculating amplitudes, since the dependence of the deformation can be isolated into an overall prefactor.

The main purpose of this article is to show that it is possible to obtain the general Leigh-Strassler deformation[2], including cubic terms with all indices equal the same value, from the star product. In section 3.2 we discuss the necessary conditions for conformal deformations of $\mathcal{N} = 4$ SYM. In Section 3.3 we consider two global $\mathbb{Z}_3$-symmetries, in order to solve an eigenvalue system with eigenvectors as a linear combination of the three chiral superfields Φ_i. The two systems are related by an element of $SU(3)$ which is also a symmetry of the $\mathcal{N} = 4$ SYM Lagrangian written in terms of $\mathcal{N} = 1$ superfields. We continue to define the star product for $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetry charges, containing three deformation parameters γ_i. The β-deformed theory is obtained by putting all parameters equal. In the the diagonal system the star product is easily evaluated. We calculate the superpotential, with ordinary multiplication replaced by the star product, in the β- and γ_i-deformed theories. The result is the general Leigh-Strassler deformed superpotential, including the terms of the form $\mathrm{Tr}\,\Phi_i^3$. In section 3.4 we compute the starproduct of two chrial superfields which are simple in the β-deformed case. In appendix 3.B we present the the results in the γ_i-deformed theory. In section 3.5 we study the tree-level amplitudes corresponding to terms in the classical Lagrangian. In the β-deformed theory we find the expected 4-point scalar interaction terms for a Leigh-Strassler deformed theory. However, in the γ_i-deformed case we obtain component terms

[1] By β-deformation we mean a one-parameter complex deformation $\beta = \beta_R + i\beta_C$. With a γ_i-deformed theory we mean a theory containing three complex parameters γ_1, γ_2 and γ_3. In the literature, a γ-deformed theory sometimes means deformations by the real part of β which is called β_R in the present work.

[2] To distinguish from the β-deformed superpotential we use the word "general" when cubic terms of the form $\mathrm{Tr}\,\Phi_i^3$ are present in the Leigh-Strassler deformed theory.

of the form $\mathrm{Tr}\,\phi_i^\dagger\phi_i^\dagger\phi_i\phi_j$, i.e with three identical indices, which are not normally considered in a Leigh-Strassler deformed theory. Their gauge invariance and supersymmetric properties have to be investigated. In Section 3.6 we extend the proof in [7] which shows that the phase-dependence of HMV planar tree- and loop-diagrams can be computed from an effective tree-level vertex, determined only by external fields. We conclude that the proof also holds for our present theories. In the final section we compute the one-loop finiteness conditions for conformal marginal deformed $\mathcal{N}=4$ supersymmetric theories with both β- and γ_i-deformation.

3.2 Conformal deformations of $\mathcal{N}=4$ SYM

The most general renormalizable $\mathcal{N}=1$ supersymmetric action which is invariant under a gauge group G, can be written as, excluding gauge-fixing and ghost terms,[3]

$$\begin{aligned} S &= \frac{1}{16T(A)g^2}\int d^4x d^2\theta \mathrm{Tr} W^\alpha W_\alpha + \int d^4x d^2\theta d^2\bar{\theta}\Phi^\dagger_A \left(e^{2gV}\right)^A{}_B \Phi^B \\ &\quad + \int d^4x d^2\theta \mathcal{W} + \mathrm{h.c.} \end{aligned} \tag{3.1}$$

The chiral superfield Φ_A and its conjugate transform under irreducible representations R of G. The index A runs over irreducible representations R_i and the component of each irreducible representation is labeled by I, such that $A=\{i,I\}$ [10]. The vector superfield $V^A{}_B = V_a\left(T^a\right)^A{}_B$ contains the generators T^a, $a=1,\dots,\dim G$, of the gauge group G defined by $\left(T^a\right)^A{}_B = \left(T^{ai}\right)^I{}_J$. The first term in (3.1) is related to the gauge theory kinematic Lagrangian containing the gauge field A^μ and a Majorana spinor, which we call λ_4. $\mathcal{W}$ is the superpotential and is given by

$$\mathcal{W} = C_{ABC}\Phi^A\Phi^B\Phi^C\,, \tag{3.2}$$

where C_{ABC} is totally symmetric in A, B and C or equivalent totally symmetric in the pairs $\{i,I\}$, $\{j,J\}$ and $\{k,K\}$. In the following we will restrict ourselves to

$$C_{ABC} \equiv C^{i\,j\,k}_{I\,J\,K} = a^{ijk}b_{IJK} + h^{ijk}d_{IJK}\,, \tag{3.3}$$

where a^{ijk} and b_{IJK} are totally anti-symmetric and h^{ijk} and d_{IJK} are totally symmetric.

[3] We use the conventions of [9] such that the generators of the gauge group satisfy $\left[T^a_R,T^b_R\right] = if^{ab}{}_cT^c_R$ for the representation R. The adjoint representation A is given by the structure constants such that $\mathrm{ad}\,T^a_R = \left(T^a_A\right)^b{}_c = -if^{ab}{}_c$, normalized as $\mathrm{Tr}\,T^a_AT^b_A = -T(A)\delta^{ab}$.

The supercurrent $J_{\alpha\dot{\alpha}}$ of the theory has the anomaly [10, 1]

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = -\frac{1}{3}\left[\frac{\beta_g}{g} W^\beta W_\beta + (d_s - 3) + \gamma^i{}_j \left(\Phi_i \bar{D}_{\dot{\beta}} \bar{D}^{\dot{\beta}} \Phi^{\dagger j}\right)\right] . \tag{3.4}$$

where $\gamma^i{}_j$ is the anomalous dimension for Φ^i. The anomaly (3.4) is zero in a conformal theory. At one-loop we have

$$\beta_g^{(1)} = \frac{g^3}{16\pi^2}\left[\sum_i T(R_i) - 3C_2(G) + \sum_{i,j} T(R_i) \gamma^{(1)i}{}_j\right] , \tag{3.5}$$

and

$$\beta^{(1)}_{h_{ijk}} = h_{ijk}\left[(d_s - 3) - \frac{1}{2}\sum_{i,j} r_i \gamma^{(1)i}{}_j\right] . \tag{3.6}$$

The number r_i counts the number of chiral fields in each term of the superpotential with the sum $d_s = \sum_i r_i$. The anomalous dimension is [11]

$$\gamma^{(1)i}{}_j = C^{ikl}_{IKL} \bar{C}^{JKL}_{jkl} - 2g^2 T(R) \delta^i_j \delta^I_J . \tag{3.7}$$

Vanishing of the one-loop anomalous dimension also implies UV finiteness of $\mathcal{N} = 1$ SYM at two-loop level [11].

$\mathcal{N} = 4$ supersymmetric Yang-Mills in the $\mathcal{N} = 1$ superfield formulation contains three chiral superfields in the adjoint representation of the $SU(N)$ gauge group and is obtained by taking $i = 1, 2, 3$ and $I \equiv a = 1, \ldots, N^2 - 1$. Thus, if we define $\Phi^j \equiv \Phi^i_a T^a$ the structure constants are $\varepsilon_{IJK} = f_{abc}$, which can be expressed $f_{abc} = -iT(R)^{-1} \mathrm{Tr}\, T^a \left[T^b, T^c\right]$. The symmetric part d_{abc} vanishes for a real representation. The $\mathcal{N} = 4$ SYM superpotential becomes

$$\mathcal{W}_{\mathcal{N}=4} = -\frac{ig}{T(R)} \varepsilon^{ijk} \mathrm{Tr}\, \Phi_i \left[\Phi_j, \Phi_k\right] . \tag{3.8}$$

In the Wess-Zumino gauge, the $\mathcal{N} = 4$ supersymmetric Lagrangian can be written in terms of $\mathcal{N} = 1$ component fields as

$$\begin{aligned}
\mathcal{L} \;=\; \mathrm{Tr}\Bigg(&\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda_4^\dagger \bar{\sigma}^\mu \mathcal{D}_\mu \lambda_4 - i\lambda_i^\dagger \bar{\sigma}^\mu \mathcal{D}_\mu \lambda_i - \bar{\mathcal{D}}_\mu \phi_i^\dagger \mathcal{D}^\mu \phi_i \\
&- \frac{\sqrt{2} g}{T(A)} \left(\lambda_4 \left[\phi_i^\dagger, \lambda_i\right] + \lambda_4^\dagger \left[\lambda_i^\dagger, \phi_i\right]\right) \\
&- \frac{g}{T(A)} \left(\varepsilon^{ijk} \lambda_i \left[\lambda_j, \phi_k\right] + \varepsilon^{ijk} \lambda_i^\dagger \left[\lambda_j^\dagger, \phi_k^\dagger\right]\right) \\
&- \frac{g^2}{2\, T(A)^2} \left[\phi_i^\dagger, \phi_i\right] \left[\phi_j^\dagger, \phi_j\right] - \frac{2g^2}{T(A)^2} \left[\phi_i^\dagger, \phi_j^\dagger\right] \left[\phi_i, \phi_j\right]\Bigg) .
\end{aligned} \tag{3.9}$$

Conformal invariance of $\mathcal{N}=4$ SYM follows from (3.7) where $\gamma^{(1)i}{}_j=0$ since $C^{ikl}_{IKL}=gT(R)\epsilon^{ijk}f_{abc}$. This also implies that $\beta^{(1)}_{h_{ijk}}=\beta^{(1)}_g=0$.

As we will see, marginal deformations of $\mathcal{N}=4$ SYM which preserve the finiteness condition at one-loop can be obtained by replacing the ordinary multiplication between all fields by an operator called star product. The general form of coupling constants (3.3) which contains the anti-symmetric part a^{ijk} and the symmetric part h^{ijk} can be written on the form

$$\mathcal{W}=a^{ijk}\mathrm{Tr}\,\Phi_i\,[\Phi_j,\Phi_k]+h^{ijk}\mathrm{Tr}\,\Phi_i\,\{\Phi_j,\Phi_k\}\ . \tag{3.10}$$

By choosing the non-zero couplings as $a^{ijk}=\epsilon^{ijk}\lambda/6$, $h^{123}=\lambda(1-q)/6(1+q)$ and $h^{iii}=h'/2$ we obtain the general Leigh-Strassler deformation [1, 12], also known as the full Leigh-Strassler deformation [13],

$$\mathcal{W}=h\left(\mathrm{Tr}\,\Phi_1\Phi_2\Phi_3-q\mathrm{Tr}\,\Phi_1\Phi_3\Phi_2\right)+h'\left(\mathrm{Tr}\,\Phi_1^3+\mathrm{Tr}\,\Phi_2^3+\mathrm{Tr}\,\Phi_3^3\right). \tag{3.11}$$

where $h=2\lambda/(1+q)$.

In the next section we will compute the couplings h, q and h' in a star product deformed theory. In section 3.7 we will evaluate the conditions for the supercurrent in (3.4) to remain anomaly-free.

3.3 Deformations from star product

Introducing the star product has shown to be beneficial in the study of marginal deformations of $\mathcal{N}=4$ SYM [2, 7]. In general, it is not easy to compute the star product of two chiral superfields. To simplify the computation we will in this section solve an eigenvalue system. We continue to define the star product for three deformation parameters. This allows us to compute the superpotential for both β- and γ_i-deformed theories.

3.3.1 Eigenvalue system

The key idea for this work is to make use of the permutation symmetries of the superpotential to study marginal deformations of $\mathcal{N}=4$ SYM, by introducing a generalized multiplication operator between all fields, which we call "star product". When the symmetries permute a set of fields in the original so called Φ-system, it is hard to compute the star product directly. Instead, we rotate the system by an $SU(3)$ transformation into the so called Ψ-system in which the symmetries act with diagonal elements. In the Ψ-system, the star product can easily be computed.

Let us begin by choosing two symmetries of the superpotential which we denote S_1 and S_2. In the diagonal Ψ-system, the symmetries act as $U(1)\times U(1)$ transformations on the vector $\mathbf{\Psi}=(\Psi_1,\Psi_2,\Psi_3)$ of chiral superfields accordingly

$$S_i : \qquad \mathbf{\Psi} \qquad \longrightarrow \qquad \mathcal{Q}_i \mathbf{\Psi} \, , \tag{3.12}$$

where

$$\mathcal{Q}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\varphi_1} & 0 \\ 0 & 0 & e^{i\varphi_1} \end{pmatrix} \qquad \text{and} \qquad \mathcal{Q}_2 = \begin{pmatrix} e^{i\varphi_2} & 0 & 0 \\ 0 & e^{-i\varphi_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.13}$$

At this stage, φ_1 and φ_2 are arbitrary parameters. The superpotential (3.10) and also the Lagrangian (3.9) are invariant under an $SU(3)$ transformation. We introduce the vector $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ of chiral superfields such that

$$\mathbf{\Psi} = T\mathbf{\Phi} \, , \qquad\qquad T \in SU(3) \, . \tag{3.14}$$

We now demand that the symmetries S_1 and S_2 act as permutations of the Φ_i's:

$$S_i : \qquad \mathbf{\Phi} \qquad \longrightarrow \qquad \mathcal{P}_i \mathbf{\Phi} \, , \tag{3.15}$$

with

$$\mathcal{P}_1 = \begin{pmatrix} 0 & a_2 & 0 \\ 0 & 0 & a_3 \\ a_1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{P}_2 = \begin{pmatrix} 0 & 0 & b_3 \\ b_1 & 0 & 0 \\ 0 & b_2 & 0 \end{pmatrix} , \tag{3.16}$$

where the parameters a_i and b_i will be determined below. The relation between $\mathcal{P}_i$ and $\mathcal{Q}_i$ is

$$\mathcal{P}_i = T^{-1} \mathcal{Q}_i T \, . \tag{3.17}$$

For the permutation matrices to be elements of SU(3), their elements have to satisfy *i)* $a_1 a_2 a_3 = 1$ and $b_1 b_2 b_3 = 1$ and *ii)* $|a_i|^2 = 1$ and $|b_i|^2 = 1$. It then follows that $\mathcal{P}_i^3 = 1$ which is equivalent to $\mathcal{Q}_i^3 = 1$. Thus, the relation (3.17) breaks the $U(1) \times U(1)$ symmetry to $\mathbb{Z}_3 \times \mathbb{Z}_3$ with $e^{i\varphi_1} = e^{i\varphi_2} = e^{i2\pi/3}$. For simplicity we define $\alpha = e^{i2\pi/3}$ with inverse $\bar{\alpha}$. The relation $1 + \alpha + \bar{\alpha} = 0$ will be used repeatedly. As a result, the symmetries S_1 and S_2 act on the Ψ_i's as

$$\begin{aligned} S_1 : \qquad & (\Psi_1, \Psi_2, \Psi_3) \qquad \longrightarrow \qquad (\Psi_1, \bar{\alpha}\Psi_2, \alpha\Psi_3) \\ S_2 : \qquad & (\Psi_1, \Psi_2, \Psi_3) \qquad \longrightarrow \qquad (\alpha\Psi_1, \bar{\alpha}\Psi_2, \Psi_3) \, . \end{aligned} \tag{3.18}$$

These relations will be used when we compute the star product in section 3.3.3.

The most general solution to (3.17) is

$$T = \begin{pmatrix} a_1 t_1 & a_1 a_2 t_1 & t_1 \\ \alpha a_1 t_2 & \bar{\alpha} a_1 a_2 t_2 & t_2 \\ \bar{\alpha} a_1 t_3 & \alpha a_1 a_2 t_3 & t_3 \end{pmatrix} , \tag{3.19}$$

where a_i are the parameters of $\mathcal{P}_1$ and $b_i = \alpha / a_{i+1}$ in $\mathcal{P}_2$. The parameters t_1, t_2 and t_3 have to satisfy *i)* $3 t_1 t_2 t_3 a_1^2 a_2 (\bar{\alpha} - \alpha) = 1$ and *ii)* $|t_i|^2 = 1/3$ for

$T \in SU(3)$. These requirements are fulfilled for (including the conditions for $\mathcal{P}_i \in SU(3)$, see below (3.17))

$$\begin{aligned} a_1 &= e^{i\theta_1}\,, & a_2 &= e^{i\theta_2} = e^{-i(\theta_1+\theta_3)}\,, & a_3 &= e^{i\theta_3}\,, \\ t_1 &= \frac{e^{i\rho_1}}{\sqrt{3}}\,, & t_2 &= \frac{e^{i\rho_2}}{\sqrt{3}} = \frac{ie^{i(\theta_3-\theta_1-\rho_1-\rho_3)}}{\sqrt{3}}\,, & t_3 &= \frac{e^{i\rho_3}}{\sqrt{3}}\,. \end{aligned} \tag{3.20}$$

The transfer matrix becomes

$$T = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i(\theta_1+\rho_1)} & e^{-i(\theta_3-\rho_1)} & e^{i\rho_1} \\ \alpha i e^{i(\theta_3-\rho_1-\rho_3)} & \bar{\alpha} i e^{-i(\theta_1+\rho_1+\rho_3)} & i e^{i(\theta_3-\theta_1-\rho_1-\rho_3)} \\ \bar{\alpha} e^{i(\theta_1+\rho_3)} & \alpha e^{-i(\theta_3-\rho_3)} & e^{i\rho_3} \end{pmatrix}. \tag{3.21}$$

If we denote the part of the elements in (3.21) by t_{ij} which are dependent of the phases θ_i and ρ_i, then we can write

$$\Psi_i = \sum_j \alpha^{(i+2)j} t_{ij}\, \Phi_j = \sum_j \alpha^{(i+2)j} e^{i\rho_i} \prod_{\tilde{j}}^{j} e^{i\theta_{\tilde{j}}}\, \Phi_j\,. \tag{3.22}$$

This compact form will be useful in the coming sections. The permutation matrices (3.16) are

$$\begin{aligned} \mathcal{P}_1 &= \begin{pmatrix} 0 & e^{-i(\theta_1+\theta_3)} & 0 \\ 0 & 0 & e^{i\theta_3} \\ e^{i\theta_1} & 0 & 0 \end{pmatrix} \\ \mathcal{P}_2 &= \alpha \begin{pmatrix} 0 & 0 & e^{-i\theta_1} \\ e^{i(\theta_1+\theta_3)} & 0 & 0 \\ 0 & e^{-i\theta_3} & 0 \end{pmatrix}. \end{aligned} \tag{3.23}$$

The transfer matrix (3.21) contains four independent parameters. Two of parameters, θ_1 and θ_3, are inherited from the permutation symmetry in (3.23). The remaining two parameters, ρ_1 and ρ_3, are coming from the original $\mathcal{N}=4$ SYM $SU(4)$ R-symmetry. It is interesting to note that there does not exist a matrix T which takes $\mathcal{Q}_i$ to $\mathcal{P}_i$ (see (3.17)) for continuous parameters. As we will see in the next section, the surviving discrete $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry will let us define the star product, which is especially simple to compute in the Ψ-system. Transforming to the Φ-system induces extra cubic terms, of the form $\mathrm{Tr}\,\Phi_i^3$, to the superpotential which correspond to terms in the general Leigh-Strassler deformed theory.

3.3.2 Definition of star product

We define the star product between two fields Ψ_i and Ψ_j as, in analogy to [2],

$$\Psi_i \star \Psi_j = e^{i \det \widetilde{Q}_{ij}}\, \Psi_i \cdot \Psi_j\,, \tag{3.24}$$

where $\Psi_i \cdot \Psi_j$ is an ordinary product and the determinant is defined as

$$\det \widetilde{Q}_{ij} = \begin{vmatrix} \widetilde{Q}_i^1 & \widetilde{Q}_i^2 \\ \widetilde{Q}_j^1 & \widetilde{Q}_j^2 \end{vmatrix} = \begin{vmatrix} \tilde{\gamma}_i Q_i^1 & \tilde{\gamma}_i Q_i^2 \\ \tilde{\gamma}_j Q_j^1 & \tilde{\gamma}_j Q_j^2 \end{vmatrix} = \tilde{\gamma}_i \tilde{\gamma}_j \det Q_{ij} \,. \tag{3.25}$$

(Q_i^1, Q_i^2) are the $S_1 \times S_2$ charges of the fields for the symmetries S_1 and S_2 of the corresponding superpotential. It will be convenient to rewrite the three deformation parameters $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ as

$$\gamma_{2(i+j)} = \tilde{\gamma}_i \tilde{\gamma}_j, \qquad 2(i+j) \bmod 3 \,, \tag{3.26}$$

so that $\gamma_1 = \tilde{\gamma}_2\tilde{\gamma}_3$, $\gamma_2 = \tilde{\gamma}_3\tilde{\gamma}_1$ and $\gamma_3 = \tilde{\gamma}_1\tilde{\gamma}_2$. Note that the deformation parameters $\tilde{\gamma}_i\tilde{\gamma}_i$ also exist. Since they always occur in the combination $\tilde{\gamma}_i\tilde{\gamma}_i \det Q_{ii}$ where $\det Q_{ii} = 0$, the deformations $\tilde{\gamma}_i\tilde{\gamma}_i$ do not have to be accounted for in calculations.

A deformed multiplication law, such as (3.24), is usually denoted $\star$ and called "star product". Non-commutative field theories are often obtained by replacing the ordinary point-wise product of fields by the Moyal star product, which is defined by a bidifferential operator over some manifold. In the present context, the star product may be viewed as generalized couplings between fields. This is a convenient way to study marginal deformations of supersymmetric $\mathcal{N} = 4$ theories.

In order to prove that the star product is associative we have to assume that the elementary fields are defined by (3.24) and (3.25) with arbitrary parameters $\tilde{\gamma}_i$ and that a composite field of n elementary fields is characterized by the additive property $(\widetilde{Q}^1_{ij\ldots n}, \widetilde{Q}^2_{ij\ldots n})$ where

$$\widetilde{Q}^{1,2}_{ij\ldots n} = \widetilde{Q}^{1,2}_i + \widetilde{Q}^{1,2}_j + \cdots + \widetilde{Q}^{1,2}_n \,. \tag{3.27}$$

We can now compute the triple star product

$$\begin{aligned} \Psi_i \star \Psi_j \star \Psi_k &= e^{i \det \widetilde{Q}_{jk}} \Psi_i \star (\Psi_j \cdot \Psi_k) \\ &= e^{i(\det \widetilde{Q}_{ij} + \det \widetilde{Q}_{jk} + \det \widetilde{Q}_{ik})} \Psi_i \cdot \Psi_j \cdot \Psi_k \,. \end{aligned} \tag{3.28}$$

The computation of the star product in (3.28) is associative. The proof is given in appendix 3.A. To keep the permutation symmetry of the trace operator also in a star product defined theory we use the short-hand notation

$$\begin{aligned} \operatorname{Tr} \Psi_i \star \Psi_j &\equiv \frac{1}{2} (\operatorname{Tr} \Psi_i \star \Psi_j + \operatorname{Tr} \Psi_j \star \Psi_i) \qquad (3.29) \\ &= \frac{1}{2} e^{\gamma_{2(i+j)} \det Q_{ij}} \operatorname{Tr} \Psi_i \cdot \Psi_j + \frac{1}{2} e^{-\gamma_{2(i+j)} \det Q_{ij}} \operatorname{Tr} \Psi_j \cdot \Psi_i \,. \end{aligned}$$

In other words, we must symmetrize the trace explicitly before replacing the ordinary multiplication with the star product. The trace for the triple star product is

$$\begin{aligned} \mathrm{Tr}\,\Psi_i \star \Psi_j \star \Psi_k &= \frac{1}{3}\, e^{i(\gamma_k \det Q_{ij} + \gamma_i \det Q_{jk} + \gamma_j \det Q_{ik})} \\ &\quad \times \left[e^{2i\gamma_i \det Q_{kj}} + e^{2i\gamma_j \det Q_{ik}} + e^{2i\gamma_k \det Q_{ji}} \right] \mathrm{Tr}\,\Psi_i \Psi_j \Psi_k \,. \end{aligned} \tag{3.30}$$

When all deformations parameters are equal we obtain the so-called β-deformed theory with $\beta = \gamma_i$. If not, we have the three-parameter γ_i-deformed theory. In section 3.4 we will compute the star product $\Phi_i \star \Phi_j$ of two β-deformed chiral superfields in the Φ-system. The general results for the γ_i-deformed theory are presented in appendix 3.B.

3.3.3 Superpotential in the one-parameter deformed theory

The β-deformed theory is obtained by setting all γ_i's equal in (3.31). We use the notation $\beta = \gamma_i$. From (3.18) we find that the superfields Ψ_i in the superpotential have charges

$$\begin{aligned} \Psi_1 &: \quad \left(Q_1^{S_1}, Q_1^{S_2}\right) = (0,1) \\ \Psi_2 &: \quad \left(Q_2^{S_1}, Q_2^{S_2}\right) = (-1,-1) \\ \Psi_3 &: \quad \left(Q_3^{S_1}, Q_3^{S_2}\right) = (1,0)\ . \end{aligned} \tag{3.31}$$

In the Ψ-system it is easy to evaluate the star product. From (3.31) and (3.31) we find

$$\begin{aligned} \mathcal{W} &= \mathrm{Tr}\,\Psi_1 \star \Psi_2 \star \Psi_3 - \mathrm{Tr}\,\Psi_1 \star \Psi_3 \star \Psi_2 \\ &= e^{i\beta}\, \mathrm{Tr}\,\Psi_1 \cdot \Psi_2 \cdot \Psi_3 - e^{-i\beta}\, \mathrm{Tr}\,\Psi_1 \cdot \Psi_3 \cdot \Psi_2 \,. \end{aligned} \tag{3.32}$$

Since the superpotential transforms as the determinant of the $SU(3)$ T-matrix in (3.21), we have

$$\mathcal{W} = \mathrm{Tr}\,\Psi_1 \star [\Psi_2 \overset{\star}{,} \Psi_3] = \mathrm{Tr}\,\Phi_1 \star [\Phi_2 \overset{\star}{,} \Phi_3]\ . \tag{3.33}$$

If we use the relation (3.22) between Ψ and Φ we find

$$\Psi_i \Psi_j \Psi_k = \sum_{l,\,m,\,n} \alpha^{(i+2)l+(j+2)m+(k+2)n} t_{il} t_{jm} t_{kn} \Phi_l \Phi_m \Phi_n \,. \tag{3.34}$$

Performing the trace gives

$$\begin{aligned} \mathrm{Tr}\,\Psi_i \Psi_j \Psi_k &= \frac{1}{3} \sum_{l,\,m,\,n} \bar{\alpha}^{l+m+n} \big(\alpha^{il+jm+kn} + \alpha^{kl+im+jn} + \alpha^{jl+km+in} \big) \\ &\quad \times t_{il} t_{jm} t_{kn} \mathrm{Tr}\,\Phi_l \Phi_m \Phi_n . \end{aligned} \tag{3.35}$$

To relate to the superpotential we compute

$$\begin{aligned}\mathrm{Tr}\,\Psi_1\Psi_2\Psi_3 &= \frac{1}{3}\sum_{l,\,m,\,n}\alpha^{n-l}\left(1+\alpha^{l+m+n}+\bar{\alpha}^{l+m+n}\right)\\ &\quad\times t_{1l}t_{2m}t_{3n}\,\mathrm{Tr}\,\Phi_l\Phi_m\Phi_n\,, \end{aligned}\tag{3.36}$$

which is zero unless $l+m+n=0$ mod 3. This implies that the only possible terms are

$$\begin{aligned}\mathrm{Tr}\,\Psi_1\Psi_2\Psi_3 &= \frac{i}{\sqrt{3}}\left[\bar{\alpha}\mathrm{Tr}\,\Phi_1\Phi_2\Phi_3+\alpha\mathrm{Tr}\,\Phi_1\Phi_3\Phi_2\right.\\ &\quad\left.+\,e^{i(2\theta_1+\theta_3)}\mathrm{Tr}\,\Phi_1^3+e^{-i(\theta_1+2\theta_3)}\mathrm{Tr}\,\Phi_2^3+e^{-i(\theta_1-\theta_3)}\mathrm{Tr}\,\Phi_3^3\right]\,.\end{aligned}\tag{3.37}$$

In a similar way, we can compute the remaining part of the superpotential (3.32). The superpotential is invariant under $SU(3)$ so that the phases θ_i can be transformed away by the field redefinition

$$\Phi_i \longrightarrow e^{i(\theta_{i+1}-\theta_i)/3}\Phi_i \tag{3.38}$$

Using (3.32), (3.33), (3.38) and (3.38) gives the β-deformed superpotential

$$\begin{aligned}\mathrm{Tr}\,\Phi_1\star[\Phi_2\,\overset{\star}{,}\,\Phi_3] &= \frac{-2}{\sqrt{3}}\left[\sin(\beta-\frac{2\pi}{3})\mathrm{Tr}\,\Phi_1\Phi_2\Phi_3+\sin(\beta+\frac{2\pi}{3})\mathrm{Tr}\,\Phi_1\Phi_3\Phi_2\right.\\ &+\quad\left.\sin\beta\left(\mathrm{Tr}\,\Phi_1^3+\mathrm{Tr}\,\Phi_2^3+\mathrm{Tr}\,\Phi_3^3\right)\right]\,.\end{aligned}\tag{3.39}$$

3.3.4 Superpotential in the three-parameter deformed theory

In this section we let the three deformation parameters be arbitrary. In a similar way as in the previous section we compute

$$\begin{aligned}\mathrm{Tr}\,\Psi_1\star\Psi_2\star\Psi_3 &= \frac{1}{3}\sum_{i,\,j,\,k}\left(e^{ix}\alpha^{k-i}+e^{iy}\alpha^{j-k}+e^{iz}\alpha^{i-j}\right)\\ &\quad\times t_{1i}t_{2j}t_{3k}\,\mathrm{Tr}\,\Phi_i\,\Phi_j\,\Phi_k\,,\end{aligned}\tag{3.40}$$

and

$$\begin{aligned}\mathrm{Tr}\,\Psi_1\star\Psi_3\star\Psi_2 &= \frac{1}{3}\sum_{i,\,j,\,k}\left(e^{-ix}\bar{\alpha}^{k-i}+e^{-iy}\bar{\alpha}^{j-k}+e^{-iz}\bar{\alpha}^{i-j}\right)\\ &\quad\times t_{1i}t_{3j}t_{2k}\,\mathrm{Tr}\,\Phi_i\,\Phi_j\,\Phi_k\,,\end{aligned}\tag{3.41}$$

where we have introduced

$$x=\gamma_2+\gamma_3-\gamma_1\,,\qquad y=\gamma_3+\gamma_1-\gamma_2\qquad\text{and}\qquad z=\gamma_1+\gamma_2-\gamma_3\,. \tag{3.42}$$

Using (3.33) then gives the superpotential

$$\mathcal{W} = \mathrm{Tr}\,\Phi_1 \star [\Phi_2 \overset{\star}{,} \Phi_3] = \frac{2i}{3} \sum_{i,\,j,\,k} P_{i,\,j,\,k}(x, y, z) t_{1i} t_{2j} t_{3k} \mathrm{Tr}\,\Phi_i\,\Phi_j\,\Phi_k\,, \tag{3.43}$$

where

$$P_{i,\,j,\,k}(x, y, z) = \sin\left(x + (k - i)u\right) + \sin\left(y + (j - k)u\right) + \sin\left(z + (i - j)u\right)\,. \tag{3.44}$$

with $u = 2\pi/3$. Explicitly the terms are

$$\begin{aligned} P_{i,\,i,\,i}(x, y, z) &= \sin(x) + \sin(y) + \sin(z)\,, \\ P_{i,\,i+1,\,i+2}(x, y, z) &= \sin(x - u) + \sin(y - u) + \sin(z - u)\,, \\ P_{i,\,i+2,\,i+1}(x, y, z) &= \sin(x + u) + \sin(y + u) + \sin(z + u)\,. \end{aligned} \tag{3.45}$$

The indices are modulus three. All other terms vanish for any value of x, y and z, due to the cyclic property of the trace operator. The P-functions[4] satisfy the identity

$$P_{i,\,i,\,i}(x, y, z) + P_{i,\,i+1,\,i+2}(x, y, z) + P_{i,\,i+2,\,i+1}(x, y, z) = 0\,. \tag{3.46}$$

Finally, after using the field redefinition (3.38), the γ_i-deformed superpotential becomes

$$\begin{aligned} \mathrm{Tr}\,\Phi_1 \star [\Phi_2 \overset{\star}{,} \Phi_3] &= \frac{-2}{\sqrt{3}} \left[P_{1,2,3}(x, y, z) \mathrm{Tr}\,\Phi_1\Phi_2\Phi_3 + P_{1,3,2}(x, y, z) \mathrm{Tr}\,\Phi_1\Phi_3\Phi_2 \right. \\ &+ \left. P_{1,1,1}(x, y, z) \left(\mathrm{Tr}\,\Phi_1^3 + \mathrm{Tr}\,\Phi_2^3 + \mathrm{Tr}\,\Phi_3^3\right)\right]. \end{aligned} \tag{3.47}$$

The superpotential (3.47) is of the form of the general Leigh-Strassler deformation (3.11) which can be seen by defining

$$h = \frac{-2}{\sqrt{3}} P_{1,2,3}(x, y, z)\,, \qquad q = -\frac{P_{1,3,2}(x, y, z)}{P_{1,2,3}(x, y, z)}\,, \qquad h' = \frac{-2}{\sqrt{3}} P_{1,1,1}(x, y, z)\,. \tag{3.48}$$

3.4 Star product of composite chiral superfields

It is straightforward to compute the star product of two chiral superfields in the Φ-system. These relations are useful when evaluating Feynman diagrams. To begin, we recall (3.22) with inverse

$$\Phi_i = \sum_j \bar{\alpha}^{(i+2)j} t^*_{ji}\,\Psi_j = \sum_j \bar{\alpha}^{(i+2)j} e^{-i(\rho_j + \sum_i \theta_i)}\,\Psi_j\,. \tag{3.49}$$

[4]These functions are not arbitrary named, since the level-set surfaces (3.45) belongs to the class of triply periodic minimal surfaces and are known in the literature as Schwartz's P-surfaces.

which gives the star product

$$\begin{aligned}\Phi_i \star \Phi_j &= \frac{1}{9}\sum_{k,l,m,n} \alpha^{(k+2)(m-i)+(l+2)(n-j)} e^{i\gamma_{2(k+l)}\det Q_{kl}} \\ &\times e^{i(\sum_{\tilde{m}}^{m}\theta_{\tilde{m}}+\sum_{\tilde{n}}^{n}\theta_{\tilde{n}}-\sum_{\tilde{i}}^{i}\theta_{\tilde{i}}-\sum_{\tilde{j}}^{j}\theta_{\tilde{j}})}\,\Phi_m\Phi_n \\ &= \frac{1}{9}\sum_{k,m,n}\alpha^{(k-1)(m+n-i-j)}\left(1+\alpha^{n-j}e^{i\gamma_{k+2}}+\alpha^{m-i}e^{-i\gamma_{k+2}}\right) \\ &\times e^{i(\sum_{\tilde{m}}^{m}\theta_{\tilde{m}}+\sum_{\tilde{n}}^{n}\theta_{\tilde{n}}-\sum_{\tilde{i}}^{i}\theta_{\tilde{i}}-\sum_{\tilde{j}}^{j}\theta_{\tilde{j}})}\,\Phi_m\Phi_n\,. \end{aligned} \tag{3.50}$$

In appendix 3.B we present the explicit expressions for the star product in the γ_i-deformed case. In the β-deformed case the expression (3.50) is considerable simplified. All terms are zero unless $i+j-m-n=0$ mod 3 which gives the expressions

$$\begin{aligned}\Phi_i \star \Phi_i &= \frac{1}{3}\Big[(1+2\cos\beta)\,\Phi_i\Phi_i \\ &+\left(1+2\cos(\beta-\frac{2\pi}{3})\right)e^{i(\theta_1-\theta_3-3\sum_{\tilde{i}}^{i}\theta_{\tilde{i}})}\Phi_{i+1}\Phi_{i+2} \\ &+\left(1+2\cos(\beta+\frac{2\pi}{3})\right)e^{i(\theta_1-\theta_3-3\sum_{\tilde{i}}^{i}\theta_{\tilde{i}})}\Phi_{i+2}\Phi_{i+1}\Big]\,, \\ \Phi_i\star\Phi_{i+1} &= \frac{1}{3}\left[(1+2\cos\beta)\,\Phi_i\Phi_{i+1}+\left(1+2\cos(\beta-\frac{2\pi}{3})\right)\Phi_{i+1}\Phi_i\right. \\ &\left.+\left(1+2\cos(\beta+\frac{2\pi}{3})\right)e^{-i(\theta_1-\theta_3-3\sum_{\tilde{i}}^{i+2}\theta_{\tilde{i}})}\Phi_{i+2}\Phi_{i+2}\right]\,, \\ \Phi_{i+1}\star\Phi_i &= \frac{1}{3}\left[(1+2\cos\beta)\,\Phi_{i+1}\Phi_i+\left(1+2\cos(\beta+\frac{2\pi}{3})\right)\Phi_i\Phi_{i+1}\right. \\ &\left.+\left(1+2\cos(\beta-\frac{2\pi}{3})\right)e^{-i(\theta_1-\theta_3-3\sum_{\tilde{i}}^{i+2}\theta_{\tilde{i}})}\Phi_{i+2}\Phi_{i+2}\right]\,. \end{aligned} \tag{3.51}$$

3.5 Tree-level amplitudes from star product

To begin, we replace the ordinary multiplication between all component fields in the Lagrangian (3.9) by the star product. From (3.31) we find that the component fields have the charges

$$\begin{array}{lcl} \psi_1, \lambda_1 & : & \left(Q_1^{S_1}, Q_1^{S_2}\right) = (0,1) \\ \psi_2, \lambda_2 & : & \left(Q_2^{S_1}, Q_2^{S_2}\right) = (-1,-1) \\ \psi_3, \lambda_3 & : & \left(Q_3^{S_1}, Q_3^{S_2}\right) = (1,0) \\ A^\mu, \lambda_4 & : & \left(Q_4^{S_1}, Q_4^{S_2}\right) = (0,0) \ . \end{array} \tag{3.52}$$

The part

$$\begin{aligned} \mathcal{L}_{inv} \ = \ -\mathrm{Tr}\Bigg(& \frac{\sqrt{2}g}{T(A)} \left(\lambda_4 \left[\phi_i^\dagger, \lambda_i\right] + \bar{\lambda}_4 \left[\lambda_i^\dagger, \phi_i\right]\right) \\ & + \frac{g^2}{2\,T(A)^2} \left[\phi_i^\dagger, \phi_i\right] \left[\phi_j^\dagger, \phi_j\right]\Bigg) \ , \end{aligned} \tag{3.53}$$

of the Lagrangian (3.9) is unchanged when replacing the normal multiplication with the star product. The reasons are that the gluino λ_4 and its conjugate from the vector multiplet are neutral and that the combinations $\lambda_i^\dagger \phi_i$ and $\phi_i^\dagger \phi_i$, with sum over i, are phase-independent.

The terms in the Lagrangian (3.9) that are not invariant under the star product are

$$\begin{aligned} \mathcal{L}_\star \ = \ -\frac{g}{T(A)} \mathrm{Tr}\Big(& \varepsilon^{ijk} \lambda_i \star [\lambda_j \overset{\star}{,} \phi_k] + \varepsilon^{ijk} \lambda_i \star \left[\lambda_j^\dagger \overset{\star}{,} \phi_k^\dagger\right] \\ & + \frac{2g}{T(A)} \left[\phi_i^\dagger \overset{\star}{,} \phi_j^\dagger\right] \star \left[\phi_i \overset{\star}{,} \phi_j\right]\Big) \ . \end{aligned} \tag{3.54}$$

Since the Lagrangian (3.9), and naturally (3.54), is invariant under the transformation (3.21) we are free to express our fields in the ψ-system. From a generalization of the triple star product (3.28) it is easy to evaluate the star product (3.24) to express

$$\begin{aligned} \sum_{i,j} \mathrm{Tr}\left[\phi_i^\dagger \overset{\star}{,} \phi_j^\dagger\right] \star \left[\phi_i \overset{\star}{,} \phi_j\right] \ = \ & 2 \sum_{i,j,k,l} Q^{ijkl}(\gamma_1, \gamma_2, \gamma_3) \\ & \times \prod_{\tilde{k},\tilde{l},\tilde{i},\tilde{j}}^{k,l,i,j} e^{i(\theta_{\tilde{k}} + \theta_{\tilde{l}} - \theta_{\tilde{i}} - \theta_{\tilde{j}})} \mathrm{Tr}\, \phi_i^\dagger \phi_j^\dagger \phi_k \phi_l \ , \end{aligned} \tag{3.55}$$

where we have defined

$$Q^{ijkl} = \sum_m \left[2\cos\left(2\gamma_{m+2} - \frac{2\pi n_1}{3}\right) - (1 + \cos 2\gamma_{m+2}) \cos\frac{2\pi n_2}{3}\right] \alpha^{(m+1)n_3} \ , \tag{3.56}$$

with

$$n_1 = i - j - k + l, \qquad n_2 = i - j + k - l \qquad \text{and} \qquad n_3 = -i - j + k + l\,. \quad (3.57)$$

We can see from (3.56) and (3.56) that interaction terms $\phi_i^\dagger \phi_j^\dagger \phi_k \phi_l$ are allowed for any combination of the indices, in the γ_i-deformed theory. That is, we may have terms with two, three or four indices of the same value. However, in the β-deformed theory, all terms are proportional to the factor $1+\alpha^{i+j-k-l}+\bar{\alpha}^{i+j-k-l}$ which is zero unless $i+j-k-l = 0 \bmod 3$. As a consequence, terms with three indices of the same value vanish. In the non-deformed theory, terms with three or four indices of the same value vanish since the interaction is a product of two commutators. Interaction terms with three indices identical are in general not considered in the context of marginal deformations of $\mathcal{N} = 4$ SYM. Properties of gauge invariance and supersymmetry have to be investigated.

The four-scalar interaction (3.56) of the F-term can be obtained from

$$\mathcal{L}_F = \left(\frac{\partial \mathcal{W}_\star}{\partial \phi_i}\right)^\dagger \star \left(\frac{\partial \mathcal{W}_\star}{\partial \phi_i}\right). \quad (3.58)$$

Replacing the star product between the derivatives by an ordinary multiplication, might at first thought give rise to a new theory without terms with three indices of the same value. However, calculations shows that the new couplings are

$$Q_{new}^{ijkl} = 2\sum_m \left[\cos(2\gamma_{m+2} - 2\pi n_1/3) - \cos(2\pi n_2/3)\right] \alpha^{(m+1)n_3}\,, \quad (3.59)$$

which still contain terms with three identical indices. In obtaining (3.59), the trace is not symmetrized since there is an ambiguity how to perform the symmetrization. It might be possible to overcome this ambiguity by evaluating the star product before defining $\Phi^j \equiv \Phi_a^i T^a$ from which it follows that the structure constants f^{abc} are related to the trace operator. This would make (3.59) a valid relation. In the present context, the general rule is that all multiplication of fields should be replaced by the star product, as in (3.58).

In deriving (3.56), and also (3.59), we have assumed the deformation parameters γ_i to be real. To introduce complex variables we can go back to the definition $\gamma_{2(i+j)} = \tilde{\gamma}_i \tilde{\gamma}_j$, see (3.26), with $\tilde{\gamma}_i = \tilde{\gamma}_i^R + i\tilde{\gamma}_i^C$ where $\tilde{\gamma}_i^R$ and $\tilde{\gamma}_i^C$ are real. This leaves us with the deformations

$$\begin{aligned}
\tilde{\gamma}_i \tilde{\gamma}_{i+1} &= \tilde{\gamma}_i^R \tilde{\gamma}_{i+1}^R - \tilde{\gamma}_i^C \tilde{\gamma}_{i+1}^C + i\left(\tilde{\gamma}_i^R \tilde{\gamma}_{i+1}^C + \tilde{\gamma}_i^C \tilde{\gamma}_{i+1}^R\right) \\
&\equiv \gamma_{i+2}^{R-} + i\gamma_{i+2}^{C+} \\
\tilde{\gamma}_i^* \tilde{\gamma}_{i+1} &= \tilde{\gamma}_i^R \tilde{\gamma}_{i+1}^R + \tilde{\gamma}_i^C \tilde{\gamma}_{i+1}^C + i\left(\tilde{\gamma}_i^R \tilde{\gamma}_{i+1}^C - \tilde{\gamma}_i^C \tilde{\gamma}_{i+1}^R\right) \\
&\equiv \gamma_{i+2}^{R+} + i\gamma_{i+2}^{C-}\,,
\end{aligned} \quad (3.60)$$

in addition to their complex conjugate. In (3.60) there is no obvious way how to separate the real and imaginary part from our original definition of γ_i without introducing extra deformations, corresponding to $\tilde{\gamma}_i^* \tilde{\gamma}_{i+1}$. This complicates the study of the real and complex part of the theory, but might at the same time open up for other interesting possibilities to consider. For complex deformations we find the couplings to be

$$\begin{aligned} Q^{ijkl} &= \sum_m \left[\cos\left(2\gamma^{R-}_{m+2} - un_1\right)\cosh 2\gamma^{C-}_{m+2} + \cos\left(2\gamma^{R+}_{i+2} - un_1\right)\cosh 2\gamma^{C+}_{i+2} \right. \\ &- \left. \cosh\left(2\gamma^{C+}_{m+2} - iun_2\right) - \cos 2\gamma^{R-}_{m+2}\cos un_2\right]\alpha^{(m+1)n_3}\,, \end{aligned} \tag{3.61}$$

where we have used $u = 2\pi/3$. If we let $\gamma^{R+}_{m+2} = \gamma^{R-}_{m+2}$ in (3.60) and (3.61), we obtain the real γ_i-deformed theory with couplings (3.56), as expected.

To compute the star product of the first term in (3.54), we can make use of the transformation (3.21) and the field redefinition (3.38) for the component fields ϕ_i and λ_i. We find

$$\begin{aligned} \varepsilon^{ijk}\mathrm{Tr}\,\lambda_i \star [\lambda_j \,\overset{\star}{,}\, \phi_k] &= \frac{2i}{3}\Big(P_{i,i+1,i+2}(x,y,z)\mathrm{Tr}\left[\lambda_i\lambda_{i+1}\phi_{i+2} - \lambda_i\phi_{i+1}\lambda_{i+2}\right] \\ &+ P_{i,i+2,i+1}(x,y,z)\mathrm{Tr}\left[\lambda_i\lambda_{i+2}\phi_{i+1} - \lambda_i\phi_{i+2}\lambda_{i+1}\right] \\ &+ P_{1,1,1}(x,y,z)\left(\mathrm{Tr}\lambda_1\left[\lambda_1,\phi_1\right]\right. \\ &+ \left.\mathrm{Tr}\lambda_2\left[\lambda_2,\phi_2\right] + \mathrm{Tr}\lambda_3\left[\lambda_3,\phi_3\right]\right)\Big)\,, \end{aligned} \tag{3.62}$$

where we have used the same notation and definitions as in the equations (3.42) and (3.45). The conjugate term can be computed in a similar way and equals

$$\begin{aligned} \varepsilon^{ijk}\mathrm{Tr}\,\lambda_i^\dagger \star \left[\lambda_j^\dagger \,\overset{\star}{,}\, \phi_k^\dagger\right] &= \frac{2i}{3}\Big(P_{i,i+2,i+1}(x^*,y^*,z^*) \\ &+ \times\mathrm{Tr}\left[\lambda_i^\dagger\lambda_{i+1}^\dagger\phi_{i+2}^\dagger - \lambda_i^\dagger\phi_{i+1}^\dagger\lambda_{i+2}^\dagger\right] \\ &+ P_{i,i+1,i+2}(x^*,y^*,z^*)\mathrm{Tr}\left[\lambda_i^\dagger\lambda_{i+2}^\dagger\phi_{i+1}^\dagger - \lambda_i^\dagger\phi_{i+2}^\dagger\lambda_{i+1}^\dagger\right] \\ &+ P_{1,1,1}(x^*,y^*,z^*)\Big(\mathrm{Tr}\lambda_1^\dagger\left[\lambda_1^\dagger,\phi_1^\dagger\right] + \mathrm{Tr}\lambda_2^\dagger\left[\lambda_2^\dagger,\phi_2^\dagger\right] \\ &+ \mathrm{Tr}\lambda_3^\dagger\left[\lambda_3^\dagger,\phi_3^\dagger\right]\Big)\Big)\,, \end{aligned} \tag{3.63}$$

where again the fields have been redefined

3.6 Phase dependence of amplitudes from star product

To compute n-point loop, or just even tree-level, amplitudes is a tedious work. Organizing the Feynman diagrams by decomposed momentum and helicity, instead of momentum and polarized spin, has shown to dramatically reduce their complexity. These MHV diagrams share an iterative structure for computing higher loops [8]. Evaluating HMV amplitudes in a star product deformed theory shows the strength of the procedure.

In [7] it was shown in a β-deformed theory not containing terms $\phi_i^{\dagger 2}\phi_i^2$ that an arbitrary HMV planar tree or loop amplitude has a β-deformed phase factor which can be read off from a single effective vertex. This vertex is only dependent on the external fields and not on the internal structure. In this section we will show that the results found in [7] also hold for our present β- and γ_i-deformed theories. In doing so, we will briefly extend the proof in [7].

The statement is that the deformation dependence for a general n-point HMV planar, tree or loop, amplitude $\mathcal{A}_n(F_1,\ldots,F_n)$ is entirely determined by the configuration of the external fields $F_1,\ldots,F_n$, so that

$$\mathcal{A}_n(F_1,\ldots,F_n): \qquad \mathrm{Tr}\,(F_1\star F_1\ldots\star F_n) = [\mathrm{phase}(\gamma)]\,\mathrm{Tr}\,(F_1F_1\ldots F_n)\ . \tag{3.64}$$

Let us start by considering a general HMV planar tree amplitude. Since an HMV diagram consists of fused vertices of opposite helicity, each propagator is proportional to $F_I^\dagger\star F_I$, with sum over I, which is phase independent due to opposite charges. This means that the internal structure is phase independent. A result which is true for both the β- and the γ_i-deformed theory. Thus, the phase dependence of the amplitude lies entirely in the external fields.

The argument is the same for planar loop amplitudes. Per definition, a planar diagram has no intersecting lines. Each internal line, between two vertices, is proportional to $F_I^\dagger\star F_I$, with sum over I, which again is independent of the phase. Hence the phase dependence of a planar diagram can be computed from an effective tree-level vertex as in (3.64), determined only by external fields.

In the ψ-system, all planar amplitudes in both the β- and γ_i-deformed theories are proportional to their $\mathcal{N}=4$ counterparts. Since $\mathcal{N}=4$ SYM is a finite theory, our derived β- and γ_i-deformed theories should also share the same property of conformal invariance. Since the ψ-system is equivalent to the ϕ-system, through an $SU(3)$ transformation, we can conclude that the Leigh-Strassler deformation obtained from the star product, including diagrams with three indices of the same value, for the specific coupling constants (3.45) and (3.56), are conformal in the planar limit. In the next section we will compute the one-loop finiteness condition. The iterative structure of planar MHV amplitudes in $\mathcal{N}=4$ SYM, studied in [8], should also hold for our deformed theories since the phase dependence can be isolated for each amplitude.

3.7 One-loop finiteness condition

The one-loop finiteness condition is equivalent to the vanishing of the anomalous dimension (3.7) that was discussed in Section 3.2. If we compare (3.3) with the superpotential (3.43) we find that

$$C^{ijk}_{abc} = \frac{1}{2} P_{i,j,k}(x,y,z) t_{1i} t_{2j} t_{3k} \left(f^{abc} + d^{abc} \right) . \tag{3.65}$$

The antisymmetric property of f^{abc} then gives

$$\begin{aligned} C^{ikl}_{acd} \bar{C}^{bcd}_{jkl} &= \frac{1}{4} \sum_i \Big[\left| P_{i,i+1,i+2} - P_{i,i+2,i+1} \right|^2 f^{acd} f_{bcd} \\ &+ \left| P_{i,i+1,i+2} + P_{i,i+2,i+1} \right|^2 d^{acd} d_{bcd} + \left| P_{i,i,i} \right|^2 d^{acd} d_{bcd} \Big] . \end{aligned} \tag{3.66}$$

Using $f^{acd} f_{bcd} = 2N$ and $d^{acd} d_{bcd} = 2N - 8/N$ and explicitly write the P - functions in (3.45), we find the one-loop finiteness condition to be

$$\begin{aligned} g^2_{\gamma_i} &= \frac{3 \left| h_{\gamma_i} \right|^2}{4} \Big[3 \left| \cos x + \cos y + \cos z \right|^2 \\ &+ 2 \left| \sin x + \sin y + \sin z \right|^2 \left(1 - \frac{4}{N^2} \right) \Big] . \end{aligned} \tag{3.67}$$

This simplifies to

$$g^2_\beta = \frac{27 \left| h_\beta \right|^2}{4} \left[3 \left| \cos \beta \right|^2 + 2 \left| \sin \beta \right|^2 \left(1 - \frac{4}{N^2} \right) \right] . \tag{3.68}$$

in the β-deformed theory. The β-deformed theory studied in [7] showed that a complex deformation of the form $\beta = \beta_R + i\beta_C$ gives the one-loop finiteness condition $g^2 \propto |h|^2 \cosh 2\beta_C$ in the large-N limit. Feynman supergraph calculations showed that this planar equivalence with the $\mathcal{N} = 4$ SYM theory holds up to four loops.

In the present β-deformed theory[5], we instead get the planar equivalence

$$g^2_\beta \propto \left| h_\beta \right|^2 \left(2 \cosh 2\beta_C + \sinh^2 \beta_C + \cos^2 \beta_R \right) , \tag{3.69}$$

which is dependent on the parameter β_R. It would be interesting to understand the underlying reason for this dependence in a supergraph formalism.

3.8 Summary and discussion

We have shown that it is possible to obtain the general Leigh-Strassler deformation, including terms of the form $\mathrm{Tr}\, \Phi_i^3$, from the definition (3.24) of the star

[5] Note that here we only have $\beta = \tilde{\beta}\tilde{\beta}$ and $\beta^* = \tilde{\beta}^* \tilde{\beta}^*$. When computing the one-loop conditions, terms as $\tilde{\beta}\tilde{\beta}^*$ are not present, so it is possible to define $\beta = \beta_R + i\beta_C$ where $\beta_R = \beta^{R-}$ and $\beta_C = \beta^{C+}$, with notation as in (3.60).

product. The superpotential has been computed for the β-deformed theory in (3.39) and for the γ_i-deformed theory in (3.47). The analysis was based on two equivalent systems of chiral superfields which we have called the Ψ- and the Φ-system, related by an $SU(3)$ transformation. The latter system corresponds to charges in an off-diagonal matrix obtained from an $SU(3)$ transformation of the diagonal $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetry charges. In the diagonal Ψ-system the star product is easily evaluated.

When we computed the tree-level amplitudes corresponding to terms in the classical Lagrangian we found the expected Leigh-Strassler deformed terms for a β-deformed theory. However, in the γ_i-deformed case, the four-scalar interaction of the F-term contained terms of the form $\mathrm{Tr}\, \phi_i^\dagger \phi_j^\dagger \phi_k \phi_l$ for any value of the indices. Terms with three equal indices vanish in the β-deformed theory, but are present in the γ_i-deformed case.

We have extended the proof in [7] to also cover our present theories. We concluded that for an arbitrary HMV planar tree or loop amplitudes, the phase dependence of the deformation can be computed from an effective tree-level vertex determined only by external fields, and not the internal structure. In the ψ-system (component fields) all planar amplitudes in our present theories are proportional to their $\mathcal{N} = 4$ counterparts. Since $\mathcal{N} = 4$ SYM is a finite theory our present theories should share the same properties. We also concluded that the iterative structure of MHV amplitudes in $\mathcal{N} = 4$ SYM, found in [8], should also hold for our deformed theories. In section 3.7 we computed the one-loop finiteness condition. It would be interesting to find permutation matrices (3.16) of a more general form to establish a relation between coupling constants and more general conditions for a finite theory.

The supergravity dual to the real β-deformed theory was generated in [2], by a combination of T-dualites and a shift (called TsT-transformation) on the isometries of the five-sphere part of $AdS_5 \times S^5$. The complex part of β followed from a non-trivial S-duality transformation. In [14] for bosons and including fermions in [15], it was shown that three consecutive TsT-transformations generate a three-parameter deformation of $AdS_5 \times S^5$. The dual field theory corresponds to a non-supersymmetric three-parameter marginal deformation of $\mathcal{N} = 4$ SYM. It would be interesting to understand if the three-parameter supergravity background can be obtained in a similar way, by consecutive TsT-transformations, for our present theories.

A Lax representation, which implies integrability of strings moving in the Lunin-Maldacena background [2], was also found in [14]. In [16] and [17], it was concluded that the integrability is lost in the planar limit, for complex β-deformed theories. More general Leigh-Strassler deformed theories, containing $\mathrm{Tr}\, \Phi_i^3$, where consider in [12] to study integrability. It would also be interesting to understand if the present results can be translated to a one-loop dilation operator to win insight in the integrability of marginal deformed $\mathcal{N} = 4$ SYM.

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3.A Associativity of the star product

In this appendix we will show that

$$(\Psi_i \star \Psi_j) \star \Psi_k = \Psi_i \star (\Psi_j \star \Psi_k) \ , \tag{3.70}$$

which is to say that the star product (3.24) is associative.

We begin to use the definition (3.27) for a composite field of two fields

$$\widetilde{Q}^1_{ij} \equiv \widetilde{Q}^1_i + \widetilde{Q}^1_j \ , \qquad \text{and} \qquad \widetilde{Q}^2_{ij} \equiv \widetilde{Q}^2_i + \widetilde{Q}^2_j \ , \tag{3.71}$$

so that $\Psi_i \cdot \Psi_j$ is characterized by $(\widetilde{Q}^1_{ij}, \widetilde{Q}^2_{ij})$. The triple star product becomes

$$\Psi_i \star (\Psi_j \star \Psi_k) = e^{i \det \widetilde{Q}_{jk}} \Psi_i \star (\Psi_j \cdot \Psi_k) = e^{i(\det \widetilde{Q}_{jk} + \det \widetilde{Q}_{i,jk})} \Psi_i \cdot \Psi_j \cdot \Psi_k \, , \tag{3.72}$$

where

$$\begin{aligned} \det \widetilde{Q}_{i,jk} \ &\equiv \ \begin{vmatrix} \widetilde{Q}^1_i & \widetilde{Q}^2_i \\ \widetilde{Q}^1_{jk} & \widetilde{Q}^2_{jk} \end{vmatrix} = \begin{vmatrix} \widetilde{Q}^1_i & \widetilde{Q}^2_i \\ \widetilde{Q}^1_j + \widetilde{Q}^1_k & \widetilde{Q}^2_j + \widetilde{Q}^2_k \end{vmatrix} \\ &= \ \widetilde{Q}^1_i \left(\widetilde{Q}^2_j + \widetilde{Q}^2_k \right) - \widetilde{Q}^2_i \left(\widetilde{Q}^1_j + \widetilde{Q}^1_k \right) \\ &= \ \widetilde{Q}^1_i \widetilde{Q}^2_j - \widetilde{Q}^2_i \widetilde{Q}^1_j + \widetilde{Q}^1_i \widetilde{Q}^2_k - \widetilde{Q}^2_i \widetilde{Q}^1_k \\ &= \ \det \widetilde{Q}_{ij} + \det \widetilde{Q}_{ik} \, . \end{aligned} \tag{3.73}$$

Thus, we have

$$\Psi_i \star (\Psi_j \star \Psi_k) = e^{i(\det \widetilde{Q}_{ij} + \det \widetilde{Q}_{jk} + \det \widetilde{Q}_{ik})} \Psi_i \cdot \Psi_j \cdot \Psi_k \, . \tag{3.74}$$

To prove associativity we also have to compute

$$\begin{aligned} (\Psi_i \star \Psi_j) \star \Psi_k \ &= \ e^{i \det \widetilde{Q}_{ij}} \, (\Psi_i \cdot \Psi_j) \star \Psi_k \\ &= \ e^{i(\det \widetilde{Q}_{jk} + \det \widetilde{Q}_{ij,k})} \Psi_i \cdot \Psi_j \cdot \Psi_k \, , \end{aligned} \tag{3.75}$$

where

$$\begin{aligned} \det \widetilde{Q}_{ij,k} \ &\equiv \ \begin{vmatrix} \widetilde{Q}^1_{ij} & \widetilde{Q}^2_{ij} \\ \widetilde{Q}^1_k & \widetilde{Q}^2_k \end{vmatrix} = \begin{vmatrix} \widetilde{Q}^1_i + \widetilde{Q}^1_j & \widetilde{Q}^2_i + \widetilde{Q}^2_j \\ \widetilde{Q}^1_k & \widetilde{Q}^2_k \end{vmatrix} \\ &= \ \det \widetilde{Q}_{ik} + \det \widetilde{Q}_{jk} \, . \end{aligned} \tag{3.76}$$

This means that

$$(\Psi_i \star \Psi_j) \star \Psi_k = e^{i(\det \widetilde{Q}_{ij} + \det \widetilde{Q}_{jk} + \det \widetilde{Q}_{ik})} \Psi_i \cdot \Psi_j \cdot \Psi_k \,. \tag{3.77}$$

Comparing (3.74) and (3.77) proves the associativity (3.70) of the star product.

3.B Star product in γ_i-deformed theory

In this appendix we present the results of star product evaluation of two chiral superfields. We us the same notation as in section 3.4. In the γ_i-deformed case we find

$$\begin{aligned}\Phi_i \star \Phi_i \;=\; & \frac{1}{9}\sum_{j,k}\Bigg[\alpha^{(k-1)(i-j)}(1+2\cos\gamma_k)\prod_{\tilde{i},\tilde{j}}^{i,j} e^{-2i(\theta_{\tilde{i}}-\theta_{\tilde{j}})}\Phi_j\Phi_j \\ & +\alpha^{(k-1)(i-j+1)}\prod_{\tilde{i},\tilde{j}}^{i,j} e^{-i(2\theta_{\tilde{i}}+\theta_{\tilde{j}})}\Big((1+2\cos(\gamma_k-u))\Phi_j\Phi_{j+1} \\ & +(1+2\cos(\gamma_k+u))\Phi_{j+1}\Phi_j\Big)\Bigg]\,,\end{aligned} \tag{3.78}$$

$$\begin{aligned}\Phi_i \star \Phi_{i+1} \;=\; & \frac{1}{9}\sum_{j,k}\Bigg[\alpha^{(k-1)(i-j-1)}(1+2\cos(\gamma_k+u))\prod_{\tilde{i},\tilde{j}}^{i,j} e^{i(\theta_{\tilde{i}+2}-2\theta_{\tilde{j}})}\Phi_j\Phi_j \\ & +\alpha^{(k-1)(i-j)}\prod_{\tilde{i},\tilde{j}}^{i,j} e^{i(\theta_{\tilde{i}+2}-\theta_{\tilde{j}+2})}\Big((1+2\cos\gamma_k)\Phi_j\Phi_{j+1} \\ & +(1+2\cos(\gamma_k-u))\Phi_{j+1}\Phi_j\Big)\Bigg]\,,\end{aligned} \tag{3.79}$$

$$\begin{aligned}\Phi_{i+1} \star \Phi_i \;=\; & \frac{1}{9}\sum_{j,k}\Bigg[\alpha^{(k-1)(i-j-1)}(1+2\cos(\gamma_k-u))\prod_{\tilde{i},\tilde{j}}^{i,j} e^{i(\theta_{\tilde{i}+2}-2\theta_{\tilde{j}})}\Phi_j\Phi_j \\ & +\alpha^{(k-1)(i-j)}\prod_{\tilde{i},\tilde{j}}^{i,j} e^{i(\theta_{\tilde{i}+2}-\theta_{\tilde{j}+2})}\Big((1+2\cos(\gamma_k+u))\Phi_j\Phi_{j+1} \\ & +(1+2\cos\gamma_k)\Phi_{j+1}\Phi_j\Big)\Bigg]\,.\end{aligned} \tag{3.80}$$

For products involving conjugate superfields we find

$$\Phi_i \star \Phi_i^\dagger = \frac{1}{9}\sum_{j,k}\Bigg[\left(3+2\cos\left(\gamma_k-\frac{2\pi}{3}(i-j)\right)\right)\Phi_j\Phi_j^\dagger \tag{3.81}$$

$$+2\bar{\alpha}^{k-1}\cos\left(\gamma_k-\frac{2\pi}{3}(i-j+1)\right)\prod_{\tilde{j}}^{j}e^{i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_j\Phi_{j+1}^\dagger$$

$$+2\,\alpha^{k-1}\cos\left(\gamma_k-\frac{2\pi}{3}(i-j+1)\right)\prod_{\tilde{j}}^{j}e^{-i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_{j+1}\Phi_j^\dagger\Bigg]\,,$$

$$\Phi_i \star \Phi_{i+1}^\dagger = \frac{1}{9}\prod_{\tilde{i}}^{i}e^{-i(\theta_{\tilde{i}}-\theta_{\tilde{i}+1})}\sum_{j,k}\Bigg[2\alpha^{k-1}\cos\left(\gamma_k-\frac{2\pi}{3}(i-j-1)\right)\Phi_j\Phi_j^\dagger$$

$$+\left(3+2\cos\left(\gamma_k-\frac{2\pi}{3}(i-j)\right)\right)\prod_{\tilde{j}}^{j}e^{i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_j\Phi_{j+1}^\dagger$$

$$+2\,\bar{\alpha}^{k-1}\cos\left(\gamma_k-\frac{2\pi}{3}(i-j)\right)\prod_{\tilde{j}}^{j}e^{-i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_{j+1}\Phi_j^\dagger\Bigg]\,, \tag{3.82}$$

$$\Phi_{i+1} \star \Phi_i^\dagger = \frac{1}{9}\prod_{\tilde{i}}^{i}e^{i(\theta_{\tilde{i}}-\theta_{\tilde{i}+1})}\sum_{j,k}\Bigg[2\bar{\alpha}^{k-1}\cos\left(\gamma_k-\frac{2\pi}{3}(i-j-1)\right)\Phi_j\Phi_j^\dagger$$

$$+2\alpha^{k-1}\cos\left(\gamma_k-\frac{2\pi}{3}(i-j)\right)\prod_{\tilde{j}}^{j}e^{i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_j\Phi_{j+1}^\dagger$$

$$+\left(3+2\,\cos\left(\gamma_k-\frac{2\pi}{3}(i-j)\right)\right)\prod_{\tilde{j}}^{j}e^{-i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_{j+1}\Phi_j^\dagger\Bigg]\,, \tag{3.83}$$

$$\Phi_i^\dagger \star \Phi_i = \frac{1}{9}\sum_{j,k}\Bigg[\left(3+2\,\cos\left(\gamma_k+\frac{2\pi}{3}(i-j)\right)\right)\Phi_j^\dagger\Phi_j$$

$$+2\alpha^{k-1}\cos\left(\gamma_k+\frac{2\pi}{3}(i-j+1)\right)\prod_{\tilde{j}}^{j}e^{-i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_j^\dagger\Phi_{j+1}$$

$$+2\bar{\alpha}^{k-1}\cos\left(\gamma_k+\frac{2\pi}{3}(i-j+1)\right)\prod_{\tilde{j}}^{j}e^{i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_{j+1}^\dagger\Phi_j\Bigg]\,, \tag{3.84}$$

$$\Phi_i^\dagger \star \Phi_{i+1} = \frac{1}{9}\prod_{\tilde{i}}^{i} e^{i(\theta_{\tilde{i}}-\theta_{\tilde{i}+1})}\sum_{j,k}\Bigg[2\bar{\alpha}^{k-1}\cos\left(\gamma_k+\frac{2\pi}{3}(i-j-1)\right)e^{-i\theta_3}\Phi_j^\dagger\Phi_j$$
$$+\left(3+2\cos\left(\gamma_k+\frac{2\pi}{3}(i-j)\right)\right)\prod_{\tilde{j}}^{j}e^{-i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_j^\dagger\Phi_{j+1}$$
$$+2\alpha^{k-1}\cos\left(\gamma_k+\frac{2\pi}{3}(i-j)\right)\prod_{\tilde{j}}^{j}e^{i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_{j+1}^\dagger\Phi_j\Bigg], \tag{3.85}$$

$$\Phi_{i+1}^\dagger \star \Phi_i = \frac{1}{9}\prod_{\tilde{i}}^{i} e^{-i(\theta_{\tilde{i}}-\theta_{\tilde{i}+1})}\sum_{j,k}\Bigg[2\alpha^{k-1}\cos\left(\gamma_k+\frac{2\pi}{3}(i-j-1)\right)e^{i\theta_1}\Phi_j^\dagger\Phi_j$$
$$+2\bar{\alpha}^{k-1}\cos\left(\gamma_k+\frac{2\pi}{3}(i-j)\right)\prod_{\tilde{j}}^{j}e^{-i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_j^\dagger\Phi_{j+1}$$
$$+\left(3+2\cos\left(\gamma_k+\frac{2\pi}{3}(i-j)\right)\right)\prod_{\tilde{j}}^{j}e^{i(\theta_{\tilde{j}}-\theta_{\tilde{j}+1})}\Phi_{j+1}^\dagger\Phi_j\Bigg]. \tag{3.86}$$

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