LU TP 01-34
DFPD-01/TH/25
CPT-2001/P.4256
October 2001

The anomalous chiral Lagrangian of order p^6

arXiv:hep-ph/0110400 v1 31 Oct 2001

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PACS: 11.30.Rd, 12.39.Fe, 13.75.-n, 14.70.Bh, 13.60.Le
Keywords: Chiral Lagrangians, Nonperturbative Effects,
Spontaneous Symmetry Breaking, QCD.

Abstract

We construct the effective chiral Lagrangian for chiral perturbation theory in the mesonic odd-intrinsic-parity sector at order p^6. The Lagrangian contains 24 in principle measurable terms and no contact terms for the general case of N_f light flavors, 23 terms for three and five for two flavors. In the two flavor case we need a total of 13 terms if an external singlet vector field is included. We discuss and implement the methods used to reduce to a minimal set. The infinite parts needed for renormalization are calculated and presented as well.

1 Motivation

Effective field theory methods are widely used in physics. Many rely on the spontaneous breaking of an internal continuous symmetry, represented by a compact connected Lie group, G, to a subgroup, H. In the breakdown "massless excitations" appear, which are usually referred to as Goldstone boson modes, $\pi^a(x)$. They parametrize the coset space G/H in terms of a general spacetime dependent G transformation $U(\pi)$ that transforms non-linearly under G and linearly under H. To be more definite, under a global transformation $g \in G$ one maps $\pi \to \pi'$ via

$$gU(\pi) \to U(\pi')h(\pi, g)\,, \tag{1.1}$$

where $h(\pi, g)$ is an element of the unbroken subgroup H [1]. In order to parametrize the low-energy dynamics of a physical system one constructs the most general G-invariant lagrangian density as a sum of monomials, which is an invariant product of covariant derivatives defined on the subalgebra corresponding to H, $\mathcal{H}$ [1]

$$\mathcal{L}(U^{-1}D_\mu^{\mathcal{H}}U, \ldots)\,. \tag{1.2}$$

External fields and explicit symmetry breaking effects can be included as well. However this procedure does not lead to the most general G-invariant action S. As is well-known [2], operators in the lagrangian density that are not invariant under G [$SU(3)_L \times SU(3)_R$ for the QCD case] can lead to a G-invariant action if their variation under G is a total derivative. These operators form the anomalous sector and they arise already at the classical level in the effective field theory.

The power of an effective field theory as given in (1.2) is that its construction only relies on the symmetries of the initial group, broken subgroup and some parameter [derivative] expansion. The latter, with more or less phenomenological insight, is related to the physical problem under study. It is thus clear that the success of the approach depends on the construction of the complete string of operators which form the Lagrangian. The general recipe for the construction is based on writing all possible terms allowed by the continuous and discrete symmetries of the physical problem [3]. The omission of any term would lead to an inconsistent parametrization of the low-energy dynamics invalidating the effective approach. On the other side a parametrization with redundant operators will make the identification of the relevant operators in a physical problem difficult. Therefore it is convenient to obtain the minimal structure of the lagrangian density. Since a long time ago it is known that the anomalous Lagrangian has a clear and nice geometrical interpretation [4]. For QCD-like theories, at leading order in four spacetime dimensions and in the absence of external currents, it is constrained to contain a single term [5] fixed entirely by the single generator of the fifth de Rham cohomology group [6]. Several classifications of the relevant operators at next-to-leading order have been performed [7, 8, 9], but no agreement was reached regarding the number of independent operators involved[1]. We clarify this issue, in particular supplementing the previous analyses

[1] While we were preparing this manuscript Ref. [10] appeared using a different basis.

with a geometrical missing ingredient: the Bianchi identities. Together with the construction of a minimal basis at next-to-leading order we provide the divergent part needed to cancel the ultra-violet behaviour of the loop graphs. This together with the full list and infinities for the even-intrinsic-parity sector of [11] completes the construction of the full p^6 mesonic Lagrangian. We use here the standard power counting where quark masses count as p^2 [12]. For an alternative counting see e.g. [13].

2 Leading order Lagrangian construction

We shall focus the analysis entirely on 4-dimensional field theories where the free gauge group G is coupled vectorially to N_f massless Dirac fermions. All of them transform according to some irreducible representation r of the gauge group. Under the assumption of maximal breaking of chiral symmetry with simultaneous preservation of maximal flavour symmetry there are three allowed scenarios of spontaneous symmetry breaking [14] depending on the representation r. We shall assume that the pattern is given by

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V \tag{2.1}$$

which corresponds to the complex representation for the fermions if $N_f = 2$.

The leading-order anomalous Lagrangian in the absence of external sources is reduced to only a single term [4]. In the case of a spontaneous symmetry breaking given by (2.1) the homotopy group is trivial and a smooth interpolating field between the 4-dimensional Minkowski spacetime and a 5-dimensional ball, B_5, can be found, allowing then to write the action as

$$s[x] = \int_{B_5} d^4x\, dt\, \mathcal{L}_1\,, \tag{2.2}$$

where $\mathcal{L}_1$ is a G-invariant density and its form is restricted by integrability conditions. Using Poincaré's lemma, (2.2) can be cast in terms of a closed 5-form on G/H. The terms that cannot be reduced to four-dimensional integrals are directly obtained by the generators of the fifth de Rham cohomology group $H^5(G/H;\mathbf{R})$. In the case where the coset subgroup is a simple Lie group, H^5 has a single generator

$$\Omega_5 = \frac{-i}{240\pi^2}\langle (U^{-1}dU)^5\rangle\,, \tag{2.3}$$

which is the Wess-Zumino-Witten term. $\langle A\rangle$ corresponds to the trace over the matrix A.

In order to describe phenomenologically relevant processes, the Goldstone boson modes need to be coupled to external gauge invariant sources. This obviously increases the number of possible structures. All the new structures are reduced to exterior derivatives of an invariant 4-form. Such exterior derivatives in terms of the initial 5-form yield terms in (2.2) that can be written as four-dimensional integrals of a G-invariant density. Explicitly the full action at next-to-leading order can be written as [4]

$$S[U,\ell,r]_{\rm WZW} = -\frac{iN_C}{240\pi^2}\int d\sigma^{ijklm}\left\langle \Sigma_i^L\Sigma_j^L\Sigma_k^L\Sigma_l^L\Sigma_m^L\right\rangle$$

$$- \frac{iN_C}{48\pi^2}\int d^4x\,\varepsilon_{\mu\nu\alpha\beta}\left(W(U,\ell,r)^{\mu\nu\alpha\beta} - W(\mathbf{1},\ell,r)^{\mu\nu\alpha\beta}\right), \tag{2.4}$$

$$\begin{aligned} W(U,\ell,r)_{\mu\nu\alpha\beta} = \Big\langle & U\ell_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta + \frac{1}{4}U\ell_\mu U^\dagger r_\nu U\ell_\alpha U^\dagger r_\beta \\ & + iU\partial_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta + i\partial_\mu r_\nu U\ell_\alpha U^\dagger r_\beta - i\Sigma^L_\mu\ell_\nu U^\dagger r_\alpha U\ell_\beta \\ & + \Sigma^L_\mu U^\dagger\partial_\nu r_\alpha U\ell_\beta - \Sigma^L_\mu\Sigma^L_\nu U^\dagger r_\alpha U\ell_\beta + \Sigma^L_\mu\ell_\nu\partial_\alpha\ell_\beta + \Sigma^L_\mu\partial_\nu\ell_\alpha\ell_\beta \\ & - i\Sigma^L_\mu\ell_\nu\ell_\alpha\ell_\beta + \frac{1}{2}\Sigma^L_\mu\ell_\nu\Sigma^L_\alpha\ell_\beta - i\Sigma^L_\mu\Sigma^L_\nu\Sigma^L_\alpha\ell_\beta\Big\rangle - (L\leftrightarrow R), \end{aligned} \tag{2.5}$$

where

$$\Sigma^L_\mu = U^\dagger\partial_\mu U\,, \qquad \Sigma^R_\mu = U\partial_\mu U^\dagger\,, \tag{2.6}$$

and $(L\leftrightarrow R)$ stands for the interchanges $U\leftrightarrow U^\dagger$, $\ell_\mu\leftrightarrow r_\mu$ and $\Sigma^L_\mu\leftrightarrow\Sigma^R_\mu$. The left and right sources, ℓ_μ and r_μ respectively, are defined in terms of the vector (v_μ) and scalar (a_μ) ones as $r_\mu = v_\mu + a_\mu, \ell_\mu = v_\mu - a_\mu$.

3 Renormalization

In quantum field theory the corrections to the Born amplitude will lead to unphysical ultraviolet (UV) divergences. Those divergences should be at most polynomials in the external momenta and/or masses, thus all non-analytical divergences should cancel with each other. To define finite quantities one needs the inclusion of polynomial counter-terms that render any observable free of UV divergences. In order to obtain the full structure of the needed poles arising at one loop level in dimensional regularization we follow the standard procedure and consider the fluctuations around the classical solution of the equation of motion for the Goldstone boson matrix[2] $\bar{U}$

$$\frac{\delta S_2}{\delta U} = 0 \quad \Rightarrow \quad \bar{U}\,. \tag{3.1}$$

This allows to write the expansion

$$U = u(1 + i\xi - \frac{1}{2}\xi^2 + \ldots)u\,, \tag{3.2}$$

where $u^2(x) = \bar{U}(\pi(x))$ and $\xi(x)$ is a traceless hermitian matrix. Eq. (3.1) defines the equation of motion (EOM) in terms of the covariant derivative of the $u(x)$ fields as

$$\nabla^\mu u_\mu = \frac{1}{2}h^\mu_\mu = \frac{i}{2}\left(\chi_- - \frac{1}{n}\langle\chi_-\rangle\right), \tag{3.3}$$

where

$$\begin{aligned} u_\mu &= i\left(u^\dagger\left(\partial_\mu - ir_\mu\right)u - u\left(\partial_\mu - i\ell_\mu\right)u^\dagger\right), \\ \chi_\pm &= u^\dagger\chi u^\dagger \pm u\chi^\dagger u\,, \end{aligned}$$

[2] The subindex in the action functional refers in what follows to the chiral power.

and the covariant derivative

$$\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X]\,,$$

for any operator X, is defined in terms of the connection

$$\Gamma_\mu = \frac{1}{2}\left(u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - i\ell_\mu)u^\dagger\right)\,.$$

We are interested in the second variation of the WZW action, ξ^2 terms. Expanding the functional action up to this order

$$S[U,\mathbf{j}] = S_2[\bar{U},\mathbf{j}] + \left(\frac{1}{2}\xi_i \frac{\delta S_2[\bar{U},\mathbf{j}]}{\delta \bar{U}_i(x_1)\delta \bar{U}_j(x_2)}\xi_j + S_4[\bar{U},\mathbf{j}]\right)$$

it is easy to verify that the second variation of the integral over the five-dimensional manifold, can be expressed as an integral over the ordinary four-dimensional spacetime (the integrand is a total derivative) [15, 16]. This gives us a hint about the possible operators that can appear at the next chiral order. The result of the various terms combine in a chirally covariant form,

$$\begin{aligned}
\delta S_{\rm WZW} \;=\; & \frac{iN_c}{96\pi^2}\int d^4x\,\epsilon^{\mu\nu\alpha\beta}\Big\{\langle(\xi\nabla_\mu\xi - \nabla_\mu\xi\xi)\left[\frac{i}{8}u_\nu u_\alpha u_\beta + \frac{1}{2}u_\nu f_{+\alpha\beta} + \frac{1}{2}f_{+\nu\alpha}u_\beta\right]\rangle \\
& + i\langle(\xi u_\mu\nabla_\nu\xi - \nabla_\nu\xi u_\mu\xi)\left[u_\alpha u_\beta + \frac{1}{2}f_{+\alpha\beta}\right]\rangle \\
& + \frac{i}{4}\langle\xi\xi\,[u_\mu u_\nu, f_{-\alpha\beta}]\rangle - \frac{1}{8}\langle\xi\xi\,[f_{-\mu\nu}, f_{+\alpha\beta}]\rangle \\
& - \frac{i}{4}\langle\xi u_\mu\xi\,\{u_\nu, f_{-\alpha\beta}\}\rangle\Big\},
\end{aligned} \tag{3.4}$$

where the operators

$$f_\pm^{\mu\nu} = uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u$$

are defined in terms of the non-abelian field strengths

$$F_L^{\mu\nu} = \partial^\mu\ell^\nu - \partial^\nu\ell^\mu - i[\ell^\mu,\ell^\nu]\,,\quad F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]\,. \tag{3.5}$$

It is now straightforward to compute in dimensional regularization, the divergent part of the action in terms of an arbitrary number of flavours and colours

$$\begin{aligned}
Z^{\rm WZ\infty}_{\rm 1-loop} \;=\; & \frac{-1}{16\pi^2(d-4)}\frac{N_cN_f}{1152\pi^2F^2}\Big\{4O_1^W + \left(-3 + \frac{6}{N_f^2}\right)O_2^W - \frac{6}{N_f}O_3^W \\
& - 2O_4^W + 4O_5^W + \frac{8}{N_f}O_6^W + \left(-\frac{3}{2} + \frac{6}{N_f^2}\right)O_{11}^W - 2O_{12}^W - 10O_{13}^W \\
& - 3O_{14}^W + O_{15}^W + 2O_{16}^W + O_{17}^W + \frac{6}{N_f}O_{18}^W \\
& - 4O_{19^W} - O_{20}^W + 5O_{21}^W + 4O_{22}^W - \frac{6}{N_f}O_{24}^W\Big\}\,.
\end{aligned} \tag{3.6}$$

The operators O_i^W are defined in Table 2. This agrees with the known result [8, 16]. Notice however, that for the case $N_f = 2$, the divergent part vanishes, due to the Cayley-Hamilton relations. The reason for this is that in the SU(2) case, there are no anomalies in the absence of a singlet vector source, i.e. the homotopy group is reduced to the trivial element. However, the physically interesting situation is when we have a singlet vector source in the formalism. In this last case the initial symmetry is extended to include electromagnetic effects. The initial symmetry group Eq. (2.1) is then modified to

$$SU(2)_L \times SU(2)_R \times U(1)_V \, . \tag{3.7}$$

Bearing in mind that the quark charge matrix, $Q = \mathrm{diag}(2/3, -1/3)$, is not a generator of $SU(2)_L \times SU(2)_R$ the anomaly fails to vanish. Allowing for such external sources, the divergences for the two-flavour case are

$$Z^{\mathrm{WZ}\infty}_{N_f=2} = \frac{-1}{16\pi^2(d-4)} \frac{N_c}{1152\pi^2 F^2} \left\{ 3\, o_6^W + o_7^W - \frac{1}{2}\, o_8^W - 2\, o_9^W - 2\, o_{10}^W + 8\, o_{11}^W - 4\, o_{12}^W - 4\, o_{13}^W \right\} , \tag{3.8}$$

where the operators o_i^W are listed in Table 3, and $o_6^W, \ldots, o_{13}^W$ represent the additional structures required by the inclusion of the singlet vector source. Since the p^4 term only involves singlet currents in this case, it is no surprise that the infinity can also be written in terms of the extra operators only.

4 Next-to-Leading order Lagrangian for generic N_f

In the previous section we have computed the divergent part of the WZW Lagrangian at next-to-leading order. As it has been mentioned already, for any phenomenological purpose, it is crucial to work out a minimal set of operators that reproduces the low-energy dynamics. For this purpose the following list of building blocks are sufficient at next-to-leading order

$$u_\mu \, , \quad h_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu \, , \quad f_{+\mu\nu} \, , \quad f_{-\mu\nu} = \nabla_\nu u_\mu - \nabla_\mu u_\nu \, , \quad \chi_+ \, , \quad \chi_- \, . \tag{4.1}$$

All others can be reduced to these. The choice of basis is motivated first of all by the operators arising in Eqs. (3.6), (3.8), and second because of its relative simplicity for reducing the number of terms. For constructing the Lagrangian, besides the continuous symmetries, one has to implement the discrete symmetries. For the case of QCD: P (parity), C (charge conjugation) and h.c. (hermiticity). The transformation of the basic construction blocks under these is in Table 1. Hermitian conjugation merely determines the presence of a global i factor but the use of C turned out to be quite restrictive in combination with P. In addition to this we have in the anomalous sector the presence of an $\epsilon^{\mu\nu\alpha\beta}$ tensor which is odd under parity.

We now sketch the arguments used during the construction:

i) If a derivative acting on $\chi_\pm$ appears in a given operator, partial integration (PI) always allows to remove it. The presence of $\epsilon^{\mu\nu\alpha\beta}$ allows for at most one power of $\chi_\pm$.

ii) Bianchi Identities (BI) are used to remove all terms with more than two-derivatives. They

operator	P	C	h.c.
u_μ	$-\varepsilon(\mu)u_\mu$	u_μ^T	u_μ
$h_{\mu\nu}$	$-\varepsilon(\mu)\varepsilon(\nu)h_{\mu\nu}$	$h_{\mu\nu}^T$	$h_{\mu\nu}$
$\chi_\pm$	$\pm\chi_\pm$	$\chi_\pm^T$	$\pm\chi_\pm$
$f_\pm^{\mu\nu}$	$\pm\varepsilon(\mu)\varepsilon(\nu)f_\pm^{\mu\nu}$	$\mp f_\pm^{\mu\nu T}$	$f_\pm^{\mu\nu}$

Table 1: P, C and hermiticity properties of the basic operators.

can all be rewritten into terms with $f_{-\mu\nu}$.
iii) If only one field strength is present in the operator, we can remove all the extra derivatives acting on it by PI.
iv) The EOM and commutators allow to remove $\nabla^\mu h_{\mu\nu}$. Using in addition commutators and antisymmetry properties of the indices, we can always remove other terms with an extra derivative on a h.
v) Terms involving the external fields only for the N_f-flavour case cannot be constructed. Terms with one χ or $\chi^\dagger$ are obviously not chiral invariant, terms with two field strengths and two extra derivatives can always be related to terms with 3 field strengths using partial integration and the BI. And finally, terms with 3 field strengths are forbidden since the C and P transformations clash.

Using these rules the full list of operators for the N_f-flavours case contains 57 monomials. Further reduction requires a more extensive study of partial integrations and the use of the lowest order EOM, Eq. (3.3), the Bianchi identities and the Schouten identity [17].

The use of the EOM is equivalent, at lowest order, to a field redefinition, because the generating functional at $\mathcal{O}(p^6)$ contains the classical lagrangian density at $\mathcal{O}(p^6)$, see the proof in [11] as well as the discussion in [18].

The BI yield two relations. The first one follows from the BI of the field strength tensor $\Gamma_{\mu\nu}$

$$\nabla_\mu\Gamma_{\nu\rho}+\nabla_\nu\Gamma_{\rho\nu}+\nabla_\rho\Gamma_{\nu\mu}=0\,,$$

and reads

$$\nabla_\mu f_{+\nu\alpha}+\nabla_\nu f_{+\alpha\mu}+\nabla_\alpha f_{+\mu\nu}=\frac{i}{2}\left([f_{-\mu\nu},u_\alpha]+[f_{-\nu\alpha},u_\mu]+[f_{-\alpha\mu},u_\nu]\right)\,, \tag{4.2}$$

while the second arises using the identity

$$f_-^{\mu\nu}=\nabla^\nu u^\mu-\nabla^\mu u^\nu\,,$$

which leads to

$$\nabla_\mu f_{-\nu\alpha}+\nabla_\nu f_{-\alpha\mu}+\nabla_\alpha f_{-\mu\nu}=\frac{i}{2}\left([f_{+\mu\nu},u_\alpha]+[f_{+\nu\alpha},u_\mu]+[f_{+\alpha\mu},u_\nu]\right)\,. \tag{4.3}$$

We shall also apply the Schouten identity [17],

$$A^\gamma\epsilon^{\mu\nu\alpha\beta}-A^\mu\epsilon^{\gamma\nu\alpha\beta}-A^\nu\epsilon^{\mu\gamma\alpha\beta}-A^\alpha\epsilon^{\mu\nu\gamma\beta}-A^\beta\epsilon^{\mu\nu\alpha\gamma}=0\,, \tag{4.4}$$

which holds for any operator A and can in principle be applied twice to the terms with 6 indices.

In the generic $SU(N_f)_L \times SU(N_f)_R$ case, without the inclusion of a vector singlet field, there are only 33 linearly independent relations which follow from the use of PI, BI and Schouten identities. This leaves us with 24 independent monomials contributing to the N_f-flavour Lagrangian

$$\mathcal{L}_6^W = \sum_{i=1}^{24} K_i^W O_i^W, \tag{4.5}$$

where the K_i^W are coupling constants. The divergences can be subtracted using the usual modified $\overline{\rm MS}$ scheme

$$K_i^W = K_i^{Wr} + \eta_i^{(N_f)} \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left(\ln(4\pi) + \gamma + 1 \right) \right\}, \tag{4.6}$$

where the subtraction coefficients $\eta_i^{(N_f)}$ can be read from Eq. (3.6). The operators O_i^W are listed in Table 2 using the notation

$$[a,b,c] = abc - cba \quad \{a,b,c\} = abc + cba\,.$$

The subtraction coefficients are repeated there for completeness.

5 A minimal set for the Lagrangian with $N_f = 2, 3$

Besides the previous arguments one can make use of the Cayley-Hamilton (CH) relation to reduce the number of operators for the 3 and 2-flavour case. Those relations follow directly from the characteristic polynomial of any matrix. Their use is rather common and we refer to any basic text book in linear algebra or to [9, 11].

In the three flavour case the use of the CH includes only one additional relation with respect to the generic case

$$0 \;=\; 2O_{16}^W \;+\; 2O_{17}^W \;+\; 2\,O_{18}^W \;+\; O_{24}^W\,. \tag{5.1}$$

Thus the number of independent operators for the case $N_f = 3$ is 23. They are listed in Table 2 and they form the lagrangian density

$$\mathcal{L}_6^W = \sum_{i=1}^{23} C_i^W O_i^W\,.$$

As in the previous case the infinities can be subtracted using

$$C_i^W = C_i^{Wr} + \eta_i^{(3)} \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left(\ln(4\pi) + \gamma + 1 \right) \right\} \tag{5.2}$$

where the coefficients $\eta_i^{(3)}$ can be read directly from Eq. (3.6) after using the additional relation (5.1).

monomial (O_i^W)	i N_f-flavours	$384\pi^2 F^2 \eta_i^{(N_f)}$	i 3-flavours	$384\pi^2 F^2 \eta_i^{(3)}$
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- u_\mu u_\nu u_\alpha u_\beta\rangle$	1	$4N_f$	1	12
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ [f_{-\mu\nu}, u_\alpha u_\beta]\rangle$	2	$\left(-3N_f + \frac{6}{N_f}\right)$	2	-7
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ u_\mu\rangle\langle u_\nu f_{-\alpha\beta}\rangle$	3	-6	3	-6
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_- \{f_{+\mu\nu}, u_\alpha u_\beta\}\rangle$	4	$-2N_f$	4	-6
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_- u_\mu f_{+\nu\alpha} u_\beta\rangle$	5	$4N_f$	5	12
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_-\rangle\langle f_{+\mu\nu} u_\alpha u_\beta\rangle$	6	8	6	8
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{+\mu\nu} f_{+\alpha\beta}\rangle$	7	0	7	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_-\rangle\langle f_{+\mu\nu} f_{+\alpha\beta}\rangle$	8	0	8	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{-\mu\nu} f_{-\alpha\beta}\rangle$	9	0	9	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_-\rangle\langle f_{-\mu\nu} f_{-\alpha\beta}\rangle$	10	0	10	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ [f_{+\mu\nu}, f_{-\alpha\beta}]\rangle$	11	$\left(-\frac{3N_f}{2} + \frac{6}{N_f}\right)$	11	$-\frac{9}{2}$
$\epsilon^{\mu\nu\alpha\beta}\langle h_{\gamma\mu} [u_\gamma, u_\nu u_\alpha u_\beta]\rangle$	12	$-2N_f$	12	-6
$i\epsilon^{\mu\nu\alpha\beta}\langle h_{\gamma\mu} \{f_{+\gamma\nu}, u_\alpha u_\beta\}\rangle$	13	$-10N_f$	13	-30
$i\epsilon^{\mu\nu\alpha\beta}\langle h_{\gamma\mu} [f_{+\nu\alpha}, u_\gamma, u_\beta]\rangle$	14	$-3N_f$	14	-9
$i\epsilon^{\mu\nu\alpha\beta}\langle h_{\gamma\mu} [u_\gamma, f_{+\nu\alpha}, u_\beta]\rangle$	15	N_f	15	3
$\epsilon^{\mu\nu\alpha\beta}\langle f_{-\gamma\mu} [u_\gamma, u_\nu u_\alpha u_\beta]\rangle$	16	$2N_f$	16	18
$\epsilon^{\mu\nu\alpha\beta}\langle f_{-\mu\nu} [u_\gamma u_\gamma, u_\alpha u_\beta]\rangle$	17	N_f	17	15
$\epsilon^{\mu\nu\alpha\beta}\langle f_{-\mu\nu} u_\alpha\rangle\langle u_\gamma u_\gamma u_\beta\rangle$	18	6	18	18
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu} \{f_{-\gamma\nu}, u_\alpha u_\beta\}\rangle$	19	$-4N_f$	19	-12
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu} [f_{-\nu\alpha}, u_\gamma, u_\beta]\rangle$	20	$-N_f$	20	-3
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu} [u_\beta, f_{-\nu\alpha}, u_\gamma]\rangle$	21	$5N_f$	21	15
$\epsilon^{\mu\nu\alpha\beta}\langle u_\mu \{\nabla_\gamma f_{+\gamma\nu}, f_{+\alpha\beta}\}\rangle$	22	$4N_f$	22	12
$\epsilon^{\mu\nu\alpha\beta}\langle u_\mu \{\nabla_\gamma f_{-\gamma\nu}, f_{-\alpha\beta}\}\rangle$	23	0	23	0
$\epsilon^{\mu\nu\alpha\beta}\langle f_{-\mu\nu} \{u_\gamma, u_\alpha\}\rangle\langle u_\gamma u_\beta\rangle$	24	-6	-	-

Table 2: The list of operators O_i^W of the p^6 odd intrinsic parity or anomalous chiral Lagrangian. For N_f flavours all 24 are relevant, for 3 flavours O_{24}^W can be dropped. The renormalization coefficients $\eta_i^{N_f}$ are listed as well. The flavour trace is denoted by $\langle\ldots\rangle$ and we use the notation $[a,b,c] = abc - cba$ and $\{a,b,c\} = abc + cba$.

monomial (o_i^W)	i 2-flavour	$384\pi^2 F^2 \eta_i^{(2)}$
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ \, [f_{-\mu\nu}, u_\alpha u_\beta]\rangle$	1	0
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_- \, \{f_{+\mu\nu}, u_\alpha u_\beta\}\rangle$	2	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{+\mu\nu} f_{+\alpha\beta}\rangle$	3	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{-\mu\nu} f_{-\alpha\beta}\rangle$	4	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ \, [f_{+\mu\nu}, f_{-\alpha\beta}]\rangle$	5	0
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle\chi_- u_\alpha u_\beta\rangle$	6	3
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\chi_-\rangle$	7	1
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\rangle\langle\chi_-\rangle$	8	$-\frac{1}{2}$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu}\rangle\langle h_{\gamma\nu} u_\alpha u_\beta\rangle$	9	-2
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu}\rangle\langle f_{-\gamma\nu} u_\alpha u_\beta\rangle$	10	-2
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\gamma\alpha} h_{\gamma\beta}\rangle$	11	8
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\gamma\alpha} f_{-\gamma\beta}\rangle$	12	-4
$\epsilon^{\mu\nu\alpha\beta}\langle\nabla_\gamma f_{+\gamma\mu}\rangle\langle f_{+\nu\alpha} u_\beta\rangle$	13	-4

Table 3: The operators o_i^W and the divergent piece $\eta_i^{(2)}$. Notation as in Table 2. The first five are for traceless external currents. The last 8 are needed if a singlet external vector current is present.

In the two flavour case, the CH relations turn out to be very constraining and the list of 24 monomials in Table 2 can be reduced up to 5. These are the first five operators $o_1^W, \ldots, o_5^W$ listed in Table 3. In addition to those operators the physically interesting case must include the singlet vector source, which is not present in the $SU(N_f)_L \times SU(N_f)_R$ formalism [c.f. (3.7)] thus allowing a traceless $U(1)$ operator, i.e. $\langle f_+^{\mu\nu}\rangle \neq 0$. In this last case, symmetry requirements allow the construction of an additional set of 28 monomials. The use of PI, BI and CH reduces the additional terms up to 8 linear independent operators. The Lagrangian in this case is given by

$$\mathcal{L}_6^W = \sum_{i=1}^{13} c_i^W o_i^W \,, \tag{5.3}$$

where the minimal list of operators o_i is given in Table 3.

As in the general case the infinities can be subtracted using

$$c_i^W = c_i^{Wr} + \eta_i^{(2)} \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left(\ln(4\pi) + \gamma + 1 \right) \right\} \,, \tag{5.4}$$

where the $\eta_i^{(2)}$ coefficients are listed in Table 3, they follow from Eq. (3.4) and the use of relations between these operators when we restrict to the $SU(2)$ algebra and directly from Eq. (3.8).

6 Conclusions

In this work we have determined the minimal anomalous chiral Lagrangian at order p^6. The minimal set of terms is 24 in the general flavour case, 23 for three flavours and 13 for the two-flavour case including for the latter the singlet vector field since it is physically relevant. We have also recalculated the infinite parts for the general flavour case which were known earlier [19, 16] and determined the infinities in the two-flavour case in the presence of a singlet vector field.

Acknowledgements
PT was supported by the EU network EURODAPHNE, EC–Contract No. ERBFMRX–CT980169. LG acknowledges partial support from European Program HPRN-CT-2000-00149. JB thanks the Institute for Nuclear Theory at the University of Washington for hospitality and the Department of Energy for partial support during the completion of this work.

Note added
When completing this work, Ref. [10] appeared. The authors work in a different basis but we fully agree on the number of terms needed. Our notation was chosen to match the one of [11].

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