

Random Feynman Graphs

B. Söderberg

Complex Systems Division, Dept. of Theoretical Physics, Lund University, Sweden

Abstract. We investigate a class of random graph ensembles based on the Feynman graphs of multidimensional integrals, representing statistical-mechanical partition functions. We show that the resulting ensembles of random graphs strongly resemble those defined in random graphs with hidden color, generalizing the known relation of the Feynman graphs of simple one-dimensional integrals to random graphs with a given degree distribution.

Keywords: graph theory; random graph; phase transition; critical phenomena; percolation; Feynman graph

PACS: 02.50.-r, 64.60.-i, 89.75.Fb

INTRODUCTION

For a long time, the concept of random graphs was synonymous with the classic **ER**-model, which has been extensively studied ever since it was introduced in the sixties [1, 2, 3]. In the last decades, however, driven by the steadily increasing availability of data on large real-world networks (e.g. various social, biological, and information-technological networks), a large variety of more general models have been considered; see e.g. [4] for a fairly up-to-date review.

One of the more commonly studied classes of models is *random graphs with a given degree distribution*, or *the configuration model* [5, 6, 7], here to be referred to as **DRG** (for “degree-driven random graphs”). There, one considers a maximally random graph with N vertices, restricted such that its degree sequence, $\{m_i, i = 1 \dots N\}$, is consistent with a specified degree distribution, $\{p_m\}$. Once the degrees of individual vertices are fixed, edges are formed by means of a random pairing of the entire set of stubs (half-edges), assuming an even total stub-count. This leads in general to a multigraph, with multiple edges and/or self-connections. The resulting graph ensembles are known to lack non-trivial edge correlations.

Recently, extensions of these models based on adding so called *hidden color*, have been studied. In a colored extension of the ER-model, Inhomogeneous Random Graphs (IRG), vertices were assumed to carry color variables, allowed to affect edge formation, which was shown to result in a class of models allowing for non-trivial edge correlations [8], while degree distributions were limited to mixtures of Poissonians.

Similarly, colored extensions of the DRG class of models have been considered, where not vertices but half-edges (*stubs*) were considered to carry a color variable, represented e.g. by an integer variable $c \in [1 \dots K]$, allowed to affect edge probabilities. The degree of a vertex amounts to its stub count; since stubs are colored it is natural to consider the concept of a *colored degree*, in the form of a vector $\mathbf{m} = \{m_c, c = 1 \dots K\}$, where K is

the number of possible colors, and m_c is the number of stubs of a definite color c . The conventional degree m then is obtained simply by summing up the components of its colored degree, $m = \sum_{c=1}^K m_c$. Thus, in the **CDRG** (for *Colored DRG*) class of models [9, 10, 11], random graphs with a given *colored degree distribution* are considered, where a sequence of colored degrees is chosen consistently with a specified distribution. The color is allowed to affect the edge distribution by introducing a color-dependent bias in the random pairing of stubs. The color can either be considered as an observable characteristic, or as invisible once it has done its job (hidden color).

In this article, we will study models of random graphs based on the *Feynman graphs* associated with a class of simple statistical mechanics models, where they represent the deviations of certain integrals from a Gaussian approximation [12]. Relations between DRG models and the Feynman graphs of certain simple univariate integrals are known, and have been explored e.g. in refs. [13, 14].

We will show that such relations are not unique for univariate integrals; indeed, *any* integral of whatever dimension, representing the partition function of a statistical-mechanical system from a large class of models, will define an ensemble of random graphs related to an associated CDRG model, and this is the main result of this article.

One thing to be gained from such relations is an alternative, and perhaps more natural, way to define the associated DRG or CDRG random graph ensembles, in terms of random Feynman Graphs.

In addition, as always when relations are found between different areas of science, one may expect conceptual gains, where phenomena in one area can be simply understood in terms of well-known phenomena in the other. Thus, the relations raise questions, e.g., as to how critical parameter values in the random graph models are mirrored in terms of phase transitions in the associated statistical physics models.

RANDOM FEYNMAN GRAPHS - UNIVARIATE CASE

In field theory and statistical physics, *Feynman graphs* provide a standard method for organizing the perturbative expansion of an integral Z representing some field theoretical or statistical physics model, around an approximate value as given by a Gaussian approximation to the integrand (see e.g. [12]).

Sometimes this approximation is based on a saddle point, an extremal point of the integrand, with the Gaussian chosen to approximate the integrand well in the neighborhood of the saddle point. However, any decomposition of the integrand into the product of a Gaussian factor and a non-trivial remainder will do.

In the spirit of refs. [13, 14], we will consider the possibility of defining ensembles of random graphs – *random Feynman graphs* – based on the Feynman graphs for such a system, with the statistical weight of each distinct graph taken to be proportional to the corresponding term in the perturbative expansion of Z . Much of the contents of this section concerns known results [13, 14], included to make the article reasonably self-contained.

Feynman Graphs for a Simple Integral

In the simplest case, the integral would be over a single real variable, x , say, with an integrand in the form of the exponential of an action $S(x)$, that can be written as the sum of a simple quadratic term $S_0(x)$ that defines the *unperturbed action*, and a non-trivial *perturbation* $V(x)$, typically a more complicated function. Thus, consider the (formal) integral

$$Z = \int e^{S(x)} dx = \int e^{S_0(x)+V(x)} dx = \int e^{-bx^2/2} e^{V(x)} dx \quad (1)$$

where we restrict ourselves to $b > 0$, and to a perturbation V with only non-negative Taylor coefficients,

$$V(x) = \sum_n v_n \frac{x^n}{n!}, \text{ with } v_n \geq 0, \text{ for } n = 0, 1, \dots \quad (2)$$

The Feynman graphs for such an integral represent different terms in the perturbative expansion of Z around the Gaussian approximation, $Z_0 = \int e^{S_0(x)} dx = \int e^{-bx^2/2} dx$.

To see this, first note that considering $V(x)$ as a perturbation means that it is to be considered as small in some sense, and so it is natural to write Z as the sum of a sequence of contributions, obtained by expanding the non-trivial factor $e^{V(x)} = \sum_n V(x)^n/n!$, yielding

$$Z = \sum_N \frac{Z_N}{N!}, \text{ with } Z_N = \int e^{-bx^2/2} V(x)^N dx. \quad (3)$$

Now, each of the N factors of $V(x)$ is to be associated with a distinct *vertex*, labelled $i = 1, \dots, N$. Next, by Taylor-expanding each factor of $V(x)$, Z_N decomposes into a sum of contributions,

$$Z_N = \sum_{\{m_i\}} Z_{\{m_i\}}, \text{ with} \quad (4)$$

$$Z_{\{m_i\}} = \prod_i \frac{v_{m_i}}{m_i!} \int e^{-bx^2/2} \prod_i x^{m_i} dx \quad (5)$$

where the sequence $\{m_i, i = 0, \dots, N\}$ represents the chosen terms $\{v_{m_i} x^{m_i}/m_i!\}$ in the respective Taylor expansions.

Now each factor x^{m_i} in the integrand represents a *vertex* of *degree* m_i , with each single factor of x representing a *stub*. Each term is now easy to evaluate. Denoting by M the total degree $M = \sum_i m_i$, we have $\int e^{-bx^2/2} x^M dx = \varepsilon_M b^{-M/2} (M-1)!!$, where ε_M is a 0/1-variable enforcing the restriction to even M ;

$$\varepsilon_M \equiv \frac{1 + (-1)^M}{2}. \quad (6)$$

The resulting factor of $(M-1)!!$ has an obvious combinatorial interpretation as the *total number of distinct pairings* of the M factors of x . In each pairing, each paired couple of x -factors can be associated with an *edge* between the corresponding vertices i, j .

By considering the m_i distinct factors of x associated with the stubs of a vertex as indistinguishable, pairings can be naturally grouped into equivalence classes. Each such class of equivalent pairings can be associated with a distinct graph, a *Feynman graph* γ , representing a specific contribution to Z , or rather to $\hat{Z} = Z/Z_0$.

$$\hat{Z} = \sum_{\gamma} Z_{\gamma}. \quad (7)$$

The contribution Z_{γ} to the total \hat{Z} from a arbitrary Feynman graph γ defines the *value* of the graph, which is given by the following *Feynman rules*.

1. Each vertex i is associated with a factor of v_{m_i} , where m_i is its degree;
2. Each edge is associated with a factor of b^{-1} ;
3. The vertex and edge factors are multiplied together, and the result divided by the proper edge symmetry factor S_{γ} (and of course by $N!$).

The edge symmetry factor S_{γ} is the usual one, accounting for the order of the edge symmetry group, with a factor of $n!$ for each m -fold edge between distinct vertices, and a factor of $2^n n!$ for each vertex with n self-connections (where the 2^n comes from the symmetry under flipping edges involved in self-connections).

The above rules apply to *labelled graphs*, where vertices are considered distinguishable, while edges are indistinguishable, as are their directions. In many cases one considers instead *unlabelled graphs*; the only difference is that then, also vertices are considered indistinguishable. As a result, the $1/N!$ factor will be replaced by the correct vertex symmetry factor of the particular graph considered, which together with the edge symmetry factor yields a *total symmetry* factor \hat{S}_{γ} .

We will, however, focus on labelled graphs, that have distinguishable vertices.

Saddlepoints and Degree Distributions

Following refs. [13, 14], we now wish to interpret the Feynman graphs associated with a particular integral as *random graphs*, with a statistical weight proportional to the corresponding contribution to \hat{Z} , as given by the Feynman rules above.

Then it is interesting to investigate what kind of random graph ensembles this leads to. For a fixed N , it is obvious that the probability for a certain *degree sequence*, $\{m_i\}$, is given by

$$P\{m_i\} = \frac{Z_{\{m_i\}}}{Z_N} \quad (8)$$

If we wish to focus on the degree of a single vertex, all we have to do is to sum over the $N - 1$ other degrees, to obtain the single vertex *degree distribution*

$$p_m = \frac{v_m \int e^{-bx^2/2} x^m V(x)^{N-1} dx}{m! \int e^{-bx^2/2} V(x)^N dx} \quad (9)$$

having the *generating function*

$$H(z) \equiv \sum_m p_m z^m = \frac{\int e^{-bx^2/2} V(xz) V(x)^{N-1} dx}{\int e^{-bx^2/2} V(x)^N dx} \quad (10)$$

Now, assume for large N that Z_N can be evaluated in a saddlepoint approximation, obtained from the neighborhood of a real, positive saddle point $x = x_0$, satisfying the saddle point condition corresponding to an extremal value of the “effective action”, $S_N(x) = -bx^2/2 + N \log V(x)$,

$$S'_N(x_0) \equiv bx_0 - NV'(x_0)/V(x_0) = 0 \quad (11)$$

To leading order, the saddle point approximation yields

$$Z_N \propto e^{-x_0^2/2} V_0^N, \quad (12)$$

with $V_0 = V(x_0)$, yielding

$$p_m \approx \frac{v_m x_0^m}{m! V_0} \Leftrightarrow H(z) \approx \frac{V(x_0 z)}{V_0}. \quad (13)$$

Thus, the perturbation $V(x)$ has an obvious relation to the generating function for the single-vertex degree distribution of Feynman graphs; this has been pointed out and explored to some extent, e.g. in ref. [13], where the perturbation was set to H from the start. Note, however, that *any* perturbation V with non-negative Taylor coefficients will do; by rescaling the integration variable, V can be recast into the form of H .

We can also consider the joint distribution of the degrees of a small number of vertices. Thus, for the pair distribution, we obtain the nicely factorizable bivariate generating function,

$$H_2(z_1, z_2) \approx \frac{V(x_0 z_1) V(x_0 z_2)}{V(x_0)^2} \approx H(z_1) H(z_2) \quad (14)$$

This factorization generalizes to larger sets, and indicates the approximate *independence* of the degree distributions of distinct vertices.

To clear the view, let us change the integration variable to one rescaled by x_0 : with $y = x/x_0$ and $V_0 = V(x_0)$, the saddle point condition (11) yields $b = NV'(x_0)/x_0 V_0$, and we obtain, apart from an uninteresting constant,

$$Z_N = \int \exp\left(-\frac{N\bar{m}}{2} y^2\right) H(y)^N dy \quad (15)$$

with an integrand designed to have an extremum at $y = 1$; \bar{m} is defined as the expected average degree, as given by $\bar{m} = H'(1)$, while $H(1) \equiv \sum_m p_m = 1$ of course is assumed. Note that the integrand has the form of a fixed factor taken to the N th power, indicating that for large N , the saddle point approximation should be OK (provided the correct saddle point is used).

In what follows, we will assume we have performed such a rescaling to a natural variable y , and that Z_N is defined by eq. (15), with $H(z) = \sum_m p_m z^m$ taken to generate a *desired degree distribution* $\{p_m\}$.

Similarities and Differences to DRG

Based on the expression eq. (15) for Z_N , let us compare the resulting ensemble of *Random Feynman Graphs* (RFG) and compare it to the corresponding DRG model with a degree distribution $\{p_m\}$ given by the one generated by $H(x)$.

Let us first note that Feynman graphs are based on Gaussian integrals, where the value $(M-1)!!$ of the integral $\int x^M e^{-bx^2/2} dx$ for an even M obviously stems from the number of distinct pairings of the M factors of x in x^M . Thus, for a given degree sequence $\{m_i\}$ yielding an even total stub count $M = \sum_i m_i$, the distribution over compatible Feynman graphs is obviously given by random stub pairing, just as in DRG. Thus, the *restrictions to a fixed degree sequence* of a DRG model and the associated RFG model are obviously identical, and so it is enough to compare the distributions over degree sequences, $P\{m_i\}$. In DRG, this factorizes as $\prod_i p_{m_i}$, with the small modification that M must be even.

For RFG, we obtain, with $M \equiv \sum_i m_i$,

$$P\{m_i\} = \prod_i p_{m_i} \frac{\int e^{-\frac{N\bar{m}}{2}x^2} x^M dx}{\int e^{-\frac{N\bar{m}}{2}x^2} H(x)^N dx} \approx \prod_i p_{m_i} (M-1)!! (N\bar{m})^{-M/2} \frac{\int e^{-\frac{N\bar{m}}{2}x^2} x^M dx}{\int e^{-\frac{N\bar{m}}{2}x^2} H(x)^N dx} \quad (16)$$

which, apart from a factor depending only on M , is identical to the DRG expression. This implies that the *restrictions to a fixed total stub count M* of a DRG model and the associated RFG model are *identical*, and so it is enough to compare the distributions over total stub counts, P_M .

The M -distribution is simplest expressed in terms of its generating function, $U(z) = \sum_M P_M z^M$. For DRG, it is essentially given by $U(z) \sim H(z)^N$. The trivial requirement of an even M yields the slightly modified expression

$$U^{\text{DRG}}(z) = \frac{H(z)^N + H(-z)^N}{1 + H(-1)^N} \quad (17)$$

corresponding to the distribution

$$P_M^{\text{DRG}} \propto \varepsilon_M \sum_{\{m_i\}} \delta_{M, \sum m_i} \prod_i p_{m_i} \quad (18)$$

For RFG, on the other hand, the result is obviously given by $P_M^{\text{RFG}} = \frac{Z_{NM}}{Z_N^M}$, yielding the generating function

$$U^{\text{RFG}}(z) = \frac{\int e^{-\frac{N\bar{m}}{2}x^2} H(xz)^N dx}{\int e^{-\frac{N\bar{m}}{2}x^2} H(x)^N dx} \quad (19)$$

The resulting M -distribution can be written as

$$P_M^{\text{RFG}} \propto (M-1)!! (N\bar{m})^{-M/2} P_M^{\text{DRG}}, \quad (20)$$

and that is essentially the *only* difference between the models.

Thus, upon clamping the total stub count M to an even number close to the expected value $\bar{M} = N\bar{m}$, a pair of associated DRG and RFG models are identical [13, 14]. Note

that the relative factor in eq. (20), $(N\bar{m})^{-M/2} P_M^{\text{DRG}}$, is stationary for the expected value $M = N\bar{m}$, which shows that the agreement for clamped M is not very sensitive to the precise clamped value, as long as it is not too far from the expected value \bar{M} .

Validity of the Saddle Point Approximation

At the basis of the above discussion is the assumption that the saddle point contributions really dominates Z_N for large N , which is not obviously always the case. A necessary condition is that $x_0 = 1$ defines a *local maximum* of the effective action $S_{\text{eff}}(x) \propto -\bar{m}x^2/2 + \log H(x)$, yielding the rather weak requirement

$$S_{\text{eff}}''(x=1) \propto \langle m(m-2) \rangle - \langle m \rangle^2 < 0 \Leftrightarrow \langle m^2 \rangle - \langle m \rangle^2 < 2\langle m \rangle \quad (21)$$

This does not appear to correspond to any obvious property of the associated graph ensemble.

In case this condition is not fulfilled, the contributions from $x \approx 1$ will fail to dominate Z_N for large N . A possible scenario then is that a competing saddle point $x_0 \neq 1$ will take over; assume it is on the positive real axis. Then we can always define a *rescaled* integration variable $\tilde{x} = x/x_0$, such that the proper saddle point is moved to $\tilde{x} = 1$. The resulting degree distribution $\{\tilde{p}_m\}$ will have a geometric factor as compared to the expected one,

$$\tilde{H}(\tilde{x}) = \frac{H(x_0\tilde{x})}{H(x_0)} \Leftrightarrow \tilde{p}_m = \frac{x_0^m p_m}{H(x_0)}. \quad (22)$$

Similarly, if we consider the full Z instead of merely the restriction Z_N to a fixed order N , we have, in the rescaled variables,

$$Z \propto \int e^{N_0 s(x)} dx, \quad \text{with} \quad (23)$$

$$s(x) = -\bar{m} \frac{x^2}{2} + H(x) \quad (24)$$

where the parameter N_0 is to be taken as a *desired value* of N . Then $s'(1) = 0$, and we have a saddle point at $x = 1$, just as for Z_N with $N = N_0$. This time, however, the requirement of a local maximum, $s''(1) < 0$, is stronger; furthermore, it has an obvious interpretation: it becomes the well-known condition of *subcriticality*, i.e. the non-existence of a *giant component* in the associated DRG model,

$$-\bar{m} + H''(x) < 0 \Leftrightarrow \langle m(m-2) \rangle < 0 \quad (25)$$

It is also interesting, for a fixed large N_0 , to compare the full Z to the restrictions Z_N for different N in the neighborhood of N_0 , and to the double restrictions Z_{NM} for different M in the neighborhood of $M_0 = N_0\bar{m}$. Of course, Z will almost always define a divergent integral, but we can assume it to be formally dominated by the saddle point value, and compare that to the contributions from different N and M . We have

$$Z = \int e^{-N_0\bar{m}x^2/2 + N_0H(x)} dx \quad (26)$$

$$Z_N = \frac{N_0^N}{N!} \int e^{-N_0 \bar{m} x^2 / 2} H(x)^N dx \quad (27)$$

$$Z_{NM} = \frac{N_0^N}{N!} \oint \frac{dz}{2\pi iz} z^{-M} H(z)^N \int e^{-N_0 \bar{m} x^2 / 2} x^M dx \quad (28)$$

$$(29)$$

where we assume Z to have a proper saddle point at $x = 1$, i.e. $\langle m(m-2) \rangle < 0$. This implies the saddle point value $e^{N_0(1-\bar{m}/2)}$ for Z , as well as for Z_N with $N = N_0$. For Z_N with a slightly different N , the value will to lowest order not change – the prefactor is obviously stationary, and due to the dominance of the saddle point $x = 1$, a slightly different power of $H(1) = 1$ makes no difference.

Thus, Z_N as a function of N will have an extremum at $N = N_0$; furthermore, the corresponding saddle point value coincides with that of Z . In a similar way, Z_{NM} as a function of M for fixed $N = N_0$, will have an extremum for $M = N\bar{m}$, and the saddle point value coincides with that of Z_N . This indicates that under certain circumstances, Z gets its most important contributions from $N \sim N_0$ and $M \sim N_0\bar{m}$, as it should.

RANDOM FEYNMAN GRAPHS - MULTIVARIATE CASE

Now let us investigate instead the slightly more complicated case of a multivariate integral, representing a statistical mechanical partition function for a K -dimensional variable $\mathbf{x} = \{x_a, a = 1 \dots K\}$. Thus, consider the K -dimensional integral

$$Z = \int e^{S(\mathbf{x})} d^K \mathbf{x} = \int e^{S_0(\mathbf{x})+V(\mathbf{x})} d^K \mathbf{x} = \int e^{-\frac{1}{2} \mathbf{x}^\top \mathbf{B} \mathbf{x}} e^{V(\mathbf{x})} d^K \mathbf{x} \quad (30)$$

with an action $S(\mathbf{x})$ in the form of an unperturbed quadratic action $S_0(\mathbf{x}) = -\sum_{ab} x_a B_{ab} x_b / 2$ plus a perturbation $V(\mathbf{x})$. We will restrict our attention to cases where (i) the inverse \mathbf{B}^{-1} of the symmetric matrix \mathbf{B} has no negative elements, and (ii) the perturbation $V(\mathbf{x})$ has only non-negative multivariate Taylor coefficients, i.e.

$$V(\mathbf{x}) = \sum_{\{m_a\}} v_{\{m_a\}} \prod_a \frac{x_a^{m_a}}{m_a!} = \sum_{\mathbf{m}} v_{\mathbf{m}} \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!}, \quad (31)$$

$$\text{with } v_{\mathbf{m}} \geq 0 \quad (32)$$

in terms of the multivariate power $\mathbf{m} = \{m_a, a = 1 \dots K\}$, and $\mathbf{m}! \equiv \prod_a m_a!$.

Feynman Graphs for a Multivariate Integral

The partition function Z can be evaluated in a perturbative manner around the Gaussian approximation, in complete analogy to the univariate case, with the perturbative expansion organized in the form of a sum of contributions associated with Feynman graphs, as follows.

The first step is to expand $e^{V(\mathbf{x})}$, yielding the decomposition

$$Z = \sum_N \frac{Z_N}{N!}, \text{ with} \quad (33)$$

$$Z_N = \int e^{-\frac{1}{2}\mathbf{x}^\top \mathbf{B}\mathbf{x}} V(\mathbf{x})^N d^K \mathbf{x} \quad (34)$$

Then, each of the N factors $V(\mathbf{x})$ is associated with a distinct vertex i , and Taylor-expanded to yield

$$Z_N = \sum_{\{\mathbf{m}_i\}} Z_{\{\mathbf{m}_i\}}, \text{ with} \quad (35)$$

$$Z_{\{\mathbf{m}_i\}} = \prod_{i=1}^N \frac{v_{\mathbf{m}_i}}{\mathbf{m}_i!} \int e^{-\frac{1}{2}\mathbf{x}^\top \mathbf{B}\mathbf{x}} \prod_i \mathbf{x}^{\mathbf{m}_i} d^K \mathbf{x} \quad (36)$$

For each of the factors $\mathbf{x}^{\mathbf{m}_i}$, its multivariate power \mathbf{m}_i is interpreted as the *colored degree* of vertex i ; thus, each of its m_{ia} factors x_a is associated with a stub with color a , belonging to vertex i , which has the (plain) degree $m_i = \sum_a m_{ia}$.

It is obvious that the integral in eq. (36) depends on $\{\mathbf{m}_i\}$ only via the total colored stub count $\mathbf{M} = \sum_i \mathbf{m}_i$. Again, as in the univariate case, the value can be seen as arising from all possible ways to pair the $M = \sum_a M_a$ (which again must be even) distinct \mathbf{x} -factors.

The pairings can be organized in equivalence classes, corresponding to distinct *Feynman graphs*, as usual characterized by a unique adjacency matrix $\{n_{ij}\}$. Each element n_{ij} counts the number of stubs of vertex i that are paired with a stub from vertex j . The value of $\hat{Z} = Z/Z_0$, with $Z_0 = \int e^{-\frac{1}{2}\mathbf{x}^\top \mathbf{B}\mathbf{x}} d^K \mathbf{x}$, becomes the sum over contributions from the individual Feynman graphs γ ,

$$\hat{Z} = \sum_{\gamma} Z_{\gamma} \quad (37)$$

These counts can be further decomposed into the elements of a *colored adjacency matrix* $\{n_{iajb}\}$, with elements n_{iajb} counting the number of a -colored stubs of vertex i that are paired with a b -colored stub of vertex j , such that the sum rules $m_{ia} = \sum_{jb} n_{iajb}$ and $n_{ij} = \sum_{ab} n_{iajb}$ hold.

Thus, the contribution to \hat{Z} from each graph γ – its *value* Z_{γ} – is due to a sum over pairings consistent with that graph, and is given by the following *Feynman rules* for the contribution.

1. Each stub in γ is assigned an independent color variable a .
2. Each vertex i is associated with a factor $v_{\mathbf{m}_i}$, where $\mathbf{m}_i = \{m_{ia}\}$ is its colored degree;
3. Each edge with stub colors (a, b) at its endpoints is associated with a factor of \mathbf{B}_{ab}^{-1} ;
4. The vertex and edge factors are multiplied together, the result is summed over all stub colors, and the result divided by the proper edge symmetry factor S_{γ} (and of course by $N!$).

These rules are obvious generalizations of the univariate versions.

Now we want to interpret these Feynman graphs as *random graphs* in a multivariate version of *RFG*, with a statistical weight for each distinct graph proportional to the corresponding graph value as given by the Feynman rules.

Saddlepoints and Colored Degree Distributions

We wish to investigate the properties of the resulting graph ensemble. To that end, let us investigate the degrees, or even better, the *colored degrees*, of the resulting graphs. For a fixed N , the distribution over colored degree sequences $\{\mathbf{m}_i\}$ is obviously given by

$$P\{\mathbf{m}_i\} = \frac{Z_{\{\mathbf{m}_i\}}}{Z_N} \quad (38)$$

By summing over the colored degrees of all vertices except one, the colored degree distribution of a single vertex results, and is given by

$$p_{\mathbf{m}} = \frac{v_{\mathbf{m}} \int e^{-\frac{1}{2}\mathbf{x}^T \mathbf{B} \mathbf{x}} \mathbf{x}^{\mathbf{m}} V(\mathbf{x})^{N-1} d^K \mathbf{x}}{\mathbf{m}! \int e^{-\frac{1}{2}\mathbf{x}^T \mathbf{B} \mathbf{x}} V(\mathbf{x})^N d^K \mathbf{x}} \quad (39)$$

As was the case for univariate integrals, we can hope to evaluate this with *saddle point* methods in the thermodynamic limit of $N \rightarrow \infty$. For a point \mathbf{x}_0 to be a saddle point of Z_N , it must render the integrand of Z_N extremal, corresponding to the saddle point condition

$$0 = -\mathbf{B}\mathbf{x}_0 + N \frac{\nabla V}{V} \Rightarrow \mathbf{B}\mathbf{x}_0 V(\mathbf{x}_0) = N \nabla V(\mathbf{x}_0) \quad (40)$$

For the single-vertex colored degree distribution, the saddle point approximation yields

$$p_{\mathbf{m}} = \frac{v_{\mathbf{m}} \mathbf{x}_0^{\mathbf{m}}}{\mathbf{m}! V(\mathbf{x}_0)} \quad (41)$$

with the generating function $H(\mathbf{z}) = \sum_{\mathbf{m}} p_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ given by

$$H(\mathbf{z}) = \frac{V(\mathbf{x}_0 \circ \mathbf{z})}{V(\mathbf{x}_0)} \quad (42)$$

where the ring “ \circ ” represents the componentwise multiplication of two vectors.

Assuming the saddle point \mathbf{x}_0 to have entirely positive components, $x_{0,a} > 0$, we can change integration variables to rescaled variables \mathbf{y} , given by $\mathbf{x} = \mathbf{y} \circ \mathbf{x}_0$. Accordingly, we can define $V_0 = V(\mathbf{x}_0)$, in terms of which $V(\mathbf{x}) = V(\mathbf{y} \circ \mathbf{x}_0) = V_0 H(\mathbf{y})$. We also choose to define a transformed matrix \mathbf{T} from $N\mathbf{T}^{-1} = \mathbf{x}_0 \circ \mathbf{B} \circ \mathbf{x}_0$, yielding $T_{ab} \geq 0$.

For Z_N , this yields the expression

$$Z_N \propto \int e^{-\frac{N}{2}\mathbf{y}^T \mathbf{T}^{-1} \mathbf{y}} H(\mathbf{y})^N d^K \mathbf{y} \quad (43)$$

with an integrand constructed to have a saddle point at $\mathbf{y} = \mathbf{1} = \{1, \dots, 1\}$, which implies the following constraint on the matrix \mathbf{T} .

$$\mathbf{T}^{-1}\mathbf{1} = \frac{\nabla H(\mathbf{1})}{H(\mathbf{1})} \Rightarrow \mathbf{T}\bar{\mathbf{m}} = \mathbf{1} \quad (44)$$

where we have introduced the *average colored degree*, $\bar{\mathbf{m}} = \nabla H|_{\mathbf{y}=\mathbf{1}}$, while normalization of the colored degree distribution with necessity implies $H(\mathbf{1}) = 1$.

Similarities and Differences to CDRG

Now let us assume we are using the convenient coordinates \mathbf{x} , with the saddle point asymptotically assumed to be at $\mathbf{x} = \mathbf{1}$, in terms of which Z_N is defined by eq. (43).

From the above discussion, it should be obvious that the multivariate RFG approach shows strong similarities to the corresponding CDRG model [10], which is defined in terms of a given graph size N , a color set $[1, \dots, K]$, a *colored degree distribution* $\{p_{\mathbf{m}}\}$, and a *color preference matrix*, $\mathbf{T} = \{T_{ab}\}$, with non-negative elements T_{ab} , required to fulfill the constraint eq. (44), as follows. For each vertex, its colored degree is drawn independently from the given distribution. Then a *weighted random pairing* of the full set of $M = \sum_a M_a$ colored stubs is performed, where each edge connecting colors a, b provides a weight factor $\propto T_{ab}$.

For the random Feynman graphs, the distribution over colored degree sequences, for fixed N , becomes

$$P_{\{\mathbf{m}_i\}}^{\text{RFG}} = \frac{Z_{\{\mathbf{m}_i\}}}{Z_N} = \prod_i p_{\mathbf{m}_i} \frac{\int e^{-\frac{N}{2}\mathbf{x}^\top \mathbf{T}^{-1} \mathbf{x}} \mathbf{M} d^K \mathbf{x}}{\int e^{-\frac{N}{2}\mathbf{y}^\top \mathbf{T}^{-1} \mathbf{y}} H(\mathbf{y})^N d^K \mathbf{y}} \quad (45)$$

This can be compared to the corresponding CDRG result, which of course has the factorized structure

$$P_{\{\mathbf{m}_i\}}^{\text{CDRG}} = \prod_i p_{\mathbf{m}_i} \quad (46)$$

In analogy to the univariate case, we note that the associated RFG and CDRG ensembles agree on (1) the distribution over colored graphs conditional on a fixed colored degree sequence, and (2) the distribution over colored degree sequences conditional upon a fixed total colored stub count \mathbf{M} . The former follows from the equivalence of the random pairing step involved in both ensembles, and the latter from a direct comparison of eqs. (45,46).

Thus, the two ensembles only disagree on the distribution over the total colored stub count \mathbf{M} ; once it is clamped to a certain value, the two yield identical distributions over graphs, even before summing over colors.

In agreement with the univariate case, the condition for the saddle point at $\mathbf{x} = \mathbf{1}$ to define a local maximum of the integrand for the formal integral $Z = \int e^{N_0(-\mathbf{x}^\top \mathbf{T}^{-1} \mathbf{x}/2 + H(\mathbf{x}))} d^K \mathbf{x}$ is equivalent to the condition of *subcriticality* of the associated CDRG model [10]. Also, Z is formally dominated by graphs with $N = N_0$

and $\mathbf{M} = N_0 \bar{\mathbf{m}}$, in the sense that the saddle point values of Z and the restrictions Z_N and Z_{NM} all agree to leading order, while Z_N as a function of N is stationary for $N = N_0$, as is $Z_{N_0 \mathbf{M}}$ as a function of \mathbf{M} for $\mathbf{M} = N_0 \langle \mathbf{m} \rangle$.

CONCLUSIONS

A large class of ensembles of Feynman graphs for integrals representing statistical-mechanical partition functions have been investigated as models for random graphs, with the statistical weight of a graph taken to be proportional to its associated value as a contribution to the integral.

For the case of univariate integrals, it was previously known [13] that the resulting random Feynman graph ensembles closely resemble those of random graphs with a given degree distribution, or DRG. In this article, it was found that for the case of *multivariate* integrals, the associated random Feynman graphs in a similar way resemble those of CDRG, a stub-colored extension of DRG. These findings are new, although such relations were conjectured in ref. [10]

For the restriction of the ensembles to a clamped value of the total (colored) stub count, the agreement was in fact shown to be exact, while the distribution over the total (colored) stub count was shown to differ, which leads to differences also in the resulting degree distributions. We speculate that this may be the effect of the random Feynman graph ensemble defining a kind of *annealed approximation* to the (C)DRG one, where the disorder as defined by the random (colored) degree sequence is considered *quenched*, i.e. randomly chosen from a given distribution, then clamped in the random pairing step. In the random Feynman graphs, as a contrast, both degrees and pairings fluctuate on an equal level.

The establishment of close relations between different models opens up the stage for questions as to the relation between the critical phenomena in the graph models (such as the appearance of a giant component) and those in the associated statistical-mechanical models. Some relations of this type was found, but more work is needed to further illuminate this issue.

Finally, we note that a straightforward generalization to random Feynman graph models for *directed* stub-colored graphs should be possible by considering suitably defined multidimensional integrals over a set of *complex* variables.

ACKNOWLEDGMENTS

It is a pleasure to thank the organizers of CNET2004 for the invitation, and for a well organized conference in the charming city of Aveiro. This work has in part been supported by the Swedish Foundation for Strategic Research.

REFERENCES

1. Erdős, P., and Rényi, A., *Publ. Math. Inst. Hungar. Acad. Sci.*, **5**, 17 (1960).
2. Janson, S., Łuczak, T., and Ruciński, A., *Random Graphs*, Wiley and Sons, New York, 2000.

3. Bollobás, B., *Random Graphs, 2nd ed.*, Cambridge University Press, Cambridge, 2001.
4. Dorogovtsev, S. N., and Mendes, J. F. F., *Evolution of Networks*, Oxford University Press, Oxford, 2003.
5. Bender, E. A., and Canfield, E. R., *J. Combinat. Theory, Ser. A*, **24**, 296–307 (1978).
6. Molloy, M., and Reed, B., *Rand. Struct. Alg.*, **6**, 161–180 (1995).
7. Newman, M. E. J., Strogatz, S. H., and Watts, D. J., *Phys. Rev. E*, **64**, 026118 (2001).
8. Söderberg, B., *Phys. Rev. E*, **66**, 066121 (2002).
9. Söderberg, B., *Phys. Rev. E*, **68**, 015102(R) (2003).
10. Söderberg, B., *Phys. Rev. E*, **68**, 026107 (2003).
11. Söderberg, B., *Acta Phys. Pol. B*, **34**, 5085–5102 (2003).
12. Itzykson, C., and Zuber, J.-B., *Quantum Field Theory*, McGraw-Hill, New York, 1980.
13. Burda, Z., Correia, J. D., and Krzywicki, A., *Phys. Rev. E*, **64**, 046118 (2001).
14. Dorogovtsev, S. N., Mendes, J. F. F., and Samukhin, A. N., *Nucl. Phys. B*, **666**, 396 (2003).