Supporting Text

Approximation of the Multinomial. Using Stirling’s approximation

\[ n! \approx (n/e)^n \sqrt{2\pi n} \], \hspace{1cm} [S1]

we get

\[ \binom{N}{n} \approx \frac{N^N \sqrt{2\pi N}}{\prod_{0 \leq i < m} n_i^{n_i} \sqrt{2\pi n_i}} \]. \hspace{1cm} [S2]

The right-hand side in Eq. S2 is an upper bound to the multinomial. To show this, we take advantage of the inequality

\[ e^{1/(12n+1)} (n/e)^n \sqrt{2\pi n} < n! < e^{1/(12n)} (n/e)^n \sqrt{2\pi n} \] \hspace{1cm} [S3]

for \( n \geq 1 \), see e.g., (1). Eq. S3 yields

\[ \binom{N}{n} < \frac{N^N \sqrt{2\pi N}}{\prod_{0 \leq i < m} n_i^{n_i} \sqrt{2\pi n_i}} \exp\left( \frac{1}{12N} - \sum_{0 \leq i < m} \frac{1}{12n_i + 1} \right) \]

\[ < \frac{N^N \sqrt{2\pi N}}{\prod_{0 \leq i < m} n_i^{n_i} \sqrt{2\pi n_i}} \], \hspace{1cm} [S4]

where the second inequality holds if at least two of \( n_0, \ldots, n_{m-1} \) are nonzero. If this is not the case, both sides of Eq. S2 are equal to 1.

Expressing \( \langle C_L \rangle_N \) in Terms of \( \langle \Omega_L \rangle_N \). Consider a given network, and let \( C_L \) denote the set of states, \( Q \), that represents a proper \( L \) cycle of the network. Let \( \omega_L \) denote the non-proper counterpart to \( C_L \), meaning that \( \omega_L = \bigcup_{\ell \mid L} C_L \) where \( \ell \mid L \) means that \( \ell \) divides \( L \). Then

\[ C_L = \omega_L \setminus \bigcup_{1 \leq \ell < L \atop \ell \mid L} \omega_\ell = \omega_L \setminus \bigcup_{d \text{ prime} \atop d \mid L} \omega_{L/d} \], \hspace{1cm} [S5]

because any positive \( \ell \) dividing \( L \) is also a divisor to a number of the form \( L/d \) where \( d \) is a prime.

Let \( \Omega_L \) denote the number of elements in \( \omega_L \). Then, the set theoretic principle of inclusion–exclusion, applied to Eq. S5, yields

\[ LC_L = \sum_{s \in \{0,1\}^\eta L} (-1)^s \Omega_{L/d}(s) \], \hspace{1cm} [S6]
where \( s = \sum_{i=1}^{n_L} s_i \), \( d_L(s) = \prod_{i=1}^{n_L} (d^i_L)^{s_i} \) and \( d^1_L, \ldots, d^{n_L}_L \) are the prime divisors to \( L \). For averages over randomly chosen \( N \)-node networks, we get

\[
\langle C_L \rangle_N = \frac{1}{n_L} \sum_{s \in \{0,1\}^{n_L}} (-1)^s \langle \Omega_{L/d_L(s)} \rangle_N .
\]  

[S7]

\( \langle \Omega_L \rangle_N \) is given by

\[
\langle \Omega_L \rangle_N = \sum_{n \in \mathbb{N}^m \atop n=1} \left( \begin{array}{c} N \\ n \end{array} \right) P^N_L(Q) ,
\]  

[S8]

where \( n = n_0 + \cdots + n_{m-1} \). Now, the summation can be split into

\[
\langle \Omega_L \rangle_N = \sum_{\hat{n} \in \mathbb{N}^{m-2}} \sum_{n_0, n_1 \in \mathbb{N} \atop n=1} \left( \begin{array}{c} N \\ n \end{array} \right) P^N_L(Q) ,
\]  

[S9]

where \( \hat{n} = (n_2, \ldots, n_{m-1}) \).

Calculation of the Inner Sum in the Expression for \( \langle \Omega_L \rangle_\infty \). Let

\[
B^N_L(\hat{n}) = \sum_{n_0, n_1 \in \mathbb{N} \atop n=1} \left( \begin{array}{c} N \\ n \end{array} \right) P^N_L(Q) .
\]  

[S10]

Then,

\[
\langle \Omega_L \rangle_N = \sum_{\hat{n} \in \mathbb{N}^{m-2}} B^N_L(\hat{n}) ,
\]  

[S11]

where

\[
B^N_L(\hat{n}) = N! \sum_{n_0, n_1 \in \mathbb{N} \atop n=1} \prod_{0 \leq i < m \atop n_i \neq 0} \frac{[A^i_L(n/N)]^{n_i}}{n_i!} .
\]  

[S12]

To calculate \( B^\infty_L(\hat{n}) \equiv \lim_{N \to \infty} B^N_L(\hat{n}) \), we apply Stirling’s formula to \( N!, n_0!, \) and \( n_1! \), which yields

\[
B^N_L(\hat{n}) \approx \frac{\sqrt{2\pi N^N}}{\sqrt{2\pi e^h}} \sum_{0 < n_0, n_1 \atop n=1} \frac{[A^0_L(n/N)]^{n_0}}{\sqrt{n_0 n_0^0}} \frac{[A^1_L(n/N)]^{n_1}}{\sqrt{n_1 n_1^1}} \prod_{2 \leq i < m \atop n_i \neq 0} \frac{[A^i_L(n/N)]^{n_i}}{n_i!} ,
\]  

[S13]

where the terms with \( n_0 = 0 \) or \( n_1 = 0 \) are ignored. Next, we approximate the sum in Eq. S13 by an integral, yielding

\[
B^N_L(\hat{n}) \approx \frac{N^h}{e^h} \sqrt{\frac{2\pi}{x_0+x_1}} \frac{d\hat{f}_L(x)}{\sqrt{x_0 x_1}} ,
\]  

[S14]
where \( \hat{n} = n_2 + \cdots + n_{m-1}, x = n/N, \hat{x} = \hat{n}/N \) and

\[
\hat{f}_L(x) = x_0 \ln \frac{A_L^0(x)}{x_0} + x_1 \ln A_L^1(x) + \sum_{2 \leq i < m, n_i \neq 0} x_i \ln A_L^i(x) - \sum_{i=2}^{m-1} \ln n_i! . \tag{S15}
\]

Because the only solution to Eq. 22 (given that \( r < 1 \)) is \( x_0 = w_{eq}, x_1 = 1 - w_{eq}, \) and \( x_i = 0 \) for \( i = 2, \ldots, m-1, \) we can find the asymptotic behavior of \( B^\infty_L \) by a Taylor expansion around this point. Let \( x_0 = w_{eq}(1 - \hat{x}) + \epsilon, x_1 = (1 - w_{eq})(1 - \hat{x}) - \epsilon \) and \( x_i = \hat{x} n_i / \hat{n} \) for \( i = 2, \ldots, m-1. \) Then,

\[
A_L^0(x) = w_{eq}(1 - r\hat{x}) + \Delta r \epsilon + \alpha \epsilon^2 + O(\hat{x}^2) + O(\hat{x} \epsilon) + O(\epsilon^3) \tag{S16}
\]

\[
A_L^1(x) = (1 - w_{eq})(1 - r\hat{x}) - \Delta r \epsilon - \alpha \epsilon^2 + O(\hat{x}^2) + O(\hat{x} \epsilon) + O(\epsilon^3) \tag{S17}
\]

and

\[
A_L^i(x) = \hat{x} \cdot \nabla A_L^i + O(\hat{x}^2) + O(\hat{x} \epsilon) \tag{S18}
\]

for \( i = 2, \ldots, m-1, \) where \( \alpha \) is a constant. Here, \( O(\hat{x}^i \epsilon^j) \) stands for an arbitrary function such that the limit of \( O(\hat{x}^i \epsilon^j)/(\hat{x}^i \epsilon^j) \) is well defined as \( (\hat{x}, \epsilon) \to (0,0). \)

A Taylor expansion of Eq. S15 yields

\[
\hat{f}_L(x) = (1 - r)\hat{x} - \frac{(1 - r)^2}{w_{eq}(1 - w_{eq})} \epsilon^2 + \hat{x} \sum_{2 \leq i < m, n_i \neq 0} \frac{n_i}{\hat{n}} \ln \hat{x} \cdot \nabla A_L^i - \sum_{i=2}^{m-1} \ln n_i! + O(\hat{x}^2) + O(\hat{x} \epsilon) + O(\epsilon^3) . \tag{S19}
\]

Completion of the square in Eq. S19 yields

\[
\hat{f}_L(x) = -\frac{1}{2} \left( \frac{1 - r}{w_{eq}(1 - w_{eq})} \epsilon + O(\hat{x}) \right)^2 + (1 - r)\hat{x} + \hat{x} \sum_{2 \leq i < m, n_i \neq 0} \frac{n_i}{\hat{n}} \ln \hat{x} \cdot \nabla A_L^i - \sum_{i=2}^{m-1} \ln n_i! + O(\hat{x}^2) + O(\hat{x} \epsilon^2) + O(\epsilon^3) . \tag{S20}
\]

Now, we can apply the saddle point approximation to Eq. S14 (given that \( r < 1 \)) together with the relation \( \hat{x} = \hat{n}/N, \) the convergence of Stirling’s formula, and the convergence of the integral approximation. This yields

\[
B^\infty_L(\hat{n}) = \frac{1}{1 - \Delta r} e^{-r\hat{n}} \prod_{2 \leq i < m, n_i \neq 0} \frac{(\hat{n} \cdot \nabla A_L^i)^{n_i}}{n_i!} . \tag{S21}
\]
Calculation of $g^\pm_L$. With implicit summation over indices occurring both up and down, Eq. 25 can be written as

$$g^+_L = G_{v^+_L v^-_L} G_{v^+_L v^-_L} \cdots G_{v^+_L v^-_L} G_{v^+_L v^-_L} \equiv \text{Tr}(G^L) \quad [S22]$$

$$g^-_L = S_{v^+_L v^-_L} G_{v^+_L v^-_L} G_{v^+_L v^-_L} \cdots G_{v^+_L v^-_L} G_{v^+_L v^-_L} \equiv \text{Tr}(S G^L), \quad [S23]$$

where

$$S^\mu_\nu^+ \equiv \delta_\nu^+ \delta_\nu^- \mu^+ \quad [S24]$$

and the trace operator is defined as

$$\text{Tr}(A) \equiv A^\mu_\nu^+ \nu^- . \quad [S25]$$

$\delta$ denotes the Kronecker-delta, meaning that $\delta_\nu^\mu = 1$ if $\nu = \mu$ and 0 otherwise.

By transforming $G$ in a suitable way, $g^\pm_L$ can be calculated in a closed form. Central to this transformation are the tensors $M$ and $\tilde{M}$ that extract moments and combinatorial moments according to the definitions

$$M_{\kappa^+ \kappa^-}^{\mu^+ \mu^-} \equiv (\mu^+)^{\kappa^+} (\mu^-)^{\kappa^-} \quad [S26]$$

and

$$\tilde{M}_{\kappa^+ \kappa^-}^{\mu^+ \mu^-} \equiv \mu^+ (\mu^+ - 1) \cdots (\mu^+ - \kappa^+ + 1) \mu^- (\mu^- - 1) \cdots (\mu^- - \kappa^- + 1). \quad [S27]$$

Eq. S26 is interpreted with the convention that $0^0 = 1$ to handle the cases where $\mu^+ = \kappa^+ = 0$ or $\mu^- = \kappa^- = 0$. Henceforth, this convention is used to handle similar special cases conveniently. $\tilde{M}$ is triangular in the sense that $\kappa^+ \leq \mu^+$ and $\kappa^- \leq \mu^-$ for all nonzero elements $\tilde{M}_{\kappa^+ \kappa^-}^{\mu^+ \mu^-}$. Hence, $\tilde{M}$ has an inverse that obeys $\mu^+ \leq \kappa^+$ and $\mu^- \leq \kappa^-$ for all nonzero elements $(\tilde{M}^{-1})_{\mu^+ \mu^-}^{\kappa^+ \kappa^-}$.

Letting $\tilde{M}$ act on $G$ yields

$$\tilde{M}_{\kappa^+ \kappa^-}^{\mu^+ \mu^-} G^{\mu^+ \mu^-} = \sum_{\mu^+ = \kappa^+} \sum_{\mu^- = \kappa^-} \exp(-\tilde{\nu}^+)^{\mu^+} (\tilde{\nu}^-)^{\mu^-} \exp(-\tilde{\nu}^-)^{\mu^+} (\tilde{\nu}^-)^{\mu^-} \quad [S28]$$

$$= (\tilde{\nu}^+)^{\kappa^+} (\tilde{\nu}^-)^{\kappa^-} . \quad [S29]$$

where $\tilde{\nu}^\pm \equiv r^{\nu^\pm} + r^{1\nu^\mp}$.

Let $C_{\lambda^+ \lambda^-}^{\kappa^+ \kappa^-}$ denote the coefficients of the formal expansion of $(z^+ + z^-)^{\lambda^+} (z^+ - z^-)^{\lambda^-}$ in such a way that

$$(z^+ + z^-)^{\lambda^+} (z^+ - z^-)^{\lambda^-} \equiv \sum_{\kappa^+ \kappa^- \in \mathbb{N}} C_{\lambda^+ \lambda^-}^{\kappa^+ \kappa^-} (z^+)^{\kappa^+} (z^-)^{\kappa^-} . \quad [S30]$$

Note that $C$ is block diagonal in the sense that $\kappa^+ + \kappa^- = \lambda^+ + \lambda^-$ for all nonzero elements. The inverse of $C$ is given by

$$(C^{-1})_{\kappa^+ \kappa^-}^{\lambda^+ \lambda^-} = 2^{\kappa^+ - \kappa^-} C_{\kappa^+ \kappa^-}^{\lambda^+ \lambda^-} . \quad [S31]$$
To see this, we consider that $C^2$ yields the coefficients of the formal expansion of the expression
\[
[(z^+ + z^-) + (z^+ - z^-)]^{\lambda^+}[(z^+ + z^-) - (z^+ - z^-)]^{\lambda^-} \equiv 2^{\lambda^+ + \lambda^-} (z^+)^{\lambda^+} (z^-)^{\lambda^-}. \tag{S32}
\]
Thus,
\[
C_{\lambda^+ + \lambda^-}^{\kappa^+ + \kappa^-} = 2^{\lambda^+ + \lambda^-} \delta_{\lambda^+ + \lambda^-}^{\kappa^+ + \kappa^-}, \tag{S33}
\]
which means that the tensor on the left-hand side in Eq. S31 is the inverse of $C$.

Letting $C$ act on $M$ and $\tilde{M}G$ yields
\[
C_{\kappa^+ + \kappa^-}^{\lambda^+ \lambda^-} M_{\mu^+ \mu^-}^{\kappa^+ \kappa^-} = (\mu^+ + \mu^-)^{\lambda^+} (\mu^+ - \mu^-)^{\lambda^-}, \tag{S34}
\]
\[
C_{\kappa^+ \kappa^-}^{\lambda^+ \lambda^-} \tilde{M}_{\mu^+ \mu^-}^{\kappa^+ \kappa^-} G_{\mu^+ \mu^-}^{\lambda^+ \lambda^-} = (\tilde{v}^+ + \tilde{v}^-)^{\lambda^+} (\tilde{v}^+ - \tilde{v}^-)^{\lambda^-} = v^{\lambda^+} (\Delta v)^{\lambda^-} (\nu^+ + \nu^-)^{\lambda^+} (\nu^+ - \nu^-)^{\lambda^-}. \tag{S35}
\]

Hence,
\[
\tilde{C} M G = D C M \tag{S37}
\]
and
\[
\tilde{C} M G \tilde{M}^{-1} C^{-1} = D C M \tilde{M}^{-1} C^{-1}, \tag{S38}
\]
where
\[
D_{\lambda^+ \lambda^-}^{\mu^+ \mu^-} = v^{\lambda^+} (\Delta v)^{\lambda^-} \delta_{\lambda^+ + \mu^+}^{\kappa^+} \delta_{\lambda^- - \mu^-}^{\kappa^-}. \tag{S39}
\]
Because $\tilde{M}^{-1}$ is triangular with the lower indices (acting to the left) less than or equal to the upper indices, right multiplication with $\tilde{M}^{-1}$ always yields a convergent result. Similarly, $C^{-1}$ is wellbehaved in the same sense, as a consequence of the block diagonal structure of $C$. Thus, both sides of the equality in Eq. S38 are welldefined.

Let $T = C M \tilde{M}^{-1} C^{-1}$, which yields
\[
\tilde{C} M G \tilde{M}^{-1} C^{-1} = D T. \tag{S40}
\]

The tensor $\tilde{M}^{-1}$ tells how to express moments in terms of combinatorial moments. Each moment can be expressed as a sum of the combinatorial moment of the same order and a linear combination of combinatorial moments of lower order. Hence,
\[
M_{\kappa^+ \kappa^-}^{\mu^+ \mu^-} (\tilde{M}^{-1}) = \delta_{\kappa^+ + \lambda^+}^{\mu^+} \delta_{\kappa^- - \lambda^-}^{\mu^-} \qquad \text{for } \lambda^+ + \lambda^- \geq \kappa^+ + \kappa^- . \tag{S41}
\]
This property is conserved as $M \tilde{M}^{-1}$ is transformed by $C$, because
\[
C_{\nu^+ \nu^-}^{\nu^+ \nu^-} = 0 \quad \text{for } \kappa^+ + \kappa^- \neq \nu^+ + \nu^-, \tag{S42}
\]
meaning that
\[
T_{\nu^+ \nu^-}^{\mu^+ \mu^-} = \delta_{\nu^+ + \mu^+}^{\mu^+} \delta_{\nu^- - \mu^-}^{\mu^-} \quad \text{for } \mu^+ + \mu^- \geq \nu^+ + \nu^-. \tag{S43}
\]
Eq. S22 can be rewritten as
\[ g^+ = \text{Tr}(G^\ell) = \text{Tr}[(\text{CMG}M^{-1}C^{-1})^\ell] = \text{Tr}[(DT)^\ell]. \] [S44] 

In order to treat Eq. S23 similarly, we need to transform GS in the same manner. To do this, we observe that
\[ (\text{CMG})_{\lambda^+\lambda^-}^{\mu^+\mu^-} S_{\mu^+\mu^-}^{\nu^+\nu^-} = r^{\lambda^+}(\Delta r)^{\lambda^-}(\nu^- + \nu^+)^{\lambda^+}(\nu^+ - \nu^-)^{\lambda^-} \] [S45]
\[ = r^{\lambda^+}(-\Delta r)^{\lambda^-}(\nu^+ + \nu^-)^{\lambda^+}(\nu^+ - \nu^-)^{\lambda^-}, \] [S46]
which means that
\[ \text{CMGS} = \hat{S}\text{CMG} \] [S47]
where
\[ \hat{S}\lambda^+\lambda_-^{\kappa^+\kappa^-} \equiv (-1)^{\lambda^-}\delta_{\lambda^+\kappa^+}\delta_{\lambda^-\kappa^-}. \] [S48]

Hence,
\[ \text{CMGSM}^{-1}C^{-1} = \hat{S}DT \] [S49]
which can be applied to Eq. S23, yielding
\[ g^- = \text{Tr}(SG^\ell) = \text{Tr}[\text{CMGSM}^{-1}C^{-1}(\text{CMG}M^{-1}C^{-1})^{\ell-1}] = \text{Tr}[\hat{S}(TD)^\ell]. \] [S50]

Because T is triangular with unitary diagonal, according to Eq. S43, whereas \( \hat{S} \) and D are diagonal, the traces in Eqs. S44 and S50 are given by
\[ g^+ = \text{Tr}[\hat{S}(TD)^\ell] \] [S51]
\[ = (D^{\mu^+\mu^-})^\ell \] [S52]
\[ = \sum_{\mu^+\mu^- \in N} r^{\ell\mu^+}(\Delta r)^{\ell\mu^-} \] [S53]
\[ = \frac{1}{1 - r^\ell \frac{1}{1 - (\Delta r)^\ell}} \] [S54]
and
\[ g^- = \text{Tr}[\hat{S}(TD)^\ell] \] [S55]
\[ = \tilde{S}^{\mu^+\mu^-}(D^{\mu^+\mu^-})^\ell \] [S56]
\[ = \sum_{\mu^+\mu^- \in N} (-1)^{\mu^-} r^{\ell\mu^+}(\Delta r)^{\ell\mu^-} \] [S57]
\[ = \frac{1}{1 - r^\ell \frac{1}{1 + (\Delta r)^\ell}}. \] [S58]

Finally, we conclude that
\[ g^\pm = \frac{1}{1 - r^\ell \frac{1}{1 \mp (\Delta r)^\ell}}. \] [S59]
**Calculation of** $\langle \Omega_L \rangle_{\infty}$. To use Eq. 29 to calculate $\langle \Omega_L \rangle_{\infty}$, we need to find the distribution of lengths and parities of the invariant sets of $L$ cycle series. For $L$ cycles, there will be invariant sets of length $\ell$ and positive parity if $\ell | L$, whereas sets of negative parity are present if $2\ell | L$. The number of invariant sets with a specific length and parity is independent of $L$ (as long as they exist at all), because the basic form of the series in such sets does not change with $L$. Only the number of basic repetitions alter. Let $J^+_\ell$ and $J^-_\ell$ denote the numbers of invariant sets of length $\ell$ with positive or negative parity, respectively.

Let $\psi^+_\ell$ and $\psi^-_\ell$ denote the sets of (infinite) time series of “true” and “false”, such that each series is identical to, or the inverse of, itself after $\ell$ time steps. Then, the set of time series that are part of an invariant set of $L$ cycles with length $\ell$ and negative parity, $J^-_\ell$, is given by

$$J^-_\ell = \psi^-_\ell \setminus \bigcup_{d \text{ odd prime}} \psi^-_{\ell/d}.$$  \[S60\]

For positive parity, we get

$$J^+_\ell = \psi^+_\ell \setminus \left( \bigcup_{d \text{ prime}} \psi^+_{\ell/d} \right) \setminus J^-_{\ell/2},$$  \[S61\]

where $J^-_{\ell/2}$ is the empty set if $\ell/2$ is not an integer.

The numbers of elements in $J^\pm_\ell$ are given by $2\ell J^\pm_\ell$, where $J^\pm_\ell$ are the numbers of invariant sets with length $\ell$. Then, the inclusion-exclusion principle yields

$$J^-_\ell = \frac{1}{2\ell} \sum_{s \in \{0,1\}^{\tilde{d}(s)}} (-1)^s 2^{\ell/d(s)},$$  \[S62\]

where $s = \sum_{i=1}^{\tilde{n}_\ell} s_i$, $\tilde{d}(s) = \prod_{i=1}^{\tilde{n}_\ell} (d^i_\ell)^{s_i}$ and $d^1_\ell, \ldots, d^{\tilde{n}_\ell}_\ell$ are the odd prime divisors to $\ell$. Similarly

$$J^+_\ell = \frac{1}{2\ell} \sum_{s_0 \in \{0,1\}} \sum_{s \in \{0,1\}^{\tilde{d}(s)}} (-1)^{s_0 + s} 2^{\ell/d(s_0,s)} - \frac{1}{2} J^-_{\ell/2},$$  \[S63\]

where $d(s_0,s) = 2^{s_0} \tilde{d}(s)$ and $J^-_{\ell/2} = 0$ if $\ell/2$ is not an integer. Insertion of Eq. S62 into Eq. S63 yields

$$J^+_\ell = \frac{1}{2\ell} \sum_{s_0 \in \{0,1\}} \sum_{s \in \{0,1\}^{\tilde{n}_\ell}} (1 + s_0)(-1)^{s_0 + s} 2^{\ell/d(s_0,s)},$$  \[S64\]

which also can be written as

$$J^+_\ell = J^-_\ell - J^-_{\ell/2}.$$  \[S65\]
Finally, we can calculate $\langle \Omega_L \rangle_\infty$ according to

$$\langle \Omega_L \rangle_\infty = (1 - r) \prod_{\ell \mid L} (g_\ell^+) \prod_{2\ell \mid L} (g_\ell^-)^{J_\ell^-}. \quad [S66]$$

The factor $(1 - r)$ instead of $1/(1 - \Delta r)$ is there to compensate for the factor $g(\rho_L^0) = g_1^+$, which is not included in Eq. 29.

**Convergence of $\langle C \rangle_\infty$.** For large $\ell$, we can use the approximations

$$\ln g_\ell^\pm = -\ln(1 - r^\ell) - \ln[1 \mp (\Delta r)^\ell] \approx r^\ell \pm (\Delta r)^\ell \quad [S67]$$

and

$$J_\ell^\pm = \frac{2^\ell}{2\ell} \quad [S68]$$

with relative error that decreases exponentially with $\ell$. Thus, the correct asymptotic behavior of $\langle \Omega_L \rangle_\infty$ is revealed by

$$\langle \Omega_L \rangle_\infty \sim \exp \left\{ \sum_{\ell \mid L} \frac{2^\ell}{2\ell} [r^\ell + (\Delta r)^\ell] + \sum_{2\ell \mid L} \frac{2^\ell}{2\ell} [r^\ell - (\Delta r)^\ell] \right\}. \quad [S69]$$

If $r > 1/2$, Eq. S69 diverges double exponentially, because the term $(2r)^L/(2L)$ will dominate the sum as $L \to \infty$. This means that the number of $L$ cycles increases very rapidly with $L$ for large $L$.

If $r < 1/2$, $|\Delta r| < 1/2$ must hold, because $|\Delta r| \leq r$. Then, the total number of states in attractors, $\langle \Omega \rangle_\infty$, will converge, because Eq. S69 then yields the convergent sum

$$\langle \Omega \rangle_\infty \sim \exp \sum_{\ell=1}^\infty \frac{(2r)^\ell}{\ell}. \quad [S70]$$

Hence, the average of the total number of attractors, $\langle C \rangle_\infty$, will converge for $r < 1/2$ and diverge for $r > 1/2$.