LU TP 05-26
hep-lat/0506004
revised
July 2005

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Abstract

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PACS: 12.38.Gc, 12.39.Fe, 11.30.Rd

Masses and Decay Constants of Pseudoscalar Mesons to Two Loops in Two-Flavor Partially Quenched Chiral Perturbation Theory

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This paper presents a first study of the masses and decay constants of the charged, or flavor-off-diagonal, pseudoscalar mesons to two loops for two flavors of sea-quarks, in Partially Quenched Chiral Perturbation Theory (PQχPT). Explicit analytical expressions up to $\mathcal{O}(p^6)$ in the momentum expansion are given. The calculations have been performed within the supersymmetric formulation of PQχPT. A numerical analysis is done to indicate the size of the corrections.

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I. INTRODUCTION

At the present time, the most promising way to derive information on low-energy hadronic observables directly from QCD is by means of numerical Lattice QCD simulations. However, it is well known that such simulations are computationally very demanding if dynamical sea-quark effects (unquenched Lattice QCD) are to be taken into account. Although much progress has been made recently, simulations with sea-quark masses that are close to the physical u, d quark masses of a few MeV are still not available, whereas sea-quark masses down to several tens of MeV are becoming possible. The physical values of the quark masses then have to be reached by extrapolation from these simulations.

A generalization of QCD where the valence quark masses are different from the sea-quark masses is possible, and is called Partially Quenched QCD (PQQCD). QCD may then be obtained as a continuous limit from this theory, in contrast to the quenched calculations where no such smooth connection exists. The reason for doing PQQCD Lattice simulations is that the valence quark masses can be varied much more efficiently than the sea-quark masses, which allows for more extensive studies of the quark mass dependence of the various observables.

The generalization of χPT to sea-quark masses different from the valence ones, i.e. to the partially quenched case [1, 2], provides a practical method for the extrapolation to physical quark masses from the lattice simulations of PQQCD. The quark mass dependence of partially quenched chiral perturbation theory (PQχPT) is explicit, and thus the limit where the sea-quark masses become equal to the valence quark masses can be taken. As a consequence, unquenched χPT [3] is included in PQχPT, and the free parameters, or low-energy constants (LEC:s), of χPT can be directly obtained from those of PQχPT [2, 4]. The need for PQχPT calculations to next-to-next-to-leading order (NNLO) has also been demonstrated by Lattice QCD simulations [5, 6].

The calculation of the charged pseudoscalar meson masses and decay constants to one loop (NLO) in PQχPT has been carried out in Refs. [2, 4, 7], and first results for the mass of a pseudoscalar meson at two loops (NNLO) in PQχPT have already appeared, for three flavors of degenerate sea-quarks [8]. The decay constants of the pseudoscalar mesons are also known to NNLO [9] for three sea-quarks. A full PQχPT calculation of the pseudoscalar meson masses for three flavors of sea-quarks is in progress [10]. However, as many Lattice QCD simulations are performed with two rather than three sea-quark flavors, such expressions for the pseudoscalar meson masses and decay constants to NNLO are also called for.

This paper presents the full calculation of the masses and decay constants of the charged, or flavor off-diagonal, pseudoscalar mesons in NNLO PQχPT, for two flavors of sea-quarks ($n_{\rm sea} = 2$). The results are characterized by the number of nondegenerate valence and sea-quarks, denoted $d_{\rm val}$ and $d_{\rm sea}$, respectively. For the decay constants of the charged pseudoscalar mesons, the maximum number of nondegenerate valence quark masses is $d_{\rm val} = 2$. The degree of quark mass degeneracy in each result is sometimes also referred to with the notation $d_{\rm val} + d_{\rm sea}$. The decay constant of the charged pion in the $SU(2)$ symmetric limit for both valence and sea-quarks thus corresponds to the $1 + 1$ case.

The analytical expressions for the NNLO shifts of the meson masses and decay constants are in general very long, but the expressions simplify considerably when the sea or valence quark masses become degenerate. In view of this, the NNLO loop results are given separately for each case of $d_{\rm val} + d_{\rm sea}$ considered. In the next sections, a short technical overview of the NNLO calculations is given, with an emphasis on the parts not covered in the previous publications [8, 9], along with the full results for the NNLO contributions to the meson masses and decay constants. Finally, a numerical analysis of the results is given, along with a concluding discussion.

II. TECHNICAL OVERVIEW

The technical aspects of the PQχPT calculations to NLO have been thoroughly covered in Refs. [4, 7]. The new parts needed for NNLO are treated in Refs. [8, 9].

For the sake of completeness, some of the issues which are directly relevant for the present two-flavor calculations are repeated here.

Most significantly, the Lagrangians of PQχPT at $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ can be directly obtained from the corresponding Lagrangians of unquenched n_F flavor χPT by replacement of traces with supertraces. This follows because the derivation of these Lagrangians in Ref. [11] only used relations which remain valid when all traces are replaced by the corresponding supertraces. For the definition of the supertraces in PQχPT see Refs [1, 2, 4]. We work here in the supersymmetric version of PQχPT without the singlet Φ_0 as discussed in Ref. [7]. The divergences of this version of PQχPT also follow directly from those of n_f flavor χPT calculated in Ref.[12], provided that n_f is set equal to the number of sea-quark flavors. This is so since all the manipulations in Ref. [12] for n_f flavors remain valid when traces are replaced with supertraces. Alternatively, this derivation of the Lagrangians of PQχPT can be argued for with the Replica method of Ref. [13].

A. Quark Masses and Low-Energy Constants

The number of independent low-energy constants (LEC:s) in unquenched and partially quenched χPT is slightly different, but the former are always linear combinations of the latter. The number of LEC:s is also dependent on the number of flavors. Table I gives an overview of the various sets of LEC:s and the number of parameters up to NNLO for the different versions of χPT and PQχPT.

TABLE I: The relevant sets of LEC:s, where the $i+j$ notation indicates the number of physically relevant (i) and contact (j) terms in the respective Lagrangians. At NLO, the latter ones are usually denoted h_i^r for $n_f = 2$ and H_i^r for $n_f = 3$. The relations between the various sets of LEC:s is discussed in the text. For simplicity, the superscript (npq) of the partially quenched LEC:s has been suppressed in most of this paper.

n_f	χPT 2	χPT 3	χPT n	PQχPT 2	PQχPT 3
LO	F, B	F_0, B_0	$F_0^{(n)}, B_0^{(n)}$	F, B	F_0, B_0
NLO	l_i^r	L_i^r	$L_i^{r(n)}$	$L_i^{r(2pq)}$	$L_i^{r(3pq)}$
$i+j$	$7+3$	$10+2$	$11+2$	$11+2$	$11+2$
NNLO	c_i^r	C_i^r	$K_i^{r(n)}$	$K_i^{r(2pq)}$	$K_i^{r(3pq)}$
$i+j$	$53+4$	$90+4$	$112+3$	$112+3$	$112+3$

At lowest order, the LEC:s of PQχPT with n flavors of sea-quarks are the same as those of n flavor unquenched χPT. At NLO with n flavors of sea-quarks, they are $L_0^{r(npq)}$ through $L_{10}^{r(npq)}$. The two possible extra terms $L_{11}^{r(npq)}$ and $L_{12}^{r(npq)}$ can be removed using field redefinitions [11, 12]. For the case of three sea-quarks, the L_i^r are simple linear combinations of the $L_i^{r(3pq)}$, and these relations can be found in Refs. [9, 11]. Similarly at NNLO, the C_i^r are linear combinations of the $K_i^{r(3pq)}$, and the corresponding relations can be found in Ref. [11]. For three flavors of sea-quarks, one simply takes $L_{11}^{r(3pq)} = L_{12}^{r(3pq)} = 0$.

The unquenched two-flavor Lagrangian as defined in the second paper of Ref. [3] differs by an L_{11}^r type term. At NLO, this makes no difference since that term does not contribute, but in order to get the correct values at NNLO for the l_i^r, c_i^r and the $L_i^{r(2pq)}, K_i^{r(2pq)}$ one should take $L_{11}^{r(2pq)} = -l_4^r/4$ and $L_{12}^{r(2pq)} = 0$. The relations between the NLO LEC:s are given in Table II, and similar linear relations between the NNLO LEC:s $K_i^{r(2pq)}$ of PQχPT and the c_i^r of unquenched two-flavor χPT may be found in Ref. [11]. The equations given in the remaining sections of this paper only deal with $n_f = 2$ PQχPT, and therefore the superscript $(2pq)$ of the $L_i^{r(2pq)}$ and the $K_i^{r(2pq)}$ have been suppressed in most of the formulas in the following sections.

TABLE II: The correspondence between the NLO low-energy constants L_i^r of two-flavor PQχPT and the l_i^r of standard two-flavor χPT. The relations relevant for the NNLO low-energy constants K_i^r are given in Ref. [11].

$n_f = 2$ χPT	$n_f = 2$ PQχPT
l_1^r	$-2\,L_0^{r(2pq)} + 4\,L_1^{r(2pq)} + 2\,L_3^{r(2pq)}$
l_2^r	$4\,L_0^{r(2pq)} + 4\,L_2^{r(2pq)}$
l_3^r	$-8\,L_4^{r(2pq)} - 4\,L_5^{r(2pq)} + 16\,L_6^{r(2pq)} + 8\,L_8^{r(2pq)}$
l_4^r	$8\,L_4^{r(2pq)} + 4\,L_5^{r(2pq)}$
l_5^r	$L_{10}^{r(2pq)}$
l_6^r	$-2\,L_9^{r(2pq)}$
l_7^r	$-16\,L_7^{r(2pq)} - 8\,L_8^{r(2pq)}$

The input quark masses m_i enter into the calculation in terms of the lowest order squared meson masses χ_i, which are defined as usual in χPT, by $\chi_i = 2B\,m_i$. In the present calculations, there are two valence inputs χ_1, χ_2, two sea inputs χ_3, χ_4, and two ghost inputs χ_5, χ_6. In order for the disconnected valence quark loops to be canceled, the masses of the ghost quarks are always equal to those of the corresponding valence quarks, such that $\chi_5 = \chi_1$ and $\chi_6 = \chi_2$. Explicitly, for $d_{\text{val}} = 1$, we have $\chi_1 = \chi_2$ and for $d_{\text{val}} = 2$, we have $\chi_1 \neq \chi_2$. Similarly, for the sea-quark sector $d_{\text{sea}} = 1$ implies $\chi_3 = \chi_4$, while for $d_{\text{sea}} = 2$ we have $\chi_3 \neq \chi_4$. The most general results (in the quark masses) for $n_{\text{sea}} = 2$ are therefore those with $d_{\text{val}} = 2$ and $d_{\text{sea}} = 2$.

B. Chiral Logarithms

The finite parts of the one-loop integrals which are encountered in the NNLO calculation of the masses and decay constants of the pseudoscalar mesons may be expressed in terms of the chiral logarithms A, B and C. Of these, the logarithms A are defined as

$$\begin{aligned}\bar{A}(\chi_i) &= -\pi_{16}\,\chi_i \log(\chi_i/\mu^2), &(1)\\ \bar{A}(\chi_i,\varepsilon) &= \bar{A}(\chi_i)^2/(2\pi_{16}\,\chi_i)\\ &+ \pi_{16}\,\chi_i\,(\pi^2/12+1/2), &(2)\end{aligned}$$

where $\pi_{16} = 1/(16\pi^2)$. Note that the chiral logarithms which depend on ε are naturally generated by the process of dimensional regularization and can be reexpressed in terms of the standard logarithms A and B. However, they have been retained in the final results for the sake of notational efficiency. The finite parts of the logarithms B are

$$\begin{aligned}\bar{B}(\chi_i,\chi_i,0) &= -\pi_{16}\left(1+\log(\chi_i/\mu^2)\right), &(3)\\ \bar{B}(\chi_i,\chi_j,0) &= -\pi_{16}\,\frac{\chi_i\log(\chi_i/\mu^2)-\chi_j\log(\chi_j/\mu^2)}{\chi_i-\chi_j},\\ \bar{B}(\chi_i,\chi_i,0,\varepsilon) &= \bar{A}(\chi_i)\bar{B}(\chi_i,\chi_i,0)/(\pi_{16}\,\chi_i) &(4)\\ &- \bar{A}(\chi_i)^2/(2\pi_{16}\,\chi_i^2)\\ &+ \pi_{16}\,(\pi^2/12+1/2),\\ \bar{B}(\chi_i,\chi_j,0,k) &= \chi_i\bar{B}(\chi_i,\chi_j,0)+\bar{A}(\chi_j), &(5)\end{aligned}$$

where the $\bar{B}(\chi_i,\chi_j,0,k)$ integrals, which are symmetric under the interchange of χ_i and χ_j, have been introduced in order to make the symmetries in the end results more explicit. Finally the remaining one-loop integral C is given by

$$\bar{C}(\chi_i,\chi_i,\chi_i,0) = -\pi_{16}/(2\chi_i). \qquad (6)$$

In all of these cases, the dependence on the subtraction scale μ has been moved into the loop integrals.

C. Two-loop Integrals

The finite two-loop integrals H^F, H_1^F, H_{21}^F that appear in the NNLO calculation of the meson masses and decay constants may be evaluated using the methods of Ref. [14]. Note that the corresponding primed integrals $H^{F'}, H_1^{F'}, H_{21}^{F'}$ indicate differentiation with respect to p^2 and appear only for the decay constants. The notation used for the H integrals in this paper is otherwise similar to that of Ref. [14], except that an extra integer argument now indicates the propagator structure, such that e.g.

$$H^F(\chi_i,\chi_j,\chi_k,p^2) \to H^F(n,\chi_i,\chi_j,\chi_k,p^2). \qquad (7)$$

The case of $n=1$ indicates that the integral consists of single propagators only, as in Ref. [14], whereas $n=2$ indicates that the first propagator appears squared and $n=3$ that the second propagator appears squared. The cases with two double propagators that can appear in the calculations are $n=5$, for which the first and second propagators appear squared, and $n=7$ for which the second and third propagators are squared. Explicit expressions for $n=1$ can be found in Ref. [14], and the other cases may be obtained by differentiation with respect to the masses of those expressions.

D. Notation for Meson Masses and Decay Constants

As explained in great detail in Refs. [4, 7, 9], the neutral meson propagators in PQχPT have a complicated structure. In order to obtain a result which can be expressed in terms of the chiral logarithms and two-loop integrals outlined in the previous subsection, that propagator has to be treated by partial fractioning, which produces expressions that consist of sums of single and double poles. The residues, which are rational functions of the sea and valence quark masses, pile up and produce very long and awkward end results. With a major effort, these can be simplified and in some cases the expressions have been compressed by more than an order of magnitude.

The residues R which are naturally generated in the calculations are, for $d_{\rm sea}=2$, the single-pole residues R^i_{jk} and the double-pole residue R^d_i. The residues simplify when pairs of sea-quark masses become degenerate, and hence the number of indices in the single-pole residue depends on the degree of degeneracy in the sea-quark masses. The residues relevant for the present two-flavor PQχPT calculations may be written in terms of the quantities $R^z_{a\ldots b}$, which are defined as

$$\begin{aligned}R^z_{ab} &= \chi_a-\chi_b,\\ R^z_{abc} &= \frac{\chi_a-\chi_b}{\chi_a-\chi_c},\\ R^z_{abcd} &= \frac{(\chi_a-\chi_b)(\chi_a-\chi_c)}{\chi_a-\chi_d}, &(8)\end{aligned}$$

and so on. Thus $R^z_{a\ldots b}$ has the same dimension as χ_i for an even number of indices and is dimensionless for an odd number of indices. It should be noted that the R^z notation may also appear independently in the final results. In such cases, the R^z have been generated by simplification procedures. For $d_{\rm sea}=2$, the residues which originate in the partial fractioning of the neutral meson propagator are

$$\begin{aligned}R^i_{jk} &= R^z_{i34jk},\\ R^d_i &= R^z_{i34\pi},\\ R^c_i &= R^i_{3\pi}+R^i_{4\pi}-R^i_{\pi\pi}, &(9)\end{aligned}$$

where the neutral pion mass in the sea-quark sector is simply given by $\chi_\pi = \chi_{34} = (\chi_3+\chi_4)/2$. Because of this trivial relation, the above residue notation is highly redundant, a fact which has been exploited in the simplification of the end results. The quantity R^v, which is defined as

$$R^v_{ijk} \;=\; R^i_{jj} + R^i_{kk} - 2R^i_{jk} \tag{10}$$

appears throughout the final results and has therefore been given a separate name. For the case of $d_{\mathrm{sea}} = 1$, all residues associated with the sea-quark sector reduce to numbers. Some nontrivial residues may still appear if the valence quarks are nondegenerate, according to

$$\begin{aligned} R^i_j &= R^z_{i3j}, \\ R^d_i &= R^z_{i3}. \end{aligned} \tag{11}$$

For convenience, R^d_i is also used for $d_{\mathrm{val}} = 1$ in order to maintain a notation similar to the more general cases.

E. Summation Conventions

The expressions for the meson masses and decay constants are symmetric, within the sea and valence quark sectors, under the interchange of quark masses, and may thus be compactified by the introduction of summation conventions which exploit these symmetries. The summations in the valence quark sector are similar to those in the three-flavor work of Ref. [9], whereas those of the sea-quark sector are considerably simpler in the two-flavor case.

The sea-quark sector of the results involves the summation index s, which may appear for the quark masses χ_i, and among the indices of the residue functions R. A simple example is $\chi_s = \chi_3 + \chi_4$. On the other hand, if the term in question consists of products of integrals, residues and quark masses, then the summation sign always applies to the term as a whole, such that e.g.

$$\bar{A}(\chi_s)\, R^1_{\eta s}\, \chi_s \;=\; \sum_{i\,=\,3,4} \bar{A}(\chi_i)\, R^1_{\eta i}\, \chi_i. \tag{12}$$

Any contribution which is written in terms of the summation index s explicitly fulfills the required symmetry properties in the sea-quark sector.

The summation indices p and q, which refer to the valence quark sector, are only needed for the case of $d_{\mathrm{val}} = 2$ and then always occur for the squared valence quark masses χ_1 and χ_2. If the index q is present, there will always be an index p and the resulting sum is over the pairs $(p,q) = (1,2)$ and $(p,q) = (2,1)$. If only p is present, the sum is over the indices 1 and 2. An example is

$$\bar{A}(\chi_p)\, R^p_{q\pi}\, \chi_p \;=\; \bar{A}(\chi_1)\, R^1_{2\pi}\, \chi_1 \;+\; [1 \leftrightarrow 2]. \tag{13}$$

Any contribution written in terms of the (p,q) notation is thus symmetric under the interchange of the valence quark masses χ_1 and χ_2.

III. TWO-FLAVOR PSEUDOSCALAR MESON DECAY CONSTANTS

The decay constants F_a of the pseudoscalar mesons are obtained from the definition

$$\langle 0 | A^\mu_a(0) | \phi_a(p) \rangle = i\sqrt{2}\, p^\mu\, F_a, \tag{14}$$

in terms of the axial current operator A^μ_a. The diagrams which contribute to that operator at $\mathcal{O}(p^6)$, or NNLO, are shown in Fig. 1. Diagrams at $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$ also contribute to Eq. (14) via the renormalization of the pseudoscalar meson wave function $\phi_a(p)$. The decay constants are functions of the valence inputs χ_1, χ_2 and the sea inputs χ_3, χ_4, which are defined in terms of the quark masses through $\chi_i = 2B\, m_i$, and of the quantities $\chi_{ij} = (\chi_i + \chi_j)/2$, which correspond to the lowest order charged meson masses. Other parameters include the decay constant in the chiral limit (F), the quark condensate in the chiral limit, via $\langle \bar{q}q \rangle = -BF^2$, and the LEC:s of $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$, i.e. the $L^{r(2pq)}_i$ and the $K^{r(2pq)}_i$, which for simplicity are denoted by L^r_i and K^r_i.

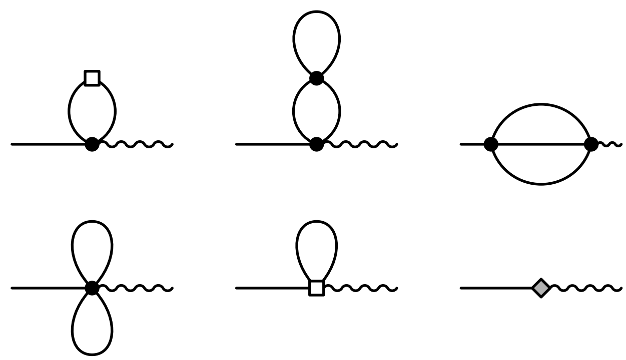

FIG. 1: Feynman diagrams at $\mathcal{O}(p^6)$ or two-loop for the matrix element of the axial current operator. Filled circles denote vertices of the $\mathcal{L}_2$ Lagrangian, whereas open squares and shaded diamonds denote vertices of the $\mathcal{L}_4$ and $\mathcal{L}_6$ Lagrangians, respectively.

The decay constants of the pseudoscalar mesons are given in the form

$$F_{\mathrm{phys}} = F\left[1 + \frac{\delta^{(4)\mathrm{vs}}}{F^2} + \frac{\delta^{(6)\mathrm{vs}}_{\mathrm{ct}} + \delta^{(6)\mathrm{vs}}_{\mathrm{loops}}}{F^4} + \mathcal{O}(p^8)\right], \tag{15}$$

where the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ contributions have been separated. The contribution from the $\mathcal{O}(p^6)$ counterterms $\delta^{(6)}_{\mathrm{ct}}$ has also been separated from the remaining part $\delta^{(6)}_{\mathrm{loops}}$, which consists mostly of the contributions from the chiral loops of $\mathcal{O}(p^6)$. It should be noted that $\delta^{(6)}_{\mathrm{loops}}$ also receives a contribution from the chiral loops and counterterms of $\mathcal{O}(p^4)$, which originate in the expansion of Eq. (14) to $\mathcal{O}(p^6)$. The superscripts (v) and (s) indicate the values of $d_{\rm val}$ and $d_{\rm sea}$, respectively. As expected, the infinities in all expressions for the decay constant have canceled. The appearance of 'unphysical' $\bar{B}$ logarithms is, in part, due to the partial quenching, and to the fact that the results have been expressed in terms of the lowest order masses rather than the full physical ones. Thus not all of them correspond to the quenched chiral logarithms which are ill-behaved in the chiral limit.

A. Results for $d_{\rm val} = 1$

The NLO result for $d_{\rm val} = 1$ is short, and will only be given for $d_{\rm sea} = 2$. The result for $d_{\rm sea} = 1$ can readily be derived from that expression. The combined NLO result (loops and counterterms) for $d_{\rm sea} = 2$, is

$$\delta^{(4)12} = 8\,L_4^r\,\chi_{34} + 4\,L_5^r\,\chi_1 + 1/2\,\bar{A}(\chi_{1s}), \qquad (16)$$

and the contribution from the $\mathcal{O}(p^6)$ counterterms to the NNLO expression for the decay constant is

$$\begin{aligned}\delta^{(6)12}_{\rm ct} &= 8\,K^r_{19}\,\chi_1^2 + 16\,K^r_{20}\,\chi_1\chi_{34} + 8\,K^r_{21}\,\chi_s^2 \\ &+ 32\,K^r_{22}\,\chi_{34}^2 + 8\,K^r_{23}\,\chi_1^2. \end{aligned} \qquad (17)$$

At NNLO, the chiral loops form, by far, the largest contribution to the decay constant. As a straightforward derivation of the results for $d_{\rm sea} = 1$ from the $d_{\rm sea} = 2$ case is tedious and complicated, the expressions for these two cases are given separately below.

$$\begin{aligned}
\delta^{(6)11}_{\rm loops} &= \pi_{16}\,L_0^r\,\left[-4\,\chi_1\chi_3 + 2\,\chi_1^2 - \chi_3^2\right] - 2\,\pi_{16}\,L_1^r\,\chi_1^2 - \pi_{16}\,L_2^r\,\left[\chi_1^2 + 3\chi_3^2\right] \\
&+ \pi_{16}\,L_3^r\,\left[-3\,\chi_1\chi_3 + 5/2\,\chi_1^2 - 1/2\,\chi_3^2\right] + \pi_{16}^2\,\left[-23/96\,\chi_1\chi_3 + 1/64\,\chi_1^2 - 47/192\,\chi_3^2\right] - 32\,L_4^rL_5^r\,\chi_1\chi_3 \\
&- 32\,L_4^{r2}\,\chi_3^2 - 8\,L_5^{r2}\,\chi_1^2 + 8\,L_{11}^{r2}\,\chi_1^2 + 6\,\bar{A}(\chi_1)\,L_0^r\,\left[\chi_1 + R_1^d\right] - 4\,\bar{A}(\chi_1)\,L_1^r\,\chi_1 - 10\,\bar{A}(\chi_1)\,L_2^r\,\chi_1 \\
&+ 6\,\bar{A}(\chi_1)\,L_3^r\,\left[\chi_1 + R_1^d\right] - 2\,\bar{A}(\chi_1)\,L_5^r\,\chi_1 - 1/2\,\bar{A}(\chi_1)\bar{B}(\chi_{13},\chi_{13},0)\,\chi_{13} + 1/8\,\bar{A}(\chi_1,\varepsilon)\,\pi_{16}\,\chi_3 \\
&- \bar{A}(\chi_{13})\,\pi_{16}\,\left[1/6\,\chi_1 + 2/3\,\chi_3\right] - 8\,\bar{A}(\chi_{13})\,L_0^r\,\chi_{13} - 20\,\bar{A}(\chi_{13})\,L_3^r\,\chi_{13} - 8\,\bar{A}(\chi_{13})\,L_4^r\,\chi_3 + 4\,\bar{A}(\chi_{13})\,L_5^r\,\chi_1 \\
&+ \bar{A}(\chi_{13},\varepsilon)\,\pi_{16}\,\chi_3 - 24\,\bar{A}(\chi_3)\,L_1^r\,\chi_3 - 6\,\bar{A}(\chi_3)\,L_2^r\,\chi_3 + 12\,\bar{A}(\chi_3)\,L_4^r\,\chi_3 + 3/8\,\bar{A}(\chi_3,\varepsilon)\,\pi_{16}\,\chi_3 \\
&+ 6\,\bar{B}(\chi_1,\chi_1,0)\,L_0^r\,R_1^d\,\chi_1 + 6\,\bar{B}(\chi_1,\chi_1,0)\,L_3^r\,R_1^d\,\chi_1 - 2\,\bar{B}(\chi_1,\chi_1,0)\,L_5^r\,R_1^d\,\chi_1 \\
&- 1/8\,\bar{B}(\chi_1,\chi_1,0,\varepsilon)\,\pi_{16}\,R_1^d\,\chi_1 - 16\,\bar{B}(\chi_{13},\chi_{13},0)\,L_4^r\,\chi_{13}\chi_3 - 8\,\bar{B}(\chi_{13},\chi_{13},0)\,L_5^r\,\chi_{13}^2 \\
&+ 32\,\bar{B}(\chi_{13},\chi_{13},0)\,L_6^r\,\chi_{13}\chi_3 + 16\,\bar{B}(\chi_{13},\chi_{13},0)\,L_8^r\,\chi_{13}^2 - 1/8\,H^F(1,\chi_1,\chi_{13},\chi_{13},\chi_1)\,\chi_3 \\
&- 3/8\,H^F(1,\chi_{13},\chi_{13},\chi_3,\chi_1)\,\chi_3 + 1/8\,H^F(2,\chi_1,\chi_{13},\chi_{13},\chi_1)\,R_1^d\,\chi_1 + 5/12\,H^{F'}(1,\chi_1,\chi_1,\chi_1,\chi_1)\,\chi_1^2 \\
&+ H^{F'}(1,\chi_1,\chi_{13},\chi_{13},\chi_1)\,\left[1/8\,\chi_1\chi_3 - 1/2\,\chi_1^2\right] + 3/8\,H^{F'}(1,\chi_{13},\chi_{13},\chi_3,\chi_1)\,\chi_1\chi_3 \\
&+ 1/2\,H^{F'}(2,\chi_1,\chi_1,\chi_1,\chi_1)\,R_1^d\,\chi_1^2 + 3/8\,H^{F'}(2,\chi_1,\chi_{13},\chi_{13},\chi_1)\,R_1^d\,\chi_1^2 \\
&+ 1/4\,H^{F'}(5,\chi_1,\chi_1,\chi_1,\chi_1)\,(R_1^d)^2\chi_1^2 - 2\,H_1^{F'}(3,\chi_{13},\chi_1,\chi_{13},\chi_1)\,R_1^d\,\chi_1^2 \\
&+ 3/8\,H_{21}^{F'}(1,\chi_1,\chi_{13},\chi_{13},\chi_1)\,\chi_1^2 + 9/8\,H_{21}^{F'}(1,\chi_3,\chi_{13},\chi_{13},\chi_1)\,\chi_1^2 \\
&- 3/8\,H_{21}^{F'}(2,\chi_1,\chi_{13},\chi_{13},\chi_1)\,R_1^d\,\chi_1^2,
\end{aligned} \qquad (18)$$

$$\begin{aligned}
\delta^{(6)12}_{\rm loops} &= \pi_{16}\,L_0^r\,\left[-4\,\chi_\pi\chi_1 - 3\,\chi_\pi^2 + 2\,\chi_1^2 + 2\,\chi_3\chi_4\right] - 2\,\pi_{16}\,L_1^r\,\chi_1^2 - \pi_{16}\,L_2^r\,\left[3\,\chi_\pi^2 + \chi_1^2\right] \\
&+ \pi_{16}\,L_3^r\,\left[-3\,\chi_\pi\chi_1 - 2\,\chi_\pi^2 + 5/2\,\chi_1^2 + 3/2\,\chi_3\chi_4\right] + \pi_{16}^2\,\left[-23/96\,\chi_\pi\chi_1 - 1/3\,\chi_\pi^2 + 1/64\,\chi_1^2 + 17/192\,\chi_3\chi_4\right] \\
&- 32\,L_4^rL_5^r\,\chi_1\chi_{34} - 32\,L_4^{r2}\,\chi_{34}^2 - 8\,L_5^{r2}\,\chi_1^2 + 8\,L_{11}^{r2}\,\chi_1^2 + 6\,\bar{A}(\chi_\pi)\,L_0^r\,R^\pi_{11}\,\chi_\pi - 8\,\bar{A}(\chi_\pi)\,L_1^r\,\chi_\pi \\
&- 2\,\bar{A}(\chi_\pi)\,L_2^r\,\chi_\pi + 6\,\bar{A}(\chi_\pi)\,L_3^r\,R^\pi_{11}\,\chi_\pi + 4\,\bar{A}(\chi_\pi)\,L_4^r\,\chi_\pi - 2\,\bar{A}(\chi_\pi)\,L_5^r\,R^\pi_{11}\,\chi_1 \\
&- 1/4\,\bar{A}(\chi_\pi)\bar{B}(\chi_{1s},\chi_{1s},0)\,R^\pi_{1s}\,\chi_{1s} + 1/8\,\bar{A}(\chi_\pi,\varepsilon)\,\pi_{16}\,R_1^c\,\chi_\pi + 6\,\bar{A}(\chi_1)\,L_0^r\,\left[R_1^c\,\chi_1 + R_1^d\right] - 4\,\bar{A}(\chi_1)\,L_1^r\,\chi_1 \\
&- 10\,\bar{A}(\chi_1)\,L_2^r\,\chi_1 + 6\,\bar{A}(\chi_1)\,L_3^r\,\left[R_1^c\,\chi_1 + R_1^d\right] - 2\,\bar{A}(\chi_1)\,L_5^r\,R_1^c\,\chi_1 - 1/4\,\bar{A}(\chi_1)\bar{B}(\chi_{1s},\chi_{1s},0)\,R^1_{s\pi}\,\chi_{1s} \\
&+ 1/8\,\bar{A}(\chi_1,\varepsilon)\,\pi_{16}\,R^1_{\pi\pi}\,\chi_{34} - 16\,\bar{A}(\chi_{34})\,L_1^r\,\chi_{34} - 4\,\bar{A}(\chi_{34})\,L_2^r\,\chi_{34} + 8\,\bar{A}(\chi_{34})\,L_4^r\,\chi_{34} \\
&+ 1/4\,\bar{A}(\chi_{34},\varepsilon)\,\pi_{16}\,\chi_{34} - \bar{A}(\chi_{1s})\,\pi_{16}\,\left[1/4\,\chi_{34} + 1/6\,\chi_{1s}\right] - 4\,\bar{A}(\chi_{1s})\,L_0^r\,\chi_{1s} - 10\,\bar{A}(\chi_{1s})\,L_3^r\,\chi_{1s} \\
&- 4\,\bar{A}(\chi_{1s})\,L_4^r\,\chi_{34} + 2\,\bar{A}(\chi_{1s})\,L_5^r\,\chi_1 + 1/4\,\bar{A}(\chi_{1s},\varepsilon)\,\pi_{16}\,\left[\chi_{34} + \chi_s\right] + 6\,\bar{B}(\chi_1,\chi_1,0)\,L_0^r\,R_1^d\,\chi_1 \\
&+ 6\,\bar{B}(\chi_1,\chi_1,0)\,L_3^r\,R_1^d\,\chi_1 - 2\,\bar{B}(\chi_1,\chi_1,0)\,L_5^r\,R_1^d\,\chi_1 - 1/8\,\bar{B}(\chi_1,\chi_1,0,\varepsilon)\,\pi_{16}\,R_1^d\,\chi_1 \\
&- 8\,\bar{B}(\chi_{1s},\chi_{1s},0)\,L_4^r\,\chi_{34}\chi_{1s} - 4\,\bar{B}(\chi_{1s},\chi_{1s},0)\,L_5^r\,\chi_{1s}^2 + 16\,\bar{B}(\chi_{1s},\chi_{1s},0)\,L_6^r\,\chi_{34}\chi_{1s} \\
&+ 8\,\bar{B}(\chi_{1s},\chi_{1s},0)\,L_8^r\,\chi_{1s}^2 + 1/16\,H^F(1,\chi_\pi,\chi_{1s},\chi_{1s},\chi_1)\,R^v_{\pi 1s}\,\chi_\pi - H^F(1,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,\left[1/8\,R^1_{s\pi}\,\chi_{1s}\right. \\
&\left.- 1/16\,R_1^c\,\chi_1\right] - 1/4\,H^F(1,\chi_{13},\chi_{14},\chi_{34},\chi_1)\,\chi_{34} + 1/16\,H^F(2,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,R_1^d\,\chi_1
\end{aligned}$$

$$+\;1/4\,H^{F'}(1,\chi_\pi,\chi_\pi,\chi_1,\chi_1)\,(R^\pi_{11})^2\chi_1^2\;+\;1/2\,H^{F'}(1,\chi_\pi,\chi_1,\chi_1,\chi_1)\,R^\pi_{11}R^c_1\,\chi_1^2$$
$$-\;H^{F'}(1,\chi_\pi,\chi_{1s},\chi_{1s},\chi_1)\,\left[1/4\,R^\pi_{11}\,\chi_1^2+1/16\,R^v_{\pi 1s}\,\chi_\pi\chi_1\right]\;+\;H^{F'}(1,\chi_1,\chi_1,\chi_1,\chi_1)\,\left[1/6\,\chi_1^2\right.$$
$$\left.+\;1/4\,(R^c_1)^2\chi_1^2\right]\;+\;H^{F'}(1,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,\left[1/8\,R^1_{s\pi}\,\chi_1\chi_{1s}-1/2\,R^1_{s\pi}\,\chi_1^2+3/16\,R^c_1\,\chi_1^2\right]$$
$$+\;1/4\,H^{F'}(1,\chi_{13},\chi_{14},\chi_{34},\chi_1)\,\chi_1\chi_{34}\;+\;1/2\,H^{F'}(2,\chi_1,\chi_\pi,\chi_1,\chi_1)\,R^\pi_{11}R^d_1\,\chi_1^2$$
$$+\;1/2\,H^{F'}(2,\chi_1,\chi_1,\chi_1,\chi_1)\,R^c_1R^d_1\,\chi_1^2\;+\;3/16\,H^{F'}(2,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,R^d_1\,\chi_1^2$$
$$+\;1/4\,H^{F'}(5,\chi_1,\chi_1,\chi_1,\chi_1)\,(R^d_1)^2\chi_1^2\;-\;1/2\,H^{F'}_1(1,\chi_\pi,\chi_{1s},\chi_{1s},\chi_1)\,R^\pi_{1s}R^z_{1s\pi}\,\chi_1^2$$
$$-\;H^{F'}_1(1,\chi_{1s},\chi_{1s},\chi_1,\chi_1)\,R^\pi_{1s}R^z_{1s\pi}\,\chi_1^2\;-\;H^{F'}_1(3,\chi_{1s},\chi_1,\chi_{1s},\chi_1)\,R^d_1\,\chi_1^2$$
$$-\;3/16\,H^{F'}_{21}(1,\chi_\pi,\chi_{1s},\chi_{1s},\chi_1)\,R^v_{\pi 1s}\,\chi_1^2\;+\;H^{F'}_{21}(1,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,\left[3/8\,R^1_{s\pi}\,\chi_1^2-3/16\,R^c_1\,\chi_1^2\right]$$
$$+\;3/4\,H^{F'}_{21}(1,\chi_{34},\chi_{13},\chi_{14},\chi_1)\,\chi_1^2\;-\;3/16\,H^{F'}_{21}(2,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,R^d_1\,\chi_1^2. \tag{19}$$

B. Results for $d_{\rm val}=2$

In general, the expressions for $d_{\rm val}=2$ are longer than those for $d_{\rm val}=1$, as the number of independent mass combinations that can appear in the integrals is significantly larger. The most general two-flavor expression for the pseudoscalar meson decay constant is the result with $d_{\rm val}=2$ and $d_{\rm sea}=2$. However, in contrast with the results for $d_{\rm val}=1$, the limit where the sea-quark masses become degenerate is nontrivial even at NLO. In the notation of Eq. (15), the NLO contribution for $d_{\rm val}=2$ and $d_{\rm sea}=2$ is

$$\begin{aligned}\delta^{(4)22} \;=\;& 8\,L^r_4\,\chi_{34}\;+\;4\,L^r_5\,\chi_{12}\;+\;\bar A(\chi_p)\,\left[1/4\,R^p_{q\pi}\right.\\ &\left.-\;1/8\,R^c_p\right]\;+\;1/4\,\bar A(\chi_{ps})\;-\;1/8\,\bar A(\chi_\pi)\,R^v_{\pi 12}\\ &-\;1/8\,\bar B(\chi_p,\chi_p,0)\,R^d_p.\end{aligned} \tag{20}$$

The NLO contribution for $d_{\rm val}=2$ and $d_{\rm sea}=1$ may be obtained from Eq. (20) by carefully taking the limit $\chi_4\to\chi_3$. Effectively this means dropping the $\bar A(\chi_\pi)$ term and replacing $\bar A(\chi_{ps})\to 2\bar A(\chi_{p3})$. Furthermore the remaining residues reduce such that $R^c_p\to 1$ and $R^p_{q\pi}\to R^p_q$. The contribution from the $\mathcal{O}(p^6)$ counterterms to the NNLO expression is

$$\begin{aligned}\delta^{(6)22}_{\rm ct} \;=\;& 4\,K^r_{19}\,\chi_p^2\;+\;16\,K^r_{20}\,\chi_{12}\chi_{34}\;+\;8\,K^r_{21}\,\chi_s^2\\ &+\;32\,K^r_{22}\,\chi_{34}^2\;+\;8\,K^r_{23}\,\chi_1\chi_2,\end{aligned} \tag{21}$$

from which the corresponding expression for $d_{\rm val}=2$ and $d_{\rm sea}=1$ may readily be inferred. The remaining NNLO contributions with $d_{\rm val}=2$ are, for $d_{\rm sea}=1$ and $d_{\rm sea}=2$,

$$\begin{aligned}\delta^{(6)21}_{\rm loops} \;=\;& \pi_{16}\,L^r_0\,\left[-2\,\chi_1\chi_2-4\,\chi_{12}\chi_3+4\,\chi_{12}^2-\chi_3^2\right]\;-\;2\,\pi_{16}\,L^r_1\,\chi_{12}^2\;-\;\pi_{16}\,L^r_2\,\left[\chi_{12}^2+3\chi_3^2\right]\\
&+\;\pi_{16}\,L^r_3\,\left[-3/2\,\chi_1\chi_2-3\,\chi_{12}\chi_3+4\,\chi_{12}^2-1/2\,\chi_3^2\right]\;+\;\pi_{16}^2\,\left[-17/192\,\chi_1\chi_2-23/96\,\chi_{12}\chi_3+5/48\,\chi_{12}^2\right.\\
&\left.-\;47/192\,\chi_3^2\right]\;-\;32\,L^r_4L^r_5\,\chi_{12}\chi_3\;-\;32\,L^{r2}_4\,\chi_3^2\;-\;8\,L^{r2}_5\,\chi_{12}^2\;+\;8\,L^{r2}_{11}\,\chi_{12}^2\;+\;\bar A(\chi_p)\,\pi_{16}\,\left[1/48\,\chi_q+1/48\,\chi_{12}\right.\\
&\left.+\;1/16\,\chi_3-1/12\,R^p_q\,\chi_{12}-1/8\,R^p_q\,\chi_3\right]\;+\;\bar A(\chi_p)\,L^r_0\,\left[\chi_p+4\,R^p_q\,\chi_p+R^d_p\right]\;+\;\bar A(\chi_p)\,L^r_3\,\left[5/2\,\chi_p+R^p_q\,\chi_p\right.\\
&\left.+\;5/2\,R^d_p\right]\;+\;\bar A(\chi_p)\,L^r_4\,\left[\chi_3-2\,R^p_q\,\chi_3\right]\;-\;\bar A(\chi_p)\,L^r_5\,\left[1/2\,\chi_{12}+R^p_q\,\chi_{12}\right]\;+\;\bar A(\chi_p)^2\,\left[9/128-1/32\,R^p_qR^q_p\right]\\
&-\;\bar A(\chi_p)\bar A(\chi_{p3})\,\left[1/16+1/12\,R^p_q\right]\;-\;\bar A(\chi_p)\bar A(\chi_{q3})\,\left[3/16+1/12\,R^p_q\right]\;+\;1/8\,\bar A(\chi_p)\bar A(\chi_{12})\\
&+\;\bar A(\chi_p)\bar B(\chi_p,\chi_p,0)\,\left[3/8\,\chi_p-1/8\,R^p_q\,\chi_p-1/32\,R^p_qR^d_p+1/64\,R^d_p\right]\;-\;\bar A(\chi_p)\bar B(\chi_q,\chi_q,0)\,\left[1/32\,R^p_qR^d_q\right.\\
&\left.-\;1/64\,R^d_q\right]\;-\;1/4\,\bar A(\chi_p)\bar B(\chi_{p3},\chi_{p3},0)\,\chi_{p3}\;-\;1/8\,\bar A(\chi_p)\bar B(\chi_1,\chi_2,0)\,R^q_p\,\chi_p\\
&+\;1/8\,\bar A(\chi_p)\bar C(\chi_p,\chi_p,\chi_p,0)\,R^d_p\,\chi_p\;-\;1/8\,\bar A(\chi_p,\varepsilon)\,\pi_{16}\,\left[\chi_p-R^p_q\,\chi_3\right]\;+\;\bar A(\chi_{p3})\,\pi_{16}\,\left[1/12\,\chi_{p3}-1/4\,\chi_{q3}\right.\\
&\left.-\;1/4\,\chi_3\right]\;-\;4\,\bar A(\chi_{p3})\,L^r_0\,\chi_{p3}\;-\;10\,\bar A(\chi_{p3})\,L^r_3\,\chi_{p3}\;-\;4\,\bar A(\chi_{p3})\,L^r_4\,\chi_3\;+\;2\,\bar A(\chi_{p3})\,L^r_5\,\chi_{12}\;-\;1/8\,\bar A(\chi_{p3})^2\\
&+\;\bar A(\chi_{p3})\bar B(\chi_p,\chi_p,0)\,\left[1/8\,\chi_p-5/8\,\chi_{p3}\right]\;-\;1/16\,\bar A(\chi_{p3})\bar B(\chi_q,\chi_q,0)\,R^d_q\;+\;1/6\,\bar A(\chi_{p3})\bar B(\chi_1,\chi_2,0)\,\chi_3\\
&+\;1/3\,\bar A(\chi_{p3})\bar B(\chi_1,\chi_2,0,k)\;+\;1/2\,\bar A(\chi_{p3},\varepsilon)\,\pi_{16}\,\chi_3\;-\;\bar A(\chi_1)\bar A(\chi_2)\,\left[1/64-1/16\,R^1_2R^2_1\right]\;-\;4\,\bar A(\chi_{12})\,L^r_1\,\chi_{12}\\
&-\;10\,\bar A(\chi_{12})\,L^r_2\,\chi_{12}\;+\;1/8\,\bar A(\chi_{12})^2\;-\;1/2\,\bar A(\chi_{12})\bar B(\chi_1,\chi_2,0,k)\;+\;1/4\,\bar A(\chi_{12},\varepsilon)\,\pi_{16}\,\chi_{12}\\
&+\;1/4\,\bar A(\chi_{13})\bar A(\chi_{23})\;-\;24\,\bar A(\chi_3)\,L^r_1\,\chi_3\;-\;6\,\bar A(\chi_3)\,L^r_2\,\chi_3\;+\;12\,\bar A(\chi_3)\,L^r_4\,\chi_3\;+\;3/16\,\bar A(\chi_3)\bar B(\chi_p,\chi_p,0)\,\chi_3\\
&-\;3/8\,\bar A(\chi_3)\bar B(\chi_1,\chi_2,0)\,\chi_3\;+\;3/8\,\bar A(\chi_3,\varepsilon)\,\pi_{16}\,\chi_3\;+\;\bar B(\chi_p,\chi_p,0)\,\pi_{16}\,\left[1/96\,R^d_p\,\chi_p+1/32\,R^d_p\,\chi_q\right.\\
&\left.+\;1/16\,R^d_p\,\chi_3\right]\;+\;\bar B(\chi_p,\chi_p,0)\,L^r_0\,R^d_p\,\chi_p\;+\;5/2\,\bar B(\chi_p,\chi_p,0)\,L^r_3\,R^d_p\chi_p\;+\;\bar B(\chi_p,\chi_p,0)\,L^r_4\,\left[2\,\chi_p\chi_3\right.\end{aligned}$$

$$
\begin{aligned}
&- 4\, R^p_q\, \chi_p\chi_3 + 3\, R^d_p\, \chi_3\big] \,+\, \bar B(\chi_p,\chi_p,0)\, L^r_5\, \big[\chi_p^2 - 2\, R^p_q\, \chi_1\chi_2 - 1/2\, R^d_p\, \chi_{12}\big] \,+\, \bar B(\chi_p,\chi_p,0)\, L^r_6\, \big[4\, \chi_3^2 \\
&- 8\, R^q_p\, \chi_p\chi_3\big] \,+\, 4\, \bar B(\chi_p,\chi_p,0)\, L^r_7\, (R^d_p)^2 \,+\, \bar B(\chi_p,\chi_p,0)\, L^r_8\, \big[2\, \chi_3^2 - 4\, R^q_p\, \chi_p^2\big] \,+\, \bar B(\chi_p,\chi_p,0)^2\, \big[1/8\, R^q_p R^d_p\, \chi_p \\
&+ 1/128\, (R^d_p)^2\big] \,-\, 1/8\, \bar B(\chi_p,\chi_p,0)\bar B(\chi_1,\chi_2,0)\, R^q_p R^d_p\, \chi_p \,+\, 1/8\, \bar B(\chi_p,\chi_p,0)\bar C(\chi_p,\chi_p,\chi_p,0)\, (R^d_p)^2\chi_p \\
&- 1/8\, \bar B(\chi_p,\chi_p,0,\varepsilon)\, \pi_{16}\, R^d_p\, \chi_{p3} \,-\, 8\, \bar B(\chi_{p3},\chi_{p3},0)\, L^r_4\, \chi_{p3}\chi_3 \,-\, 4\, \bar B(\chi_{p3},\chi_{p3},0)\, L^r_5\, \chi_{p3}^2 \\
&+ 16\, \bar B(\chi_{p3},\chi_{p3},0)\, L^r_6\, \chi_{p3}\chi_3 \,+\, 8\, \bar B(\chi_{p3},\chi_{p3},0)\, L^r_8\, \chi_{p3}^2 \,+\, 1/64\, \bar B(\chi_1,\chi_1,0)\bar B(\chi_2,\chi_2,0)\, R^d_1 R^d_2 \\
&- 8\, \bar B(\chi_1,\chi_2,0)\, L^r_7\, R^d_1 R^d_2 \,-\, 4\, \bar B(\chi_1,\chi_2,0)\, L^r_8\, R^d_1 R^d_2 \,+\, 4\, \bar C(\chi_p,\chi_p,\chi_p,0)\, L^r_4\, R^d_p\, \chi_p\chi_3 \\
&+ 2\, \bar C(\chi_p,\chi_p,\chi_p,0)\, L^r_5\, R^d_p\, \chi_p^2 \,-\, 8\, \bar C(\chi_p,\chi_p,\chi_p,0)\, L^r_6\, R^d_p\, \chi_p\chi_3 \,-\, 4\, \bar C(\chi_p,\chi_p,\chi_p,0)\, L^r_8\, R^d_p\, \chi_p^2 \\
&+ H^F(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\, \big[1/8\, \chi_p - 5/64\, \chi_{12} + 1/16\, R^p_q R^q_p\, \chi_{12}\big] \,+\, H^F(1,\chi_p,\chi_{13},\chi_{23},\chi_{12})\, \big[-1/16\, \chi_p \\
&+ 1/16\, \chi_q - 1/8\, R^p_q\, \chi_3\big] \,+\, H^F(1,\chi_1,\chi_{12},\chi_2,\chi_{12})\, \big[1/32\, \chi_{12} - 1/8\, R^1_2 R^2_1\, \chi_{12}\big] \\
&- 1/8\, H^F(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\, \chi_{12} \,-\, 3/8\, H^F(1,\chi_{13},\chi_{23},\chi_3,\chi_{12})\, \chi_3 \\
&+ H^F(2,\chi_p,\chi_p,\chi_{12},\chi_{12})\, \big[1/16\, R^p_q R^d_p\, \chi_{12} - 1/32\, R^d_p\, \chi_{12}\big] \,+\, H^F(2,\chi_p,\chi_{12},\chi_q,\chi_{12})\, \big[1/16\, R^q_p R^d_p\, \chi_{12} \\
&- 1/32\, R^d_p\, \chi_{12}\big] \,+\, 1/8\, H^F(2,\chi_p,\chi_{13},\chi_{23},\chi_{12})\, R^d_p\, \chi_{p3} \,-\, 1/64\, H^F(5,\chi_p,\chi_p,\chi_{12},\chi_{12})\, (R^d_p)^2\chi_{12} \\
&- 1/32\, H^F(5,\chi_1,\chi_2,\chi_{12},\chi_{12})\, R^d_1 R^d_2\, \chi_{12} \,-\, H^{F'}(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\, \big[1/8\, \chi_p\chi_{12} + 15/64\, \chi_{12}^2 - 3/16\, R^p_q\, \chi_{12}^2 \\
&+ 1/16\, (R^p_q)^2\chi_{12}^2\big] \,+\, H^{F'}(1,\chi_p,\chi_{13},\chi_{23},\chi_{12})\, \big[1/8\, \chi_{12}^2 + 3/8\, R^q_p\, \chi_p\chi_{12} + 1/8\, R^q_p\, \chi_q\chi_{12} - 1/8\, R^q_p\, \chi_{12}\chi_3\big] \\
&- H^{F'}(1,\chi_1,\chi_{12},\chi_2,\chi_{12})\, \big[1/32\, \chi_{12}^2 - 3/8\, R^1_2 R^2_1\, \chi_{12}^2\big] \,+\, 1/8\, H^{F'}(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\, \chi_{12}^2 \\
&+ 3/8\, H^{F'}(1,\chi_{13},\chi_{23},\chi_3,\chi_{12})\, \chi_{12}\chi_3 \,-\, H^{F'}(2,\chi_p,\chi_p,\chi_{12},\chi_{12})\, \big[1/16\, R^p_q R^d_p\, \chi_{12}^2 - 5/32\, R^d_p\, \chi_{12}^2\big] \\
&- H^{F'}(2,\chi_p,\chi_{12},\chi_q,\chi_{12})\, \big[1/16\, R^q_p R^d_p\, \chi_{12}^2 - 1/32\, R^d_p\, \chi_{12}^2\big] \,-\, 1/8\, H^{F'}(2,\chi_p,\chi_{13},\chi_{23},\chi_{12})\, R^d_p\, \chi_{p3}\chi_{12} \\
&+ 5/64\, H^{F'}(5,\chi_p,\chi_p,\chi_{12},\chi_{12})\, (R^d_p)^2\chi_{12}^2 \,+\, 1/32\, H^{F'}(5,\chi_1,\chi_2,\chi_{12},\chi_{12})\, R^d_1 R^d_2\, \chi_{12}^2 \\
&+ H^{F'}_1(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\, \big[5/4\, \chi_{12}^2 - 1/2\, R^p_q R^q_p\, \chi_{12}^2\big] \,-\, H^{F'}_1(1,\chi_p,\chi_{13},\chi_{23},\chi_{12})\, R^q_p\, \chi_{12}^2 \\
&- H^{F'}_1(1,\chi_{p3},\chi_{q3},\chi_p,\chi_{12})\, \chi_{12}^2 \,+\, H^{F'}_1(1,\chi_{12},\chi_1,\chi_2,\chi_{12})\, \big[1/4\, \chi_{12}^2 - 1/2\, R^1_2 R^2_1\, \chi_{12}^2\big] \\
&- 1/4\, H^{F'}_1(3,\chi_{12},\chi_p,\chi_p,\chi_{12})\, R^q_p R^d_p\, \chi_{12}^2 \,+\, 1/4\, H^{F'}_1(3,\chi_{12},\chi_p,\chi_q,\chi_{12})\, R^q_p R^d_p\, \chi_{12}^2 \\
&- 1/8\, H^{F'}_1(7,\chi_{12},\chi_p,\chi_p,\chi_{12})\, (R^d_p)^2\chi_{12}^2 \,-\, 3/8\, H^{F'}_{21}(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\, \chi_{12}^2 \\
&+ 3/8\, H^{F'}_{21}(1,\chi_p,\chi_{13},\chi_{23},\chi_{12})\, R^q_p\, \chi_{12}^2 \,+\, 3/8\, H^{F'}_{21}(1,\chi_{p3},\chi_{q3},\chi_p,\chi_{12})\, R^p_q\, \chi_{12}^2 \\
&- 3/8\, H^{F'}_{21}(1,\chi_{p3},\chi_{q3},\chi_q,\chi_{12})\, R^p_q\, \chi_{12}^2 \,+\, H^{F'}_{21}(1,\chi_{12},\chi_p,\chi_p,\chi_{12})\, \big[15/64\, \chi_{12}^2 - 3/16\, R^p_q R^q_p\, \chi_{12}^2\big] \\
&- H^{F'}_{21}(1,\chi_{12},\chi_1,\chi_2,\chi_{12})\, \big[3/32\, \chi_{12}^2 - 3/8\, R^1_2 R^2_1\, \chi_{12}^2\big] \,+\, 3/8\, H^{F'}_{21}(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\, \chi_{12}^2 \\
&+ 9/8\, H^{F'}_{21}(1,\chi_3,\chi_{13},\chi_{23},\chi_{12})\, \chi_{12}^2 \,-\, 3/8\, H^{F'}_{21}(3,\chi_{p3},\chi_p,\chi_{q3},\chi_{12})\, R^d_p\, \chi_{12}^2 \\
&- H^{F'}_{21}(3,\chi_{12},\chi_p,\chi_p,\chi_{12})\, \big[3/16\, R^p_q R^d_p\, \chi_{12}^2 - 3/32\, R^d_p\, \chi_{12}^2\big] \,-\, H^{F'}_{21}(3,\chi_{12},\chi_p,\chi_q,\chi_{12})\, \big[3/16\, R^q_p R^d_p\, \chi_{12}^2 \\
&- 3/32\, R^d_p\, \chi_{12}^2\big] \,+\, 3/64\, H^{F'}_{21}(7,\chi_{12},\chi_p,\chi_p,\chi_{12})\, (R^d_p)^2\chi_{12}^2 \,+\, 3/32\, H^{F'}_{21}(7,\chi_{12},\chi_1,\chi_2,\chi_{12})\, R^d_1 R^d_2\, \chi_{12}^2,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\delta^{(6)22}_{\rm loops} \;=\;& \pi_{16}\, L^r_0\, \big[-4\, \chi_\pi\chi_{12} - 3\, \chi_\pi^2 - 2\, \chi_1\chi_2 + 4\, \chi_{12}^2 + 2\, \chi_3\chi_4\big] \,-\, 2\, \pi_{16}\, L^r_1\, \chi_{12}^2 \,-\, \pi_{16}\, L^r_2\, \big[3\, \chi_\pi^2 + \chi_{12}^2\big] \\
&+ \pi_{16}\, L^r_3\, \big[-3\, \chi_\pi\chi_{12} - 2\, \chi_\pi^2 - 3/2\, \chi_1\chi_2 + 4\, \chi_{12}^2 + 3/2\, \chi_3\chi_4\big] \,+\, \pi_{16}^2\, \big[-23/96\, \chi_\pi\chi_{12} - 1/3\, \chi_\pi^2 - 17/192\, \chi_1\chi_2 \\
&+ 5/48\, \chi_{12}^2 + 17/192\, \chi_3\chi_4\big] \,-\, 32\, L^r_4 L^r_5\, \chi_{12}\chi_{34} \,-\, 32\, L^{r2}_4\, \chi_{34}^2 \,-\, 8\, L^{r2}_5\, \chi_{12}^2 \,+\, 8\, L^{r2}_{11}\, \chi_{12}^2 \\
&+ \bar A(\chi_p)\, \pi_{16}\, \big[-1/12\, R^p_{q\pi}\, \chi_{12} - 1/8\, R^p_{q\pi}\, \chi_{34} + 1/48\, R^c_p\, \chi_q + 1/48\, R^c_p\, \chi_{12} + 1/16\, R^c_p\, \chi_{34}\big] \\
&+ \bar A(\chi_p)\, L^r_0\, \big[4\, R^p_{q\pi}\, \chi_p + R^c_p\, \chi_p + R^d_p\big] \,+\, \bar A(\chi_p)\, L^r_3\, \big[R^p_{q\pi}\, \chi_p + 5/2\, R^c_p\, \chi_p + 5/2\, R^d_p\big] \\
&- \bar A(\chi_p)\, L^r_4\, \big[2\, R^p_{q\pi}\, \chi_{34} - R^c_p\, \chi_{34}\big] \,-\, \bar A(\chi_p)\, L^r_5\, \big[R^p_{q\pi}\, \chi_{12} + 1/2\, R^c_p\, \chi_{12}\big] \,+\, \bar A(\chi_p)^2\, \big[1/16 + 1/32\, (R^p_{q\pi})^2 \\
&- 1/32\, R^p_{q\pi} R^c_p + 1/128\, (R^c_p)^2\big] \,-\, \bar A(\chi_p)\bar A(\chi_{ps})\, \big[1/24\, R^p_{q\pi} + 5/48\, R^p_{s\pi} - 7/96\, R^c_p\big] \\
&- \bar A(\chi_p)\bar A(\chi_{qs})\, \big[1/24\, R^p_{q\pi} + 1/16\, R^p_{s\pi} + 1/32\, R^c_p\big] \,-\, \bar A(\chi_p)\bar A(\chi_\pi)\, \big[1/32\, R^p_{q\pi} R^v_{\pi12} - 1/64\, R^c_p R^v_{\pi12}\big] \\
&+ 1/8\, \bar A(\chi_p)\bar A(\chi_{12}) \,+\, 1/12\, \bar A(\chi_p)\bar A(\chi_{34})\, R^\pi_{pp} \,+\, \bar A(\chi_p)\bar B(\chi_p,\chi_p,0)\, \big[1/4\, \chi_p - 1/8\, R^p_{q\pi} R^c_p\, \chi_p - 1/32\, R^p_{q\pi} R^d_p \\
&+ 1/8\, (R^c_p)^2\chi_p + 1/64\, R^c_p R^d_p\big] \,+\, 1/8\, \bar A(\chi_p)\bar B(\chi_p,\chi_\pi,0)\, R^z_{q\pi p} R^c_p R^v_{\pi12}\, \chi_p \,-\, \bar A(\chi_p)\bar B(\chi_q,\chi_q,0)\, \big[1/32\, R^p_{q\pi} R^d_q
\end{aligned}
$$

$$
\begin{aligned}
&- 1/64\, R^c_p R^d_q] \;-\; 1/8\, \bar{A}(\chi_p)\bar{B}(\chi_{ps},\chi_{ps},0)\, R^p_{s\pi}\,\chi_{ps} \;-\; 1/8\, \bar{A}(\chi_p)\bar{B}(\chi_1,\chi_2,0)\, R^q_{p\pi} R^c_p\,\chi_p \\
&+ 1/8\, \bar{A}(\chi_p)\bar{C}(\chi_p,\chi_p,\chi_p,0)\, R^c_p R^d_p\,\chi_p \;+\; \bar{A}(\chi_p,\varepsilon)\,\pi_{16}\, [-1/8\,\chi_p + 1/8\, R^p_{q\pi}\,\chi_{34} + 1/8\, R^\pi_{pp}\,\chi_{34}] \\
&+ \bar{A}(\chi_{ps})\,\pi_{16}\, [1/24\,\chi_{ps} - 1/8\,\chi_{qs} - 1/8\,\chi_{34}] \;-\; 2\,\bar{A}(\chi_{ps})\, L^r_0\,\chi_{ps} \;-\; 5\,\bar{A}(\chi_{ps})\, L^r_3\,\chi_{ps} \;-\; 2\,\bar{A}(\chi_{ps})\, L^r_4\,\chi_{34} \\
&+ \bar{A}(\chi_{ps})\, L^r_5\,\chi_{12} \;-\; 1/32\,\bar{A}(\chi_{ps})^2 \;+\; \bar{A}(\chi_{ps})\bar{A}(\chi_\pi)\, [7/96\, R^\pi_{pp} - 5/48\, R^\pi_{ps} - 1/32\, R^\pi_{qq} + 5/48\, R^\pi_{qs} - 1/24\, R^\pi_{12}] \\
&+ \bar{A}(\chi_{ps})\bar{B}(\chi_p,\chi_p,0)\, [1/16\, R^p_{s\pi}\,\chi_p - 5/16\, R^p_{s\pi}\,\chi_{ps}] \;-\; \bar{A}(\chi_{ps})\bar{B}(\chi_p,\chi_\pi,0)\, [1/12\, R^\pi_{ps} R^z_{qp\pi}\,\chi_p \\
&+ 1/6\, R^\pi_{ps} R^z_{qp\pi}\,\chi_{ps}] \;-\; 1/32\,\bar{A}(\chi_{ps})\bar{B}(\chi_q,\chi_q,0)\, R^d_q \;+\; 1/12\,\bar{A}(\chi_{ps})\bar{B}(\chi_1,\chi_2,0)\, R^q_{s\pi}\,\chi_s \\
&+ 1/6\,\bar{A}(\chi_{ps})\bar{B}(\chi_1,\chi_2,0,k)\, R^q_{s\pi} \;+\; 1/8\,\bar{A}(\chi_{ps},\varepsilon)\,\pi_{16}\, [\chi_{34} + \chi_s] \;-\; 1/16\,\bar{A}(\chi_{p3})\bar{A}(\chi_{p4}) \\
&+ 1/16\,\bar{A}(\chi_{p3})\bar{A}(\chi_{q4}) \;+\; \bar{A}(\chi_\pi)\,\pi_{16}\, [1/24\, R^v_{\pi 12}\,\chi_\pi + 1/16\, R^v_{\pi 12}\,\chi_{12}] \;+\; \bar{A}(\chi_\pi)\, L^r_0\, [6\, R^\pi_{12}\,\chi_\pi + R^v_{\pi 12}\,\chi_\pi] \\
&- 8\,\bar{A}(\chi_\pi)\, L^r_1\,\chi_\pi \;-\; 2\,\bar{A}(\chi_\pi)\, L^r_2\,\chi_\pi \;+\; \bar{A}(\chi_\pi)\, L^r_3\, [6\, R^\pi_{12}\,\chi_\pi + 5/2\, R^v_{\pi 12}\,\chi_\pi] \;+\; \bar{A}(\chi_\pi)\, L^r_4\, [4\,\chi_\pi + R^v_{\pi 12}\,\chi_\pi] \\
&- \bar{A}(\chi_\pi)\, L^r_5\, [2\, R^\pi_{12}\,\chi_{12} + 1/2\, R^v_{\pi 12}\,\chi_{12}] \;+\; 1/128\,\bar{A}(\chi_\pi)^2\, (R^v_{\pi 12})^2 \;+\; 1/12\,\bar{A}(\chi_\pi)\bar{A}(\chi_{34})\, R^v_{\pi 12} \\
&+ \bar{A}(\chi_\pi)\bar{B}(\chi_p,\chi_p,0)\, [-1/8\, R^p_{q\pi} R^\pi_{pp}\,\chi_p + 3/16\, R^c_p\,\chi_p - 1/8\,(R^c_p)^2\chi_p - 1/16\, R^d_p + 1/64\, R^d_p R^v_{\pi 12}] \\
&- 1/8\,\bar{A}(\chi_\pi)\bar{B}(\chi_p,\chi_\pi,0)\, R^p_{\pi\pi} R^\pi_{12} R^z_{pq\pi}\,\chi_p \;-\; 1/8\,\bar{A}(\chi_\pi)\bar{B}(\chi_{ps},\chi_{ps},0)\, R^\pi_{ps}\,\chi_{ps} \\
&- 1/16\,\bar{A}(\chi_\pi)\bar{B}(\chi_\pi,\chi_\pi,0)\, R^v_{\pi 12}\,\chi_{34} \;-\; 1/8\,\bar{A}(\chi_\pi)\bar{B}(\chi_1,\chi_2,0)\, R^1_{\pi\pi} R^2_{\pi\pi}\,\chi_{34} \\
&+ 1/8\,\bar{A}(\chi_\pi)\bar{C}(\chi_p,\chi_p,\chi_p,0)\, R^\pi_{pp} R^d_p\,\chi_p \;+\; 1/8\,\bar{A}(\chi_\pi,\varepsilon)\,\pi_{16}\, [\chi_\pi - R^\pi_{12}\,\chi_\pi - R^v_{\pi 12}\,\chi_\pi] \\
&+ \bar{A}(\chi_1)\bar{A}(\chi_2)\, [-1/32\, R^p_{q\pi} R^c_q + 1/16\, R^1_{2\pi} R^2_{1\pi} + 1/64\, R^c_1 R^c_2] \;-\; 4\,\bar{A}(\chi_{12})\, L^r_1\,\chi_{12} \;-\; 10\,\bar{A}(\chi_{12})\, L^r_2\,\chi_{12} \\
&+ 1/8\,\bar{A}(\chi_{12})^2 \;-\; 1/2\,\bar{A}(\chi_{12})\bar{B}(\chi_1,\chi_2,0,k) \;+\; 1/4\,\bar{A}(\chi_{12},\varepsilon)\,\pi_{16}\,\chi_{12} \;-\; 16\,\bar{A}(\chi_{34})\, L^r_1\,\chi_{34} \\
&- 4\,\bar{A}(\chi_{34})\, L^r_2\,\chi_{34} \;+\; 8\,\bar{A}(\chi_{34})\, L^r_4\,\chi_{34} \;+\; 1/8\,\bar{A}(\chi_{34})\bar{B}(\chi_p,\chi_p,0)\, R^p_{\pi\pi}\,\chi_{34} \\
&+ \bar{A}(\chi_{34})\bar{B}(\chi_p,\chi_\pi,0)\, [1/12\, R^z_{q\pi p} R^v_{\pi 12}\,\chi_p - 1/6\, R^z_{q\pi p} R^v_{\pi 12}\,\chi_{34}] \;-\; 1/6\,\bar{A}(\chi_{34})\bar{B}(\chi_p,\chi_\pi,0,k)\, R^z_{q\pi p} R^v_{\pi 12} \\
&+ 1/8\,\bar{A}(\chi_{34})\bar{B}(\chi_\pi,\chi_\pi,0)\, R^v_{\pi 12}\,\chi_{34} \;+\; \bar{A}(\chi_{34})\bar{B}(\chi_1,\chi_2,0)\, [-1/4\,\chi_{34} + 1/6\, R^\pi_{12}\,\chi_{12} - 1/4\, R^\pi_{12}\,\chi_{34}] \\
&- 1/6\,\bar{A}(\chi_{34})\bar{B}(\chi_1,\chi_2,0,k)\, R^\pi_{12} \;+\; 1/4\,\bar{A}(\chi_{34},\varepsilon)\,\pi_{16}\,\chi_{34} \;+\; 1/16\,\bar{A}(\chi_{1s})\bar{A}(\chi_{2s}) \\
&+ \bar{B}(\chi_p,\chi_p,0)\,\pi_{16}\, [1/96\, R^d_p\,\chi_p + 1/32\, R^d_p\,\chi_q + 1/16\, R^d_p\,\chi_{34}] \;+\; \bar{B}(\chi_p,\chi_p,0)\, L^r_0\, R^d_p\,\chi_p \\
&+ 5/2\,\bar{B}(\chi_p,\chi_p,0)\, L^r_3\, R^d_p\,\chi_p \;+\; \bar{B}(\chi_p,\chi_p,0)\, L^r_4\, [-4\, R^p_{q\pi}\,\chi_p\chi_{34} + 2\, R^c_p\,\chi_p\chi_{34} + 3\, R^d_p\,\chi_{34}] \\
&+ \bar{B}(\chi_p,\chi_p,0)\, L^r_5\, [-2\, R^p_{q\pi}\,\chi_1\chi_2 + R^c_p\,\chi_p^2 - 1/2\, R^d_p\,\chi_{12}] \;+\; \bar{B}(\chi_p,\chi_p,0)\, L^r_6\, [8\, R^p_{q\pi}\,\chi_p\chi_{34} - 4\, R^c_p\,\chi_p\chi_{34} \\
&- 4\, R^d_p\,\chi_{34}] \;+\; 4\,\bar{B}(\chi_p,\chi_p,0)\, L^r_7\,(R^d_p)^2 \;+\; \bar{B}(\chi_p,\chi_p,0)\, L^r_8\, [4\, R^p_{q\pi}\,\chi_1\chi_2 - 2\, R^c_p\,\chi_p^2 + 2\,(R^d_p)^2] \\
&- \bar{B}(\chi_p,\chi_p,0)^2\, [1/8\, R^p_{q\pi} R^d_p\,\chi_p - 1/8\, R^c_p R^d_p\,\chi_p - 1/128\,(R^d_p)^2] \\
&+ 1/8\,\bar{B}(\chi_p,\chi_p,0)\bar{B}(\chi_p,\chi_\pi,0)\, R^z_{q\pi p} R^d_p R^v_{\pi 12}\,\chi_p \;-\; 1/8\,\bar{B}(\chi_p,\chi_p,0)\bar{B}(\chi_1,\chi_2,0)\, R^q_{p\pi} R^d_p\,\chi_p \\
&+ 1/8\,\bar{B}(\chi_p,\chi_p,0)\bar{C}(\chi_p,\chi_p,\chi_p,0)\,(R^d_p)^2\chi_p \;-\; 1/16\,\bar{B}(\chi_p,\chi_p,0,\varepsilon)\,\pi_{16}\, [R^d_p\,\chi_p + R^d_p\,\chi_{34}] \\
&+ 8\,\bar{B}(\chi_p,\chi_\pi,0)\, L^r_7\, R^z_{qp\pi} R^d_p R^z_{\pi 34p} \;+\; 4\,\bar{B}(\chi_p,\chi_\pi,0)\, L^r_8\, R^z_{qp\pi} R^d_p R^z_{\pi 34p} \;-\; 4\,\bar{B}(\chi_{ps},\chi_{ps},0)\, L^r_4\,\chi_{ps}\chi_{34} \\
&- 2\,\bar{B}(\chi_{ps},\chi_{ps},0)\, L^r_5\,\chi_{ps}^2 \;+\; 8\,\bar{B}(\chi_{ps},\chi_{ps},0)\, L^r_6\,\chi_{ps}\chi_{34} \;+\; 4\,\bar{B}(\chi_{ps},\chi_{ps},0)\, L^r_8\,\chi_{ps}^2 \\
&+ 2\,\bar{B}(\chi_\pi,\chi_\pi,0)\, L^r_4\, R^v_{\pi 12}\,\chi_\pi^2 \;+\; \bar{B}(\chi_\pi,\chi_\pi,0)\, L^r_5\, R^v_{\pi 12}\,\chi_\pi^2 \;-\; 4\,\bar{B}(\chi_\pi,\chi_\pi,0)\, L^r_6\, R^v_{\pi 12}\,\chi_\pi^2 \\
&- 4\,\bar{B}(\chi_\pi,\chi_\pi,0)\, L^r_7\,(R^z_{\pi 4})^2 R^v_{\pi 12} \;-\; \bar{B}(\chi_\pi,\chi_\pi,0)\, L^r_8\, [R^v_{\pi 12}\,\chi_3^2 + R^v_{\pi 12}\,\chi_4^2] \\
&+ 1/64\,\bar{B}(\chi_1,\chi_1,0)\bar{B}(\chi_2,\chi_2,0)\, R^d_1 R^d_2 \;-\; 8\,\bar{B}(\chi_1,\chi_2,0)\, L^r_7\, R^d_1 R^d_2 \;-\; 4\,\bar{B}(\chi_1,\chi_2,0)\, L^r_8\, R^d_1 R^d_2 \\
&+ 4\,\bar{C}(\chi_p,\chi_p,\chi_p,0)\, L^r_4\, R^d_p\,\chi_p\chi_{34} \;+\; 2\,\bar{C}(\chi_p,\chi_p,\chi_p,0)\, L^r_5\, R^d_p\,\chi_p^2 \;-\; 8\,\bar{C}(\chi_p,\chi_p,\chi_p,0)\, L^r_6\, R^d_p\,\chi_p\chi_{34} \\
&- 4\,\bar{C}(\chi_p,\chi_p,\chi_p,0)\, L^r_8\, R^d_p\,\chi_p^2 \;+\; H^F(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\, [1/8\,\chi_p - 1/16\,\chi_{12} - 1/16\,(R^p_{q\pi})^2\chi_{12} \\
&+ 1/16\, R^p_{q\pi} R^c_p\,\chi_{12} - 1/64\,(R^c_p)^2\chi_{12}] \;+\; H^F(1,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\, [-1/8\, R^p_{q\pi}\,\chi_{qs} + 1/16\, R^p_{q\pi}\,\chi_{12} \\
&- 3/32\, R^p_{s\pi}\,\chi_p + 1/32\, R^p_{s\pi}\,\chi_q + 1/16\, R^c_p\,\chi_{ps}] \;-\; 1/8\, H^F(1,\chi_{p3},\chi_{q4},\chi_{34},\chi_{12})\,\chi_{34} \\
&+ H^F(1,\chi_\pi,\chi_p,\chi_{12},\chi_{12})\, [1/16\, R^p_{q\pi} R^v_{\pi 12}\,\chi_{12} - 1/32\, R^c_p R^v_{\pi 12}\,\chi_{12}] \;-\; 1/64\, H^F(1,\chi_\pi,\chi_\pi,\chi_{12},\chi_{12})\,(R^v_{\pi 12})^2\chi_{12} \\
&+ 1/32\, H^F(1,\chi_\pi,\chi_{1s},\chi_{2s},\chi_{12})\, [-R^\pi_{ps}\,\chi_p + R^\pi_{ps}\,\chi_q + R^v_{\pi ps}\,\chi_\pi + R^v_{\pi 12}\,\chi_s] \\
&+ H^F(1,\chi_1,\chi_{12},\chi_2,\chi_{12})\, [1/16\, R^p_{q\pi} R^c_q\,\chi_{12} - 1/8\, R^1_{2\pi} R^2_{1\pi}\,\chi_{12} - 1/32\, R^c_1 R^c_2\,\chi_{12}] \\
&- 1/8\, H^F(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\,\chi_{12} \;+\; H^F(2,\chi_p,\chi_p,\chi_{12},\chi_{12})\, [1/16\, R^p_{q\pi} R^d_p\,\chi_{12} - 1/32\, R^c_p R^d_p\,\chi_{12}] \\
&- 1/32\, H^F(2,\chi_p,\chi_\pi,\chi_{12},\chi_{12})\, R^d_p R^v_{\pi 12}\,\chi_{12} \;+\; H^F(2,\chi_p,\chi_{12},\chi_q,\chi_{12})\, [1/16\, R^q_{p\pi} R^d_p\,\chi_{12} - 1/32\, R^c_q R^d_p\,\chi_{12}] \\
&+ 1/16\, H^F(2,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\, R^d_p\,\chi_{ps} \;-\; 1/64\, H^F(5,\chi_p,\chi_p,\chi_{12},\chi_{12})\,(R^d_p)^2\chi_{12}
\end{aligned}
$$

$$
\begin{aligned}
&-\, 1/32\, H^{F}(5,\chi_1,\chi_2,\chi_{12},\chi_{12})\, R_1^d R_2^d \chi_{12} \;+\; H^{F'}(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\; [-1/8\, \chi_p\chi_{12} - 3/16\, \chi_{12}^2 \\
&-\, 1/16\, (R^p_{q\pi})^2\chi_{12}^2 + 3/16\, R^p_{q\pi}R^c_p\, \chi_{12}^2 - 3/64\, (R^c_p)^2\chi_{12}^2] \;+\; H^{F'}(1,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\; [-5/16\, R^p_{q\pi}\, \chi_q\chi_{12} \\
&+\, 1/8\, R^p_{q\pi}\, \chi_{ps}\chi_{12} + 3/32\, R^p_{s\pi}\, \chi_q\chi_{12} + 7/32\, R^p_{s\pi}\, \chi_{12}\chi_s - 1/16\, R^c_p\, \chi_{ps}\chi_{12}] \\
&+\, 1/8\, H^{F'}(1,\chi_{p3},\chi_{q4},\chi_{34},\chi_{12})\, \chi_{12}\chi_{34} \;+\; H^{F'}(1,\chi_\pi,\chi_p,\chi_{12},\chi_{12})\; [1/4\, R^p_{q\pi}R^\pi_{12}\, \chi_{12}^2 - 1/16\, R^p_{q\pi}R^v_{\pi 12}\, \chi_{12}^2 \\
&+\, 1/8\, R^\pi_{pp}R^c_p\, \chi_{12}^2 + 1/32\, R^c_p R^v_{\pi 12}\, \chi_{12}^2] \;+\; H^{F'}(1,\chi_\pi,\chi_\pi,\chi_{12},\chi_{12})\; [1/4\, (R^\pi_{12})^2\chi_{12}^2 + 1/4\, R^\pi_{12}R^v_{\pi 12}\, \chi_{12}^2 \\
&+\, 5/64\, (R^v_{\pi 12})^2\chi_{12}^2] \;+\; H^{F'}(1,\chi_\pi,\chi_{1s},\chi_{2s},\chi_{12})\; [-3/32\, R^\pi_{ps}\, \chi_p\chi_{12} - 5/32\, R^\pi_{ps}\, \chi_q\chi_{12} + 1/4\, R^\pi_{12}\, \chi_{12}^2 \\
&-\, 1/32\, R^v_{\pi ps}\, \chi_\pi\chi_{12} - 1/32\, R^v_{\pi 12}\, \chi_{12}\chi_s] \;+\; H^{F'}(1,\chi_1,\chi_{12},\chi_2,\chi_{12})\; [-1/16\, R^p_{q\pi}R^c_q\, \chi_{12}^2 + 3/8\, R^1_{2\pi}R^2_{1\pi}\, \chi_{12}^2 \\
&+\, 1/32\, R^c_1R^c_2\, \chi_{12}^2] \;+\; 1/8\, H^{F'}(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\, \chi_{12}^2 \;+\; H^{F'}(2,\chi_p,\chi_p,\chi_{12},\chi_{12})\; [-1/16\, R^p_{q\pi}R^d_p\, \chi_{12}^2 \\
&+\, 5/32\, R^c_pR^d_p\, \chi_{12}^2] \;+\; H^{F'}(2,\chi_p,\chi_\pi,\chi_{12},\chi_{12})\; [1/8\, R^\pi_{pp}R^d_p\, \chi_{12}^2 + 1/32\, R^d_pR^v_{\pi 12}\, \chi_{12}^2] \\
&-\, H^{F'}(2,\chi_p,\chi_{12},\chi_q,\chi_{12})\; [1/16\, R^q_{p\pi}R^d_p\, \chi_{12}^2 - 1/32\, R^c_qR^d_p\, \chi_{12}^2] \;-\; 1/16\, H^{F'}(2,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\, R^d_p\, \chi_{ps}\chi_{12} \\
&+\, 5/64\, H^{F'}(5,\chi_p,\chi_p,\chi_{12},\chi_{12})\, (R^d_p)^2\chi_{12}^2 \;+\; 1/32\, H^{F'}(5,\chi_1,\chi_2,\chi_{12},\chi_{12})\, R^d_1R^d_2\, \chi_{12}^2 \\
&+\, H_1^{F'}(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\; [\chi_{12}^2 + 1/2\, (R^p_{q\pi})^2\, \chi_{12}^2 - 1/2\, R^p_{q\pi}R^c_p\, \chi_{12}^2 + 1/4\, (R^c_p)^2\chi_{12}^2] \\
&+\, 1/2\, H_1^{F'}(1,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\, R^p_{q\pi}R^z_{sqp}\, \chi_{12}^2 \;-\; 1/2\, H_1^{F'}(1,\chi_{ps},\chi_{qs},\chi_p,\chi_{12})\, R^p_{s\pi}\, \chi_{12}^2 \\
&-\, 1/2\, H_1^{F'}(1,\chi_{ps},\chi_{qs},\chi_\pi,\chi_{12})\, R^\pi_{12}R^z_{sp\pi}\, \chi_{12}^2 \;+\; 1/4\, H_1^{F'}(1,\chi_{12},\chi_p,\chi_\pi,\chi_{12})\; [R^p_{q\pi}R^v_{\pi 12}\, \chi_{12}^2 \\
&-\, R^\pi_{pp}R^z_{qp\pi}R^c_p\, \chi_{12}^2] \;-\; H_1^{F'}(1,\chi_{12},\chi_\pi,\chi_\pi,\chi_{12})\; [1/4\, R^\pi_{12}R^v_{\pi 12}\, \chi_{12}^2 + 1/8\, (R^v_{\pi 12})^2\chi_{12}^2] \\
&+\, H_1^{F'}(1,\chi_{12},\chi_1,\chi_2,\chi_{12})\; [1/4\, R^p_{q\pi}R^c_q\, \chi_{12}^2 - 1/2\, R^1_{2\pi}R^2_{1\pi}\, \chi_{12}^2] \;+\; 1/4\, H_1^{F'}(3,\chi_{12},\chi_p,\chi_p,\chi_{12})\; [R^p_{q\pi}R^d_p\, \chi_{12}^2 \\
&-\, R^c_pR^d_p\, \chi_{12}^2] \;+\; 1/4\, H_1^{F'}(3,\chi_{12},\chi_p,\chi_q,\chi_{12})\, R^q_{p\pi}R^d_p\, \chi_{12}^2 \;+\; 1/4\, H_1^{F'}(3,\chi_{12},\chi_p,\chi_\pi,\chi_{12})\, R^\pi_{12}R^z_{pq\pi}R^d_p\, \chi_{12}^2 \\
&-\, 1/8\, H_1^{F'}(7,\chi_{12},\chi_p,\chi_p,\chi_{12})\, (R^d_p)^2\chi_{12}^2 \;-\; 3/8\, H_{21}^{F'}(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\, \chi_{12}^2 \\
&-\, 3/16\, H_{21}^{F'}(1,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\, R^p_{q\pi}R^z_{sqp}\, \chi_{12}^2 \;+\; 3/16\, H_{21}^{F'}(1,\chi_{ps},\chi_{qs},\chi_p,\chi_{12})\; [R^p_{q\pi}\, \chi_{12}^2 + R^p_{s\pi}\, \chi_{12}^2 - R^c_p\, \chi_{12}^2] \\
&+\, 3/16\, H_{21}^{F'}(1,\chi_{ps},\chi_{qs},\chi_q,\chi_{12})\, R^q_{p\pi}R^z_{spq}\, \chi_{12}^2 \;+\; 3/16\, H_{21}^{F'}(1,\chi_{ps},\chi_{qs},\chi_\pi,\chi_{12})\, R^\pi_{12}R^z_{pq\pi}R^z_{sp\pi}\, \chi_{12}^2 \\
&-\, 3/16\, H_{21}^{F'}(1,\chi_\pi,\chi_{1s},\chi_{2s},\chi_{12})\, R^\pi_{12}R^z_{s1\pi}R^z_{s2\pi}\, \chi_{12}^2 \;+\; H_{21}^{F'}(1,\chi_{12},\chi_p,\chi_p,\chi_{12})\; [3/16\, \chi_{12}^2 + 3/16\, (R^p_{q\pi})^2\chi_{12}^2 \\
&-\, 3/16\, R^p_{q\pi}R^c_p\, \chi_{12}^2 + 3/64\, (R^c_p)^2\chi_{12}^2] \;+\; H_{21}^{F'}(1,\chi_{12},\chi_p,\chi_\pi,\chi_{12})\; [-3/16\, R^p_{q\pi}R^v_{\pi 12}\, \chi_{12}^2 + 3/32\, R^c_pR^v_{\pi 12}\, \chi_{12}^2] \\
&+\, 3/64\, H_{21}^{F'}(1,\chi_{12},\chi_\pi,\chi_\pi,\chi_{12})\, (R^v_{\pi 12})^2\chi_{12}^2 \;+\; H_{21}^{F'}(1,\chi_{12},\chi_1,\chi_2,\chi_{12})\; [-3/16\, R^p_{q\pi}R^c_q\, \chi_{12}^2 + 3/8\, R^1_{2\pi}R^2_{1\pi}\, \chi_{12}^2 \\
&+\, 3/32\, R^c_1R^c_2\, \chi_{12}^2] \;+\; 3/8\, H_{21}^{F'}(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\, \chi_{12}^2 \;+\; 3/8\, H_{21}^{F'}(1,\chi_{34},\chi_{p3},\chi_{q4},\chi_{12})\, \chi_{12}^2 \\
&-\, 3/16\, H_{21}^{F'}(3,\chi_{ps},\chi_p,\chi_{qs},\chi_{12})\, R^d_p\, \chi_{12}^2 \;-\; H_{21}^{F'}(3,\chi_{12},\chi_p,\chi_p,\chi_{12})\; [3/16\, R^p_{q\pi}R^d_p\, \chi_{12}^2 - 3/32\, R^c_pR^d_p\, \chi_{12}^2] \\
&-\, H_{21}^{F'}(3,\chi_{12},\chi_p,\chi_q,\chi_{12})\; [3/16\, R^q_{p\pi}R^d_p\, \chi_{12}^2 - 3/32\, R^c_qR^d_p\, \chi_{12}^2] \;+\; 3/32\, H_{21}^{F'}(3,\chi_{12},\chi_p,\chi_\pi,\chi_{12})\, R^d_pR^v_{\pi 12}\, \chi_{12}^2 \\
&+\, 3/64\, H_{21}^{F'}(7,\chi_{12},\chi_p,\chi_p,\chi_{12})\, (R^d_p)^2\chi_{12}^2 \;+\; 3/32\, H_{21}^{F'}(7,\chi_{12},\chi_1,\chi_2,\chi_{12})\, R^d_1R^d_2\, \chi_{12}^2.
\end{aligned}
\tag{23}
$$

IV. TWO-FLAVOR PSEUDOSCALAR MESON MASSES

The pseudoscalar meson squared masses $M^2_{\rm PS}$ are calculated from the resummation of the irreducible self-energies, according to

$$M^2_{\rm PS} = M_0^2 + \Sigma(M^2_{\rm PS}), \tag{24}$$

where M_0 is the lowest order meson mass, and Σ is the sum of the irreducible self-energy diagrams. The self-energy contributions $-i\Sigma$ at $\mathcal{O}(p^6)$, or NNLO, are shown in Fig. 2. The pseudoscalar meson mass is a function of the same mass inputs and LEC:s as outlined for the decay constants in the previous section, although the set of $\mathcal{O}(p^6)$ LEC:s that contribute to the meson mass is slightly different from that of the decay constant, in particular the number of $\mathcal{O}(p^6)$ LEC:s is larger for the meson mass. The results for the pseudoscalar meson masses are given in the form

$$M^2_{\rm PS} = M_0^2 + \frac{\delta^{(4){\rm vs}}}{F^2} + \frac{\delta^{(6){\rm vs}}_{\rm ct} + \delta^{(6){\rm vs}}_{\rm loops}}{F^4} + \mathcal{O}(p^8), \tag{25}$$

where the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ contributions have been separated. It should be noted that the lowest order mass has not been factored out of the results, as this is not meaningful for the results with $d_{\rm val} = 2$. Similarly to the treatment of the decay constant, the contribution from the $\mathcal{O}(p^6)$ counterterms has also been separated from the rest of the NNLO result, which consists mostly of the contributions from the chiral loops of $\mathcal{O}(p^6)$, although there is also a contribution from the diagrams at $\mathcal{O}(p^4)$ through the expansion of Eq. (25) to $\mathcal{O}(p^6)$. The superscripts (v) and (s) indicate the values of $d_{\rm val}$ and $d_{\rm sea}$, respectively. As expected, the infinities in all expressions for the meson mass have canceled.

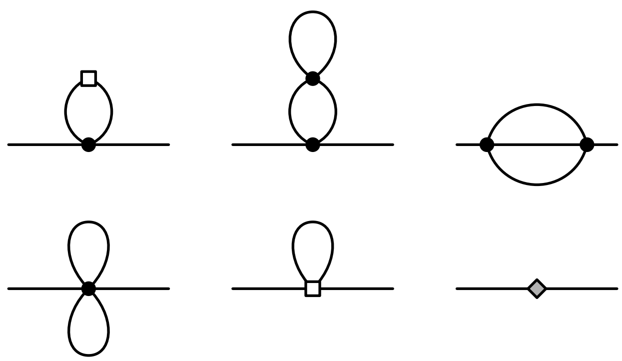

FIG. 2: Feynman diagrams at $\mathcal{O}(p^6)$ or two-loop for the irreducible self-energy contribution $-i\Sigma$ to the pseudoscalar meson mass. Filled circles denote vertices of the $\mathcal{L}_2$ Lagrangian, whereas open squares and shaded diamonds denote vertices of the $\mathcal{L}_4$ and $\mathcal{L}_6$ Lagrangians, respectively.

A. Results for $d_{\rm val} = 1$

The lowest order mass, M_0^2, is for $d_{\rm val} = 1$ equal to χ_1. The NLO, or $\mathcal{O}(p^4)$ contribution to the meson mass for $d_{\rm val} = 1$ and $d_{\rm sea} = 2$ is

$$\begin{aligned}\delta^{(4)12} \;=\; & -\,16\,L_4^r\,\chi_1\chi_{34} \,-\, 8\,L_5^r\,\chi_1^2 \\ & +\,32\,L_6^r\,\chi_1\chi_{34} \,+\, 16\,L_8^r\,\chi_1^2 \\ & -\,1/2\,\bar{A}(\chi_\pi)\,R^\pi_{11}\,\chi_1 \,-\, 1/2\,\bar{A}(\chi_1)\,R^c_1\,\chi_1 \\ & -\,1/2\,\bar{B}(\chi_1,\chi_1,0)\,R^d_1\,\chi_1, \end{aligned} \qquad (26)$$

from which the $d_{\rm sea} = 1$ case can be obtained by carefully taking the limit $\chi_4 \to \chi_3$. In that limit the $\bar{A}(\chi_\pi)$ term vanishes, and the remaining residues reduce such that $R^c_1 \to 1$, yielding the correct result for $d_{\rm val} = 1$ and $d_{\rm sea} = 1$. The tree level contribution from the $\mathcal{O}(p^6)$ Lagrangian is

$$\begin{aligned}\delta^{(6)12}_{\rm ct} \;=\; & -\,32\,K_{17}^r\,\chi_1^3 \,-\, 64\,K_{18}^r\,\chi_1^2\chi_{34} \,-\, 16\,K_{19}^r\,\chi_1^3 \\ & -\,32\,K_{20}^r\,\chi_1^2\chi_{34} \,-\, 16\,K_{21}^r\,\chi_1\chi_s^2 \\ & -\,64\,K_{22}^r\,\chi_1\chi_{34}^2 \,-\, 16\,K_{23}^r\,\chi_1^3 \,+\, 48\,K_{25}^r\,\chi_1^3 \\ & +\,K_{26}^r\,\left[16\,\chi_1\chi_s^2 + 64\,\chi_1^2\chi_{34}\right] \,+\, 192\,K_{27}^r\,\chi_1\chi_{34}^2 \\ & +\,32\,K_{39}^r\,\chi_1^3 \,+\, 64\,K_{40}^r\,\chi_1^2\chi_{34}, \end{aligned} \qquad (27)$$

from which the result for $d_{\rm sea} = 1$ may again be obtained by setting $\chi_4 \to \chi_3$. The remaining contributions to the meson mass at $\mathcal{O}(p^6)$ we again give separately for $d_{\rm sea} = 1$ and $d_{\rm sea} = 2$. These contributions are

$$\begin{aligned}\delta^{(6)11}_{\rm loops} \;=\; & \pi_{16}\,L_0^r\,\left[2\,\chi_1\chi_3^2 + 8\,\chi_1^2\chi_3 - 4\,\chi_1^3\right] \,+\, 4\,\pi_{16}\,L_1^r\,\chi_1^3 \,+\, \pi_{16}\,L_2^r\,\left[6\,\chi_1\chi_3^2 + 2\,\chi_1^3\right] \,+\, \pi_{16}\,L_3^r\,\left[\chi_1\chi_3^2 + 6\,\chi_1^2\chi_3 - 5\,\chi_1^3\right] \\ & +\,\pi_{16}^2\,\left[47/96\,\chi_1\chi_3^2 + 35/144\,\chi_1^2\chi_3 - 17/288\,\chi_1^3\right] \,+\, 256\,L_4^rL_5^r\,\chi_1^2\chi_3 \,-\, 512\,L_4^rL_6^r\,\chi_1\chi_3^2 \,-\, 256\,L_4^rL_8^r\,\chi_1^2\chi_3 \\ & +\,128\,L_4^rL_{11}^r\,\chi_1^2\chi_3 \,+\, 256\,L_4^{r2}\,\chi_1\chi_3^2 \,-\, 256\,L_5^rL_6^r\,\chi_1^2\chi_3 \,-\, 128\,L_5^rL_8^r\,\chi_1^3 \,+\, 64\,L_5^rL_{11}^r\,\chi_1^3 \,+\, 64\,L_5^{r2}\,\chi_1^3 \\ & -\,256\,L_6^rL_{11}^r\,\chi_1^2\chi_3 \,-\, 128\,L_8^rL_{11}^r\,\chi_1^3 \,-\, 12\,\bar{A}(\chi_1)\,L_0^r\,\left[\chi_1^2 + R^d_1\,\chi_1\right] \,+\, 8\,\bar{A}(\chi_1)\,L_1^r\,\chi_1^2 \,+\, 20\,\bar{A}(\chi_1)\,L_2^r\,\chi_1^2 \\ & -\,12\,\bar{A}(\chi_1)\,L_3^r\,\left[\chi_1^2 + R^d_1\,\chi_1\right] \,+\, 16\,\bar{A}(\chi_1)\,L_4^r\,\chi_1\chi_3 \,+\, \bar{A}(\chi_1)\,L_5^r\,\left[16\,\chi_1^2 + 8\,R^d_1\,\chi_1\right] \\ & +\,16\,\bar{A}(\chi_1)\,L_6^r\,\left[\chi_1^2 + R^d_1\,\chi_1\right] \,+\, 32\,\bar{A}(\chi_1)\,L_7^r\,R^d_1\,\chi_1 \,-\, 32\,\bar{A}(\chi_1)\,L_8^r\,\chi_1^2 \,+\, 5/8\,\bar{A}(\chi_1)^2\chi_1 \\ & +\,\bar{A}(\chi_1)\bar{B}(\chi_1,\chi_1,0)\,\left[3/2\,\chi_1^2 + 1/4\,R^d_1\,\chi_1\right] \,+\, 1/2\,\bar{A}(\chi_1)\bar{C}(\chi_1,\chi_1,\chi_1,0)\,R^d_1\,\chi_1^2 \,-\, \bar{A}(\chi_1,\varepsilon)\,\pi_{16}\,\left[7/4\,\chi_1^2\right. \\ & \left. -\,1/4\,R^d_1\,\chi_1\right] \,+\, 4/3\,\bar{A}(\chi_{13})\,\pi_{16}\,\chi_1\chi_3 \,+\, 16\,\bar{A}(\chi_{13})\,L_0^r\,\chi_1\chi_{13} \,+\, 40\,\bar{A}(\chi_{13})\,L_3^r\,\chi_1\chi_{13} \,-\, 32\,\bar{A}(\chi_{13})\,L_5^r\,\chi_1\chi_{13} \\ & +\,64\,\bar{A}(\chi_{13})\,L_8^r\,\chi_1\chi_{13} \,-\, \bar{A}(\chi_{13})^2\,\chi_1 \,-\, 2\,\bar{A}(\chi_{13})\bar{B}(\chi_1,\chi_1,0)\,\chi_1\chi_3 \,-\, 2\,\bar{A}(\chi_{13},\varepsilon)\,\pi_{16}\,\chi_1\chi_3 \\ & +\,48\,\bar{A}(\chi_3)\,L_1^r\,\chi_1\chi_3 \,+\, 12\,\bar{A}(\chi_3)\,L_2^r\,\chi_1\chi_3 \,-\, 48\,\bar{A}(\chi_3)\,L_4^r\,\chi_1\chi_3 \,+\, 48\,\bar{A}(\chi_3)\,L_6^r\,\chi_1\chi_3 \\ & +\,3/4\,\bar{A}(\chi_3)\bar{B}(\chi_1,\chi_1,0)\,\chi_1\chi_3 \,-\, 3/4\,\bar{A}(\chi_3,\varepsilon)\,\pi_{16}\,\chi_1\chi_3 \,-\, 12\,\bar{B}(\chi_1,\chi_1,0)\,L_0^r\,R^d_1\,\chi_1^2 \\ & -\,12\,\bar{B}(\chi_1,\chi_1,0)\,L_3^r\,R^d_1\,\chi_1^2 \,-\, \bar{B}(\chi_1,\chi_1,0)\,L_4^r\,\left[24\,\chi_1\chi_3^2 - 32\,\chi_1^2\chi_3\right] \,+\, \bar{B}(\chi_1,\chi_1,0)\,L_5^r\,\left[4\,\chi_1^3 + 24\,R^d_1\,\chi_1^2\right] \\ & +\,\bar{B}(\chi_1,\chi_1,0)\,L_6^r\,\left[32\,\chi_1\chi_3^2 - 48\,\chi_1^2\chi_3\right] \,+\, 16\,\bar{B}(\chi_1,\chi_1,0)\,L_7^r\,(R^d_1)^2\chi_1 \,-\, \bar{B}(\chi_1,\chi_1,0)\,L_8^r\,\left[8\,\chi_1^3 + 48\,R^d_1\,\chi_1^2\right. \\ & \left. -\,8\,(R^d_1)^2\chi_1\right] \,+\, \bar{B}(\chi_1,\chi_1,0)^2\,\left[1/2\,R^d_1\,\chi_1^2 + 1/8\,(R^d_1)^2\chi_1\right] \,+\, 1/2\,\bar{B}(\chi_1,\chi_1,0)\bar{C}(\chi_1,\chi_1,\chi_1,0)\,(R^d_1)^2\chi_1^2 \\ & +\,1/4\,\bar{B}(\chi_1,\chi_1,0,\varepsilon)\,\pi_{16}\,R^d_1\,\chi_1^2 \,+\, 16\,\bar{C}(\chi_1,\chi_1,\chi_1,0)\,L_4^r\,R^d_1\,\chi_1^2\chi_3 \,+\, 8\,\bar{C}(\chi_1,\chi_1,\chi_1,0)\,L_5^r\,R^d_1\,\chi_1^3 \end{aligned}$$

$$
\begin{aligned}
&-\ 32\,\bar{C}(\chi_1,\chi_1,\chi_1,0)\,L_6^r\,R_1^d\,\chi_1^2\chi_3\ -\ 16\,\bar{C}(\chi_1,\chi_1,\chi_1,0)\,L_8^r\,R_1^d\,\chi_1^3\ +\ 5/6\,H^F(1,\chi_1,\chi_1,\chi_1,\chi_1)\,\chi_1^2\\
&+\ H^F(1,\chi_1,\chi_{13},\chi_{13},\chi_1)\,\left[1/4\,\chi_1\chi_3-\chi_1^2\right]\ +\ 3/4\,H^F(1,\chi_{13},\chi_{13},\chi_3,\chi_1)\,\chi_1\chi_3\\
&+\ H^F(2,\chi_1,\chi_1,\chi_1,\chi_1)\,R_1^d\,\chi_1^2\ +\ 3/4\,H^F(2,\chi_1,\chi_{13},\chi_{13},\chi_1)\,R_1^d\,\chi_1^2\ +\ 1/2\,H^F(5,\chi_1,\chi_1,\chi_1,\chi_1)\,(R_1^d)^2\chi_1^2\\
&-\ 4\,H_1^F(3,\chi_{13},\chi_1,\chi_{13},\chi_1)\,R_1^d\,\chi_1^2\ +\ 3/4\,H_{21}^F(1,\chi_1,\chi_{13},\chi_{13},\chi_1)\,\chi_1^2\ +\ 9/4\,H_{21}^F(1,\chi_3,\chi_{13},\chi_{13},\chi_1)\,\chi_1^2\\
&-\ 3/4\,H_{21}^F(2,\chi_1,\chi_{13},\chi_{13},\chi_1)\,R_1^d\,\chi_1^2, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\delta^{(6)12}_{\mathrm{loops}}\ =\ &\pi_{16}\,L_0^r\,\left[8\,\chi_\pi\chi_1^2+6\,\chi_\pi^2\chi_1-4\,\chi_1\chi_3\chi_4-4\,\chi_1^3\right]\ +\ 4\,\pi_{16}\,L_1^r\,\chi_1^3\ +\ \pi_{16}\,L_2^r\,\left[6\,\chi_\pi^2\chi_1+2\,\chi_1^3\right]\\
&+\ \pi_{16}\,L_3^r\,\left[6\,\chi_\pi\chi_1^2+4\,\chi_\pi^2\chi_1-3\,\chi_1\chi_3\chi_4-5\,\chi_1^3\right]\ +\ \pi_{16}^2\,\left[35/144\,\chi_\pi\chi_1^2+2/3\,\chi_\pi^2\chi_1-17/96\,\chi_1\chi_3\chi_4\right.\\
&\left.-\ 17/288\,\chi_1^3\right]\ +\ 256\,L_4^rL_5^r\,\chi_1^2\chi_{34}\ -\ 512\,L_4^rL_6^r\,\chi_1\chi_{34}^2\ -\ 256\,L_4^rL_8^r\,\chi_1^2\chi_{34}\ +\ 128\,L_4^rL_{11}^r\,\chi_1^2\chi_{34}\\
&+\ 256\,L_4^{r2}\,\chi_1\chi_{34}^2\ -\ 256\,L_5^rL_6^r\,\chi_1^2\chi_{34}\ -\ 128\,L_5^rL_8^r\,\chi_1^3\ +\ 64\,L_5^rL_{11}^r\,\chi_1^3\ +\ 64\,L_5^{r2}\,\chi_1^3\ -\ 256\,L_6^rL_{11}^r\,\chi_1^2\chi_{34}\\
&-\ 128\,L_8^rL_{11}^r\,\chi_1^3\ -\ 12\,\bar{A}(\chi_\pi)\,L_0^r\,R_{11}^\pi\,\chi_\pi\chi_1\ +\ 16\,\bar{A}(\chi_\pi)\,L_1^r\,\chi_\pi\chi_1\ +\ 4\,\bar{A}(\chi_\pi)\,L_2^r\,\chi_\pi\chi_1\ -\ 12\,\bar{A}(\chi_\pi)\,L_3^r\,R_{11}^\pi\,\chi_\pi\chi_1\\
&-\ 16\,\bar{A}(\chi_\pi)\,L_4^r\,R_1^c\,\chi_\pi\chi_1\ +\ 8\,\bar{A}(\chi_\pi)\,L_5^r\,\left[R_{11}^\pi\,\chi_\pi\chi_1+R_{11}^\pi\,\chi_1^2\right]\ +\ 16\,\bar{A}(\chi_\pi)\,L_6^r\,R_1^c\,\chi_\pi\chi_1\\
&+\ 32\,\bar{A}(\chi_\pi)\,L_7^r\,R_{\pi 341}^z\,\chi_1\ -\ 32\,\bar{A}(\chi_\pi)\,L_8^r\,R_{11}^\pi\,\chi_1^2\ +\ 1/8\,\bar{A}(\chi_\pi)^2\,(R_{11}^\pi)^2\chi_1\ +\ 1/4\,\bar{A}(\chi_\pi)\bar{A}(\chi_1)\,R_{11}^\pi R_1^c\,\chi_1\\
&-\ 1/3\,\bar{A}(\chi_\pi)\bar{A}(\chi_{34})\,R_{11}^\pi\,\chi_1\ -\ 2/3\,\bar{A}(\chi_\pi)\bar{A}(\chi_{1s})\,R_{1s}^\pi R_{1s\pi}^z\,\chi_1\ -\ 1/4\,\bar{A}(\chi_\pi)\bar{B}(\chi_\pi,\chi_\pi,0)\,R_{11}^\pi\,\chi_1\chi_{34}\\
&+\ 1/2\,\bar{A}(\chi_\pi)\bar{B}(\chi_1,\chi_\pi,0)\,R_{11}^\pi R_{\pi\pi}^1\,\chi_1^2\ +\ \bar{A}(\chi_\pi)\bar{B}(\chi_1,\chi_1,0)\,\left[3/4\,R_1^c\,\chi_1^2-1/2\,(R_1^c)^2\chi_1^2-1/4\,R_1^cR_1^d\,\chi_1\right]\\
&+\ 1/2\,\bar{A}(\chi_\pi)\bar{C}(\chi_1,\chi_1,\chi_1,0)\,R_{11}^\pi R_1^d\,\chi_1^2\ -\ 1/4\,\bar{A}(\chi_\pi,\varepsilon)\,\pi_{16}\,R_1^c\,\chi_\pi\chi_1\ -\ 12\,\bar{A}(\chi_1)\,L_0^r\,\left[R_1^c\,\chi_1^2+R_1^d\,\chi_1\right]\\
&+\ 8\,\bar{A}(\chi_1)\,L_1^r\,\chi_1^2\ +\ 20\,\bar{A}(\chi_1)\,L_2^r\,\chi_1^2\ -\ 12\,\bar{A}(\chi_1)\,L_3^r\,\left[R_1^c\,\chi_1^2+R_1^d\,\chi_1\right]\ +\ 16\,\bar{A}(\chi_1)\,L_4^r\,R_1^c\,\chi_1\chi_{34}\\
&+\ \bar{A}(\chi_1)\,L_5^r\,\left[16\,R_1^c\,\chi_1^2+8\,R_1^d\,\chi_1\right]\ +\ \bar{A}(\chi_1)\,L_6^r\,\left[32\,\chi_1^2-16\,R_1^c\,\chi_1\chi_{34}\right]\ +\ 32\,\bar{A}(\chi_1)\,L_7^r\,R_1^d\,\chi_1\\
&-\ 32\,\bar{A}(\chi_1)\,L_8^r\,R_1^c\,\chi_1^2\ +\ \bar{A}(\chi_1)^2\,\left[1/2\,\chi_1+1/8\,(R_1^c)^2\chi_1\right]\ +\ 1/3\,\bar{A}(\chi_1)\bar{A}(\chi_{34})\,R_{11}^\pi\,\chi_1\\
&+\ 2/3\,\bar{A}(\chi_1)\bar{A}(\chi_{1s})\,R_{1s}^\pi R_{1s\pi}^z\,\chi_1\ +\ 1/2\,\bar{A}(\chi_1)\bar{B}(\chi_1,\chi_\pi,0)\,R_{11}^\pi R_1^c\,\chi_1^2\ +\ \bar{A}(\chi_1)\bar{B}(\chi_1,\chi_1,0)\,\left[\chi_1^2+1/2\,(R_1^c)^2\chi_1^2\right.\\
&\left.+\ 1/4\,R_1^cR_1^d\,\chi_1\right]\ +\ 1/2\,\bar{A}(\chi_1)\bar{C}(\chi_1,\chi_1,\chi_1,0)\,R_1^cR_1^d\,\chi_1^2\ -\ \bar{A}(\chi_1,\varepsilon)\,\pi_{16}\,\left[1/2\,\chi_1\chi_{34}+3/2\,\chi_1^2-1/4\,R_1^c\,\chi_1\chi_{34}\right]\\
&-\ 1/2\,\bar{A}(\chi_{13})\bar{A}(\chi_{14})\,\chi_1\ +\ 32\,\bar{A}(\chi_{34})\,L_1^r\,\chi_1\chi_{34}\ +\ 8\,\bar{A}(\chi_{34})\,L_2^r\,\chi_1\chi_{34}\ -\ 32\,\bar{A}(\chi_{34})\,L_4^r\,\chi_1\chi_{34}\\
&+\ 32\,\bar{A}(\chi_{34})\,L_6^r\,\chi_1\chi_{34}\ +\ 1/2\,\bar{A}(\chi_{34})\bar{B}(\chi_\pi,\chi_\pi,0)\,R_{11}^\pi\,\chi_\pi\chi_1\ -\ \bar{A}(\chi_{34})\bar{B}(\chi_1,\chi_\pi,0)\,\left[2/3\,R_{11}^\pi\,\chi_\pi\chi_1\right.\\
&\left.+\ 1/3\,R_{11}^\pi\,\chi_1^2\right]\ +\ 1/2\,\bar{A}(\chi_{34})\bar{B}(\chi_1,\chi_1,0)\,R_{\pi\pi}^1\,\chi_1\chi_{34}\ -\ 1/2\,\bar{A}(\chi_{34},\varepsilon)\,\pi_{16}\,\chi_1\chi_{34}\ +\ \bar{A}(\chi_{1s})\,\pi_{16}\,\left[1/2\,\chi_1\chi_{34}\right.\\
&\left.+\ 1/6\,\chi_1\chi_s\right]\ +\ 8\,\bar{A}(\chi_{1s})\,L_0^r\,\chi_1\chi_{1s}\ +\ 20\,\bar{A}(\chi_{1s})\,L_3^r\,\chi_1\chi_{1s}\ -\ 16\,\bar{A}(\chi_{1s})\,L_5^r\,\chi_1\chi_{1s}\ +\ 32\,\bar{A}(\chi_{1s})\,L_8^r\,\chi_1\chi_{1s}\\
&-\ 1/4\,\bar{A}(\chi_{1s})^2\,\chi_1\ -\ \bar{A}(\chi_{1s})\bar{B}(\chi_1,\chi_\pi,0)\,\left[2/3\,R_{1s}^\pi\,\chi_1\chi_{1s}+1/3\,R_{1s}^\pi\,\chi_1^2\right]\ -\ \bar{A}(\chi_{1s})\bar{B}(\chi_1,\chi_1,0)\,R_{s\pi}^1\,\chi_1\chi_s\\
&-\ 1/2\,\bar{A}(\chi_{1s},\varepsilon)\,\pi_{16}\,\left[\chi_1\chi_{34}+\chi_1\chi_s\right]\ +\ 8\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L_4^r\,R_{11}^\pi\,\chi_\pi^2\chi_1\ +\ 4\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L_5^r\,R_{11}^\pi\,\chi_\pi^2\chi_1\\
&-\ 16\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L_6^r\,R_{11}^\pi\,\chi_\pi^2\chi_1\ -\ 16\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L_7^r\,R_{11}^\pi(R_{\pi 4}^z)^2\chi_1\ -\ 4\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L_8^r\,\left[R_{11}^\pi\,\chi_1\chi_3^2\right.\\
&\left.+\ R_{11}^\pi\,\chi_1\chi_4^2\right]\ +\ 32\,\bar{B}(\chi_1,\chi_\pi,0)\,L_7^r\,R_{\pi 341}^zR_1^d\,\chi_1\ +\ 16\,\bar{B}(\chi_1,\chi_\pi,0)\,L_8^r\,R_{\pi 341}^zR_1^d\,\chi_1\\
&+\ 1/2\,\bar{B}(\chi_1,\chi_\pi,0)\bar{B}(\chi_1,\chi_1,0)\,R_{11}^\pi R_1^d\,\chi_1^2\ -\ 12\,\bar{B}(\chi_1,\chi_1,0)\,L_0^r\,R_1^d\,\chi_1^2\ -\ 12\,\bar{B}(\chi_1,\chi_1,0)\,L_3^r\,R_1^d\,\chi_1^2\\
&+\ \bar{B}(\chi_1,\chi_1,0)\,L_4^r\,\left[8\,R_1^c\,\chi_1^2\chi_{34}+24\,R_1^d\,\chi_1\chi_{34}\right]\ +\ \bar{B}(\chi_1,\chi_1,0)\,L_5^r\,\left[4\,R_1^c\,\chi_1^3+24\,R_1^d\,\chi_1^2\right]\\
&-\ 16\,\bar{B}(\chi_1,\chi_1,0)\,L_6^r\,\left[R_1^c\,\chi_1^2\chi_{34}+2\,R_1^d\,\chi_1\chi_{34}\right]\ +\ 16\,\bar{B}(\chi_1,\chi_1,0)\,L_7^r\,(R_1^d)^2\chi_1\\
&+\ \bar{B}(\chi_1,\chi_1,0)\,L_8^r\,\left[-8\,R_1^c\,\chi_1^3-48\,R_1^d\,\chi_1^2+8\,(R_1^d)^2\chi_1\right]\ +\ \bar{B}(\chi_1,\chi_1,0)^2\,\left[1/2\,R_1^cR_1^d\,\chi_1^2+1/8\,(R_1^d)^2\chi_1\right]\\
&+\ 1/2\,\bar{B}(\chi_1,\chi_1,0)\bar{C}(\chi_1,\chi_1,\chi_1,0)\,(R_1^d)^2\chi_1^2\ +\ 1/4\,\bar{B}(\chi_1,\chi_1,0,\varepsilon)\,\pi_{16}\,R_1^d\,\chi_1^2\ +\ 16\,\bar{C}(\chi_1,\chi_1,\chi_1,0)\,L_4^r\,R_1^d\,\chi_1^2\chi_{34}\\
&+\ 8\,\bar{C}(\chi_1,\chi_1,\chi_1,0)\,L_5^r\,R_1^d\,\chi_1^3\ -\ 32\,\bar{C}(\chi_1,\chi_1,\chi_1,0)\,L_6^r\,R_1^d\,\chi_1^2\chi_{34}\ -\ 16\,\bar{C}(\chi_1,\chi_1,\chi_1,0)\,L_8^r\,R_1^d\,\chi_1^3\\
&+\ 1/2\,H^F(1,\chi_\pi,\chi_\pi,\chi_1,\chi_1)\,(R_{11}^\pi)^2\chi_1^2\ +\ H^F(1,\chi_\pi,\chi_1,\chi_1,\chi_1)\,R_{11}^\pi R_1^c\,\chi_1^2\\
&-\ H^F(1,\chi_\pi,\chi_{1s},\chi_{1s},\chi_1)\,\left[1/2\,R_{11}^\pi\,\chi_1^2+1/8\,R_{\pi 1s}^v\,\chi_\pi\chi_1\right]\ +\ H^F(1,\chi_1,\chi_1,\chi_1,\chi_1)\,\left[1/3\,\chi_1^2+1/2\,(R_1^c)^2\chi_1^2\right]\\
&-\ H^F(1,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,\left[3/4\,R_{s\pi}^1\,\chi_1^2-3/8\,R_1^c\,\chi_1^2+1/8\,R_1^d\,\chi_1\right]\ +\ 1/2\,H^F(1,\chi_{13},\chi_{14},\chi_{34},\chi_1)\,\chi_1\chi_{34}\\
&+\ H^F(2,\chi_1,\chi_\pi,\chi_1,\chi_1)\,R_{11}^\pi R_1^d\,\chi_1^2\ +\ H^F(2,\chi_1,\chi_1,\chi_1,\chi_1)\,R_1^cR_1^d\,\chi_1^2\ +\ 3/8\,H^F(2,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,R_1^d\,\chi_1^2\\
&+\ 1/2\,H^F(5,\chi_1,\chi_1,\chi_1,\chi_1)\,(R_1^d)^2\chi_1^2\ -\ H_1^F(1,\chi_\pi,\chi_{1s},\chi_{1s},\chi_1)\,R_{1s}^\pi R_{1s\pi}^z\,\chi_1^2\\
&-\ 2\,H_1^F(1,\chi_{1s},\chi_{1s},\chi_1,\chi_1)\,R_{1s}^\pi R_{1s\pi}^z\,\chi_1^2\ -\ 2\,H_1^F(3,\chi_{1s},\chi_1,\chi_{1s},\chi_1)\,R_1^d\,\chi_1^2\\
&-\ 3/8\,H_{21}^F(1,\chi_\pi,\chi_{1s},\chi_{1s},\chi_1)\,R_{\pi 1s}^v\,\chi_1^2\ +\ H_{21}^F(1,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,\left[3/4\,R_{s\pi}^1\,\chi_1^2-3/8\,R_1^c\,\chi_1^2\right]\\
&+\ 3/2\,H_{21}^F(1,\chi_{34},\chi_{13},\chi_{14},\chi_1)\,\chi_1^2\ -\ 3/8\,H_{21}^F(2,\chi_1,\chi_{1s},\chi_{1s},\chi_1)\,R_1^d\,\chi_1^2. \qquad (29)
\end{aligned}
$$

B. Results for $d_{\rm val} = 2$

The most general expression for the pseudoscalar meson mass at $n_{\rm sea} = 2$ is the result which has $d_{\rm val} = 2$ and $d_{\rm sea} = 2$. In the notation of Eq. (25), the lowest order mass M_0^2 is equal to χ_{12}. The contribution at NLO for $d_{\rm val} = 2$ and $d_{\rm sea} = 2$ is

$$
\begin{aligned}
\delta^{(4)22} = & - 16\, L_4^r\, \chi_{12}\chi_{34} - 8\, L_5^r\, \chi_{12}^2 + 32\, L_6^r\, \chi_{12}\chi_{34} \\
& + 16\, L_8^r\, \chi_{12}^2 - 1/2\, \bar{A}(\chi_p)\, R^p_{q\pi}\, \chi_{12} \\
& - 1/2\, \bar{A}(\chi_\pi)\, R^\pi_{12}\, \chi_{12},
\end{aligned} \tag{30}
$$

from which the result for $d_{\rm val} = 2$ and $d_{\rm sea} = 1$ is easily obtained by taking the limit $\chi_4 \to \chi_3$ such that $R^p_{q\pi} \to R^p_q$ and dropping the $\bar{A}(\chi_\pi)$ term. The NNLO contribution from the $\mathcal{O}(p^6)$ Lagrangian is given by

$$
\begin{aligned}
\delta^{(6)22}_{\rm ct} = & - 32\, K_{17}^r\, \chi_{12}^3 - 64\, K_{18}^r\, \chi_{12}^2\chi_{34} - 8\, K_{19}^r\, \chi_p^2\chi_{12} \\
& - 32\, K_{20}^r\, \chi_{12}^2\chi_{34} - 16\, K_{21}^r\, \chi_{12}\chi_s^2 \\
& - 64\, K_{22}^r\, \chi_{12}\chi_{34}^2 - 16\, K_{23}^r\, \chi_1\chi_{12}\chi_2 \\
& + 24\, K_{25}^r\, \chi_p^2\chi_{12} + K_{26}^r\, \big[16\, \chi_{12}\chi_s^2 \\
& + 64\, \chi_{12}^2\chi_{34}\big] + 192\, K_{27}^r\, \chi_{12}\chi_{34}^2 + 32\, K_{39}^r\, \chi_{12}^3 \\
& + 64\, K_{40}^r\, \chi_{12}^2\chi_{34}.
\end{aligned} \tag{31}
$$

The result for $d_{\rm val} = 2$ and $d_{\rm sea} = 1$ may readily be inferred from Eq. (31) by taking the limit $\chi_4 \to \chi_3$. The loop contributions at NNLO, separately for $d_{\rm sea} = 1$ and $d_{\rm sea} = 2$, are

$$
\begin{aligned}
\delta^{(6)21}_{\rm loops} = & \pi_{16}\, L_0^r\, \big[4\,\chi_1\chi_{12}\chi_2 + 2\,\chi_{12}\chi_3^2 + 8\,\chi_{12}^2\chi_3 - 8\,\chi_{12}^3\big] + 4\,\pi_{16}\, L_1^r\, \chi_{12}^3 + \pi_{16}\, L_2^r\, \big[6\,\chi_{12}\chi_3^2 + 2\,\chi_{12}^3\big] \\
& + \pi_{16}\, L_3^r\, \big[3\,\chi_1\chi_{12}\chi_2 + \chi_{12}\chi_3^2 + 6\,\chi_{12}^2\chi_3 - 8\,\chi_{12}^3\big] + \pi_{16}^2\, \big[17/96\,\chi_1\chi_{12}\chi_2 + 47/96\,\chi_{12}\chi_3^2 + 35/144\,\chi_{12}^2\chi_3 \\
& - 17/72\,\chi_{12}^3\big] + 256\, L_4^r L_5^r\, \chi_{12}^2\chi_3 - 512\, L_4^r L_6^r\, \chi_{12}\chi_3^2 - 256\, L_4^r L_8^r\, \chi_{12}^2\chi_3 + 128\, L_4^r L_{11}^r\, \chi_{12}^2\chi_3 \\
& + 256\, L_4^{r2}\, \chi_{12}\chi_3^2 - 256\, L_5^r L_6^r\, \chi_{12}^2\chi_3 - 128\, L_5^r L_8^r\, \chi_{12}^3 + 64\, L_5^r L_{11}^r\, \chi_{12}^3 + 64\, L_5^{r2}\, \chi_{12}^3 - 256\, L_6^r L_{11}^r\, \chi_{12}^2\chi_3 \\
& - 128\, L_8^r L_{11}^r\, \chi_{12}^3 + \bar{A}(\chi_p)\,\pi_{16}\, \big[-1/24\,\chi_q\chi_{12} - 1/8\,\chi_{12}\chi_3 + 1/4\,R^p_q\,\chi_{12}\chi_3 + 1/12\,R^p_q\,\chi_{12}^2\big] \\
& - \bar{A}(\chi_p)\, L_0^r\, \big[2\,\chi_p\chi_{12} + 8\,R^p_q\,\chi_p\chi_{12} + 2\,R^d_p\,\chi_{12}\big] - \bar{A}(\chi_p)\, L_3^r\, \big[5\,\chi_p\chi_{12} + 2\,R^p_q\,\chi_p\chi_{12} + 5\,R^d_p\,\chi_{12}\big] \\
& + 16\,\bar{A}(\chi_p)\, L_4^r\, R^p_q\,\chi_{12}\chi_3 + \bar{A}(\chi_p)\, L_5^r\, \big[4\,\chi_p\chi_{12} + 8\,R^p_q\,\chi_p\chi_{12}\big] - 16\,\bar{A}(\chi_p)\, L_6^r\, R^p_q\,\chi_{12}\chi_3 \\
& + \bar{A}(\chi_p)\, L_7^r\, \big[16/3\,R^d_p\,\chi_{p3} + 32/3\,R^d_p\,\chi_{12}\big] - \bar{A}(\chi_p)\, L_8^r\, \big[8\,\chi_p\chi_{12} + 16/3\,R^p_q\,\chi_1\chi_2 + 32/3\,R^p_q\,\chi_{12}^2 + 4/3\,(R^d_p)^2\big] \\
& + \bar{A}(\chi_p)^2\, \big[5/32\,\chi_{12} + 1/8\,(R^p_q)^2\chi_{12} - 1/24\,R^q_p\,\chi_p\big] + \bar{A}(\chi_p)\bar{A}(\chi_{p3})\, \big[1/24\,\chi_p - 1/8\,\chi_q + 1/24\,R^p_q\,\chi_p \\
& + 19/72\,R^p_q\,\chi_q\big] + \bar{A}(\chi_p)\bar{A}(\chi_{q3})\, \big[1/4\,\chi_{12} + 1/6\,R^p_q\,\chi_q - 1/24\,R^q_p\,\chi_p + 1/72\,R^q_p\,\chi_q\big] \\
& + \bar{A}(\chi_p)\bar{A}(\chi_{12})\, \big[1/18\,\chi_p - 1/36\,\chi_{12}\big] - 1/8\,\bar{A}(\chi_p)\bar{A}(\chi_3)\,\chi_3 + \bar{A}(\chi_p)\bar{B}(\chi_p,\chi_p,0)\, \big[11/24\,R^p_q\,\chi_p\chi_{12} \\
& - 1/12\,R^p_q\,\chi_p^2 - 1/8\,R^p_q\,\chi_{12}^2 + 1/24\,R^p_q R^d_p\,\chi_p\big] - \bar{A}(\chi_p)\bar{B}(\chi_q,\chi_q,0)\, \big[1/24\,R^p_q R^d_q\,\chi_q + 1/16\,R^d_q\,\chi_{12}\big] \\
& + 1/6\,\bar{A}(\chi_p)\bar{B}(\chi_1,\chi_2,0)\, R^q_p\,\chi_p\chi_{12} + 1/12\,\bar{A}(\chi_p)\bar{B}(\chi_1,\chi_2,0,k)\, R^q_p\,\chi_p + 1/4\,\bar{A}(\chi_p,\varepsilon)\,\pi_{16}\,\big[\chi_p\chi_{12} \\
& - R^p_q\,\chi_{12}\chi_3\big] - \bar{A}(\chi_{p3})\,\pi_{16}\,\big[1/3\,\chi_p\chi_{12} - 2/3\,\chi_{12}\chi_3 - 1/3\,\chi_{12}^2\big] + 8\,\bar{A}(\chi_{p3})\, L_0^r\,\chi_{p3}\chi_{12} \\
& + 20\,\bar{A}(\chi_{p3})\, L_3^r\,\chi_{p3}\chi_{12} - 16\,\bar{A}(\chi_{p3})\, L_5^r\,\chi_{p3}\chi_{12} + 32\,\bar{A}(\chi_{p3})\, L_8^r\,\chi_{p3}\chi_{12} \\
& + 1/4\,\bar{A}(\chi_{p3})\bar{B}(\chi_q,\chi_q,0)\, R^d_q\,\chi_{12} - \bar{A}(\chi_{p3})\bar{B}(\chi_1,\chi_2,0)\, \big[4/9\,\chi_{p3}\chi_{12} + 1/9\,\chi_1\chi_2 + 1/9\,R^q_p\,\chi_p^2\big] \\
& + \bar{A}(\chi_{p3})\bar{B}(\chi_1,\chi_2,0,k)\, \big[-2/9\,\chi_{p3} - 2/9\,\chi_{12} + 1/9\,R^q_p\,\chi_q\big] - \bar{A}(\chi_{p3},\varepsilon)\,\pi_{16}\,\chi_{12}\chi_3 \\
& + \bar{A}(\chi_1)\bar{A}(\chi_2)\, \big[-1/16\,\chi_{12} - 1/24\,\chi_3 + 1/4\,R^1_2 R^2_1\,\chi_{12}\big] + 8\,\bar{A}(\chi_{12})\, L_1^r\,\chi_{12}^2 + 20\,\bar{A}(\chi_{12})\, L_2^r\,\chi_{12}^2 \\
& + 32\,\bar{A}(\chi_{12})\, L_6^r\,\chi_{12}^2 - 1/4\,\bar{A}(\chi_{12})^2\,\chi_{12} + \bar{A}(\chi_{12})\bar{B}(\chi_1,\chi_2,0)\, \big[1/9\,\chi_1\chi_2 + 4/9\,\chi_{12}^2\big] \\
& + 4/9\,\bar{A}(\chi_{12})\bar{B}(\chi_1,\chi_2,0,k)\,\chi_{12} - 2\,\bar{A}(\chi_{12},\varepsilon)\,\pi_{16}\,\chi_{12}^2 - \bar{A}(\chi_{13})\bar{A}(\chi_{23})\,\chi_{12} + 48\,\bar{A}(\chi_3)\, L_1^r\,\chi_{12}\chi_3 \\
& + 12\,\bar{A}(\chi_3)\, L_2^r\,\chi_{12}\chi_3 - 48\,\bar{A}(\chi_3)\, L_4^r\,\chi_{12}\chi_3 + 48\,\bar{A}(\chi_3)\, L_6^r\,\chi_{12}\chi_3 + 1/2\,\bar{A}(\chi_3)\bar{B}(\chi_1,\chi_2,0)\,\chi_{12}\chi_3 \\
& + 1/4\,\bar{A}(\chi_3)\bar{B}(\chi_1,\chi_2,0,k)\,\chi_3 - 3/4\,\bar{A}(\chi_3,\varepsilon)\,\pi_{16}\,\chi_{12}\chi_3 - \bar{B}(\chi_p,\chi_p,0)\,\pi_{16}\,\big[1/24\,R^d_p\,\chi_q\chi_{12} + 1/8\,R^d_p\,\chi_{12}\chi_3\big] \\
& - 2\,\bar{B}(\chi_p,\chi_p,0)\, L_0^r\, R^d_p\,\chi_p\chi_{12} - 5\,\bar{B}(\chi_p,\chi_p,0)\, L_3^r\, R^d_p\,\chi_p\chi_{12} + 8\,\bar{B}(\chi_p,\chi_p,0)\, L_4^r\, R^p_q\,\chi_p\chi_{12}\chi_3 \\
& + \bar{B}(\chi_p,\chi_p,0)\, L_5^r\, \big[4\,R^p_q\,\chi_p^3 + 2\,R^d_p\,\chi_1\chi_2\big] - 16\,\bar{B}(\chi_p,\chi_p,0)\, L_6^r\, R^p_q\,\chi_p\chi_{12}\chi_3 - \bar{B}(\chi_p,\chi_p,0)\, L_8^r\, \big[8\,R^p_q\,\chi_p^3 \\
& + 4\,R^d_p\,\chi_1\chi_2\big] + \bar{B}(\chi_p,\chi_p,0)^2\, \big[9/32\,R^p_q R^d_p\,\chi_p\chi_{12} - 1/32\,R^p_q R^d_p\,\chi_q\chi_{12}\big] \\
& + 1/6\,\bar{B}(\chi_p,\chi_p,0)\bar{B}(\chi_1,\chi_2,0)\, R^q_p R^d_p\,\chi_p\chi_{12} + 1/12\,\bar{B}(\chi_p,\chi_p,0)\bar{B}(\chi_1,\chi_2,0,k)\, R^q_p R^d_p\,\chi_p \\
& + 1/4\,\bar{B}(\chi_p,\chi_p,0,\varepsilon)\,\pi_{16}\, R^d_p\,\chi_{p3}\chi_{12} - 1/16\,\bar{B}(\chi_1,\chi_1,0)\bar{B}(\chi_2,\chi_2,0)\, R^d_1 R^d_2\,\chi_{12} \\
& + 32/3\,\bar{B}(\chi_1,\chi_2,0)\, L_7^r\, R^d_1 R^d_2\,\chi_{12} + 16/3\,\bar{B}(\chi_1,\chi_2,0)\, L_8^r\, R^d_1 R^d_2\,\chi_{12} + 16/3\,\bar{B}(\chi_1,\chi_2,0,k)\, L_7^r\, R^d_1 R^d_2
\end{aligned}
$$

$$
\begin{aligned}
&+\ 8/3\,\bar{B}(\chi_1,\chi_2,0,k)\,L_8^r\,R_1^dR_2^d\ +\ H^F(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\ \big[-1/4\,\chi_p\chi_{12}-15/32\,\chi_{12}^2+3/8\,R_q^p\,\chi_{12}^2\\
&-\ 1/8\,(R_q^p)^2\chi_{12}^2\big]\ +\ H^F(1,\chi_p,\chi_{13},\chi_{23},\chi_{12})\ \big[1/4\,\chi_{12}^2+3/4\,R_p^q\,\chi_p\chi_{12}+1/4\,R_p^q\,\chi_q\chi_{12}-1/4\,R_p^q\,\chi_{12}\chi_3\big]\\
&-\ H^F(1,\chi_1,\chi_{12},\chi_2,\chi_{12})\ \big[1/16\,\chi_{12}^2-3/4\,R_2^1R_1^2\,\chi_{12}^2\big]\ +\ 1/4\,H^F(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\,\chi_{12}^2\\
&+\ 3/4\,H^F(1,\chi_{13},\chi_{23},\chi_3,\chi_{12})\,\chi_{12}\chi_3\ -\ H^F(2,\chi_p,\chi_p,\chi_{12},\chi_{12})\ \big[1/8\,R_q^pR_p^d\,\chi_{12}^2-5/16\,R_p^d\,\chi_{12}^2\big]\\
&-\ H^F(2,\chi_p,\chi_{12},\chi_q,\chi_{12})\ \big[1/8\,R_p^qR_p^d\,\chi_{12}^2-1/16\,R_p^d\,\chi_{12}^2\big]\ -\ 1/4\,H^F(2,\chi_p,\chi_{13},\chi_{23},\chi_{12})\,R_p^d\,\chi_{p3}\chi_{12}\\
&+\ 5/32\,H^F(5,\chi_p,\chi_p,\chi_{12},\chi_{12})\,(R_p^d)^2\chi_{12}^2\ +\ 1/16\,H^F(5,\chi_1,\chi_2,\chi_{12},\chi_{12})\,R_1^dR_2^d\,\chi_{12}^2\\
&+\ H_1^F(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\ \big[5/2\,\chi_{12}^2-R_q^pR_p^q\,\chi_{12}^2\big]\ -\ 2\,H_1^F(1,\chi_p,\chi_{13},\chi_{23},\chi_{12})\,R_p^q\,\chi_{12}^2\\
&-\ 2\,H_1^F(1,\chi_{p3},\chi_{q3},\chi_p,\chi_{12})\,\chi_{12}^2\ +\ H_1^F(1,\chi_{12},\chi_1,\chi_2,\chi_{12})\ \big[1/2\,\chi_{12}^2-R_2^1R_1^2\,\chi_{12}^2\big]\\
&-\ 1/2\,H_1^F(3,\chi_{12},\chi_p,\chi_p,\chi_{12})\,R_p^qR_p^d\,\chi_{12}^2\ +\ 1/2\,H_1^F(3,\chi_{12},\chi_p,\chi_q,\chi_{12})\,R_p^qR_p^d\,\chi_{12}^2\\
&-\ 1/4\,H_1^F(7,\chi_{12},\chi_p,\chi_p,\chi_{12})\,(R_p^d)^2\chi_{12}^2\ -\ 3/4\,H_{21}^F(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\,\chi_{12}^2\\
&+\ 3/4\,H_{21}^F(1,\chi_p,\chi_{13},\chi_{23},\chi_{12})\,R_p^q\,\chi_{12}^2\ +\ 3/4\,H_{21}^F(1,\chi_{p3},\chi_{q3},\chi_p,\chi_{12})\,R_q^p\,\chi_{12}^2\\
&-\ 3/4\,H_{21}^F(1,\chi_{p3},\chi_{q3},\chi_q,\chi_{12})\,R_q^p\,\chi_{12}^2\ +\ H_{21}^F(1,\chi_{12},\chi_p,\chi_p,\chi_{12})\ \big[15/32\,\chi_{12}^2-3/8\,R_q^pR_p^q\,\chi_{12}^2\big]\\
&-\ H_{21}^F(1,\chi_{12},\chi_1,\chi_2,\chi_{12})\ \big[3/16\,\chi_{12}^2-3/4\,R_2^1R_1^2\,\chi_{12}^2\big]\ +\ 3/4\,H_{21}^F(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\,\chi_{12}^2\\
&+\ 9/4\,H_{21}^F(1,\chi_3,\chi_{13},\chi_{23},\chi_{12})\,\chi_{12}^2\ -\ 3/4\,H_{21}^F(3,\chi_{p3},\chi_p,\chi_{q3},\chi_{12})\,R_p^d\,\chi_{12}^2\\
&-\ H_{21}^F(3,\chi_{12},\chi_p,\chi_p,\chi_{12})\ \big[3/8\,R_q^pR_p^d\,\chi_{12}^2-3/16\,R_p^d\,\chi_{12}^2\big]\ -\ H_{21}^F(3,\chi_{12},\chi_p,\chi_q,\chi_{12})\ \big[3/8\,R_p^qR_p^d\,\chi_{12}^2\\
&-\ 3/16\,R_p^d\,\chi_{12}^2\big]\ +\ 3/32\,H_{21}^F(7,\chi_{12},\chi_p,\chi_p,\chi_{12})\,(R_p^d)^2\chi_{12}^2\ +\ 3/16\,H_{21}^F(7,\chi_{12},\chi_1,\chi_2,\chi_{12})\,R_1^dR_2^d\,\chi_{12}^2, \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
\delta^{(6)22}_{\rm loops}\ =\ &\pi_{16}\,L_0^r\ \big[8\,\chi_\pi\chi_{12}^2+6\,\chi_\pi^2\chi_{12}+4\,\chi_1\chi_{12}\chi_2-4\,\chi_{12}\chi_3\chi_4-8\,\chi_{12}^3\big]\ +\ 4\,\pi_{16}\,L_1^r\,\chi_{12}^3\ +\ \pi_{16}\,L_2^r\ \big[6\,\chi_\pi^2\chi_{12}+2\,\chi_{12}^3\big]\\
&+\ \pi_{16}\,L_3^r\ \big[6\,\chi_\pi\chi_{12}^2+4\,\chi_\pi^2\chi_{12}+3\,\chi_1\chi_{12}\chi_2-3\,\chi_{12}\chi_3\chi_4-8\,\chi_{12}^3\big]\ +\ \pi_{16}^2\ \big[35/144\,\chi_\pi\chi_{12}^2+2/3\,\chi_\pi^2\chi_{12}\\
&+\ 17/96\,\chi_1\chi_{12}\chi_2-\ 17/96\,\chi_{12}\chi_3\chi_4-17/72\,\chi_{12}^3\big]\ +\ 256\,L_4^rL_5^r\,\chi_{12}^2\chi_{34}\ -\ 512\,L_4^rL_6^r\,\chi_{12}\chi_{34}^2\\
&-\ 256\,L_4^rL_8^r\,\chi_{12}^2\chi_{34}\ +\ 128\,L_4^rL_{11}^r\,\chi_{12}^2\chi_{34}\ +\ 256\,L_4^{r2}\,\chi_{12}\chi_{34}^2\ -\ 256\,L_5^rL_6^r\,\chi_{12}^2\chi_{34}\ -\ 128\,L_5^rL_8^r\,\chi_{12}^3\\
&+\ 64\,L_5^rL_{11}^r\,\chi_{12}^3\ +\ 64\,L_5^{r2}\,\chi_{12}^3\ -\ 256\,L_6^rL_{11}^r\,\chi_{12}^2\chi_{34}\ -\ 128\,L_8^rL_{11}^r\,\chi_{12}^3\ +\ \bar{A}(\chi_p)\,\pi_{16}\ \big[1/4\,R_{q\pi}^p\,\chi_{12}\chi_{34}\\
&+\ 1/12\,R_{q\pi}^p\,\chi_{12}^2-1/24\,R_p^c\,\chi_q\chi_{12}-1/8\,R_p^c\,\chi_{12}\chi_{34}\big]\ -\ \bar{A}(\chi_p)\,L_0^r\ \big[8\,R_{q\pi}^p\,\chi_p\chi_{12}+2\,R_p^c\,\chi_p\chi_{12}+2\,R_p^d\,\chi_{12}\big]\\
&-\ \bar{A}(\chi_p)\,L_3^r\ \big[2\,R_{q\pi}^p\,\chi_p\chi_{12}+5\,R_p^c\,\chi_p\chi_{12}+5\,R_p^d\,\chi_{12}\big]\ +\ 16\,\bar{A}(\chi_p)\,L_4^r\,R_{q\pi}^p\,\chi_{12}\chi_{34}\ +\ \bar{A}(\chi_p)\,L_5^r\ \big[8\,R_{q\pi}^p\,\chi_p\chi_{12}\\
&+\ 4\,R_p^c\,\chi_p\chi_{12}\big]\ -\ 16\,\bar{A}(\chi_p)\,L_6^r\,R_{q\pi}^p\,\chi_{12}\chi_{34}\ +\ \bar{A}(\chi_p)\,L_7^r\ \big[32/3\,R_p^d\,\chi_p+16/3\,R_p^d\,\chi_q-8/3\,(R_p^d)^2\big]\\
&-\ \bar{A}(\chi_p)\,L_8^r\ \big[16/3\,R_{q\pi}^p\,\chi_1\chi_2+32/3\,R_{q\pi}^p\,\chi_{12}^2+8\,R_p^c\,\chi_p\chi_{12}+4/3\,(R_p^d)^2\big]\\
&+\ \bar{A}(\chi_p)^2\ \big[1/8\,\chi_{12}+1/8\,(R_{q\pi}^p)^2\chi_{12}+1/24\,R_{q\pi}^pR_p^c\,\chi_p-1/24\,(R_p^c)^2\chi_p+1/32\,(R_p^c)^2\chi_{12}\big]\\
&+\ \bar{A}(\chi_p)\bar{A}(\chi_{ps})\ \big[1/48\,R_{q\pi}^p\,\chi_p+19/144\,R_{q\pi}^p\,\chi_q-5/144\,R_{s\pi}^p\,\chi_p-1/16\,R_{s\pi}^p\,\chi_q+1/18\,R_p^c\,\chi_p\big]\\
&+\ \bar{A}(\chi_p)\bar{A}(\chi_{qs})\ \big[1/12\,R_{q\pi}^p\,\chi_q-1/48\,R_{s\pi}^pR_{qsp}^z\,\chi_p+1/144\,R_{s\pi}^pR_{qsp}^z\,\chi_q+1/8\,R_p^c\,\chi_{12}\big]\\
&+\ \bar{A}(\chi_p)\bar{A}(\chi_\pi)\ \big[1/24\,R_{q\pi}^pR_{pp}^\pi\,\chi_p-1/40\,R_{q\pi}^pR_{qq}^\pi\,\chi_q+13/120\,R_{q\pi}^pR_{12}^\pi\,\chi_p+1/8\,R_{q\pi}^pR_{12}^\pi\,\chi_q-1/60\,R_{\pi\pi}^pR_{qq}^\pi\,\chi_\pi\\
&-\ 1/24\,R_{qq}^\pi R_p^c\,\chi_p-1/24\,R_p^c\,\chi_\pi+1/96\,R_p^cR_{\pi12}^v\,\chi_p+1/32\,R_p^cR_{\pi12}^v\,\chi_q-1/16\,R_p^cR_{\pi12}^v\,\chi_\pi-1/12\,R_{\pi34p}^z\big]\\
&+\ \bar{A}(\chi_p)\bar{A}(\chi_{12})\ [1/18\,\chi_p-1/36\,\chi_{12}]\ -\ \bar{A}(\chi_p)\bar{A}(\chi_{34})\ \big[1/12\,R_{\pi\pi}^p\,\chi_{34}-1/9\,R_{pp}^\pi\,\chi_p\big]\\
&+\ \bar{A}(\chi_p)\bar{B}(\chi_p,\chi_p,0)\ \big[11/24\,R_{q\pi}^pR_p^c\,\chi_p\chi_{12}-1/12\,R_{q\pi}^pR_p^c\,\chi_p^2-1/8\,R_{q\pi}^pR_p^c\,\chi_{12}^2+1/24\,R_{q\pi}^pR_p^d\,\chi_p\big]\\
&+\ \bar{A}(\chi_p)\bar{B}(\chi_p,\chi_\pi,0)\ \big[1/6\,R_{12}^\pi R_p^c\,\chi_p^2+1/12\,R_{12}^\pi R_p^c\,\chi_1\chi_2\big]\ -\ \bar{A}(\chi_p)\bar{B}(\chi_q,\chi_q,0)\ \big[1/24\,R_{q\pi}^pR_q^d\,\chi_q\\
&+\ 1/16\,R_p^cR_q^d\,\chi_{12}\big]\ +\ 1/6\,\bar{A}(\chi_p)\bar{B}(\chi_1,\chi_2,0)\,R_{p\pi}^qR_p^c\,\chi_p\chi_{12}\ +\ 1/12\,\bar{A}(\chi_p)\bar{B}(\chi_1,\chi_2,0,k)\,R_{p\pi}^qR_p^c\,\chi_p\\
&+\ 1/4\,\bar{A}(\chi_p,\varepsilon)\,\pi_{16}\ \big[\chi_p\chi_{12}-R_{q\pi}^p\,\chi_{12}\chi_{34}-R_{pp}^\pi\,\chi_{12}\chi_{34}\big]\ +\ \bar{A}(\chi_{ps})\,\pi_{16}\ \big[1/6\,\chi_q\chi_{12}+1/4\,\chi_{12}\chi_{34}+1/12\,\chi_{12}\chi_s\\
&-\ 1/6\,\chi_{12}^2\big]\ +\ 4\,\bar{A}(\chi_{ps})\,L_0^r\,\chi_{ps}\chi_{12}\ +\ 10\,\bar{A}(\chi_{ps})\,L_3^r\,\chi_{ps}\chi_{12}\ -\ 8\,\bar{A}(\chi_{ps})\,L_5^r\,\chi_{ps}\chi_{12}\ +\ 16\,\bar{A}(\chi_{ps})\,L_8^r\,\chi_{ps}\chi_{12}\\
&+\ \bar{A}(\chi_{ps})\bar{A}(\chi_\pi)\ \big[1/18\,R_{pp}^\pi\,\chi_p-5/144\,R_{ps}^\pi\,\chi_p-1/16\,R_{ps}^\pi\,\chi_q+1/8\,R_{qq}^\pi\,\chi_{12}-5/48\,R_{qs}^\pi\,\chi_p\\
&-\ 19/144\,R_{qs}^\pi R_{ps\pi}^z\,\chi_q+1/48\,R_{12}^\pi\chi_p\big]\ -\ \bar{A}(\chi_{ps})\bar{B}(\chi_p,\chi_\pi,0)\ \big[2/9\,R_{qs}^\pi\,\chi_p\chi_{ps}+1/9\,R_{qs}^\pi\,\chi_p\chi_{12}\\
&+\ 1/9\,R_{qs}^\pi\,\chi_q\chi_{ps}\ +\ 1/18\,R_{qs}^\pi R_{ps\pi}^z\,\chi_1\chi_2+1/18\,R_{12}^\pi\,\chi_p^2\big]\ +\ 1/8\,\bar{A}(\chi_{ps})\bar{B}(\chi_q,\chi_q,0)\,R_q^d\,\chi_{12}\\
&-\ \bar{A}(\chi_{ps})\bar{B}(\chi_1,\chi_2,0)\ \big[1/18\,R_{p\pi}^q\,\chi_p^2+2/9\,R_{s\pi}^q\,\chi_{ps}\chi_{12}+1/18\,R_{s\pi}^q\,\chi_1\chi_2\big]
\end{aligned}
$$

$$
\begin{aligned}
&+\ \bar{A}(\chi_{ps})\bar{B}(\chi_1,\chi_2,0,k)\ \left[1/18\,R^q_{p\pi}\,\chi_q - 1/9\,R^q_{s\pi}\,\chi_{ps} - 1/9\,R^q_{s\pi}\,\chi_{12}\right]\ -\ 1/4\,\bar{A}(\chi_{ps},\varepsilon)\,\pi_{16}\,[\chi_{12}\chi_{34}+\chi_{12}\chi_s]\\
&-\ 1/4\,\bar{A}(\chi_{p3})\bar{A}(\chi_{q4})\,\chi_{12}\ -\ 1/12\,\bar{A}(\chi_\pi)\,\pi_{16}\,\left[R^v_{\pi12}\,\chi_\pi\chi_{12}\ +\ R^v_{\pi12}\,\chi^2_{12}\right]\\
&-\ \bar{A}(\chi_\pi)\,L^r_0\,[12\,R^\pi_{12}\,\chi_\pi\chi_{12}+2\,R^v_{\pi12}\,\chi_\pi\chi_{12}]\ +\ 16\,\bar{A}(\chi_\pi)\,L^r_1\,\chi_\pi\chi_{12}\ +\ 4\,\bar{A}(\chi_\pi)\,L^r_2\,\chi_\pi\chi_{12}\\
&-\ \bar{A}(\chi_\pi)\,L^r_3\,[12\,R^\pi_{12}\,\chi_\pi\chi_{12}+5\,R^v_{\pi12}\,\chi_\pi\chi_{12}]\ -\ 16\,\bar{A}(\chi_\pi)\,L^r_4\,[\chi_\pi\chi_{12}-R^\pi_{12}\,\chi_\pi\chi_{12}]\\
&+\ \bar{A}(\chi_\pi)\,L^r_5\,\left[4\,R^\pi_{pp}\,\chi_\pi\chi_{12}+8\,R^\pi_{12}\,\chi^2_{12}\right]\ +\ 16\,\bar{A}(\chi_\pi)\,L^r_6\,[\chi_\pi\chi_{12}-R^\pi_{12}\,\chi_\pi\chi_{12}]\\
&+\ \bar{A}(\chi_\pi)\,L^r_7\,\left[64/3\,R^\pi_{12}\,\chi_\pi\chi_{12}-64/3\,R^\pi_{12}\,\chi^2_{12}+16/3\,R^z_{\pi34p}\,\chi_q-8/3\,(R^z_{\pi4})^2R^v_{\pi12}\right]\\
&-\ \bar{A}(\chi_\pi)\,L^r_8\,\left[32/3\,R^\pi_{12}\,\chi_1\chi_2+64/3\,R^\pi_{12}\,\chi^2_{12}+16/3\,R^v_{\pi12}\,\chi^2_\pi+4\,R^v_{\pi12}\,\chi_1\chi_2-4/3\,R^v_{\pi12}\,\chi_3\chi_4\right]\\
&+\ \bar{A}(\chi_\pi)^2\ \left[1/8\,(R^\pi_{12})^2\chi_{12}-1/24\,R^\pi_{12}R^v_{\pi12}\,\chi_\pi+1/24\,R^\pi_{12}R^v_{\pi12}\,\chi_{12}-1/24\,R^v_{\pi12}\,\chi_\pi\right.\\
&\left.-\ 1/24\,(R^v_{\pi12})^2\chi_\pi+1/32\,(R^v_{\pi12})^2\chi_{12}\right]\ -\ \bar{A}(\chi_\pi)\bar{A}(\chi_{34})\,[1/3\,R^\pi_{12}\,\chi_{12}+1/36\,R^v_{\pi12}\,\chi_\pi]\\
&+\ \bar{A}(\chi_\pi)\bar{B}(\chi_p,\chi_p,0)\ \left[1/4\,R^p_{q\pi}R^\pi_{pp}\,\chi^2_p-1/12\,R^\pi_{pp}R^d_p\,\chi_p-1/24\,R^\pi_{qq}R^d_p\,\chi_p+1/96\,R^d_pR^v_{\pi12}\,\chi_p\right.\\
&\left.+\ 1/32\,R^d_pR^v_{\pi12}\,\chi_q-1/16\,R^d_pR^v_{\pi12}\,\chi_\pi\right]\ +\ \bar{A}(\chi_\pi)\bar{B}(\chi_p,\chi_\pi,0)\ \left[1/6\,R^p_{\pi\pi}R^\pi_{12}\,\chi_p\chi_{12}+1/12\,R^p_{\pi\pi}R^\pi_{12}\,\chi^2_p\right]\\
&-\ 1/4\,\bar{A}(\chi_\pi)\bar{B}(\chi_\pi,\chi_\pi,0)\,R^\pi_{12}\,\chi_{12}\chi_{34}\ +\ 1/6\,\bar{A}(\chi_\pi)\bar{B}(\chi_1,\chi_2,0)\,R^1_{\pi\pi}R^2_{\pi\pi}\,\chi_{12}\chi_{34}\\
&+\ 1/12\,\bar{A}(\chi_\pi)\bar{B}(\chi_1,\chi_2,0,k)\,R^1_{\pi\pi}R^2_{\pi\pi}\,\chi_{34}\ +\ \bar{A}(\chi_\pi,\varepsilon)\,\pi_{16}\,[-1/4\,\chi_\pi\chi_{12}+1/4\,R^\pi_{12}\,\chi_\pi\chi_{12}+1/4\,R^v_{\pi12}\,\chi_\pi\chi_{12}]\\
&+\ \bar{A}(\chi_1)\bar{A}(\chi_2)\ \left[-1/24\,R^p_{q\pi}R^c_q\,\chi_q+1/4\,R^1_{2\pi}R^2_{1\pi}\,\chi_{12}-1/16\,R^c_1R^c_2\,\chi_{12}\right]\ +\ 8\,\bar{A}(\chi_{12})\,L^r_1\,\chi^2_{12}\\
&+\ 20\,\bar{A}(\chi_{12})\,L^r_2\,\chi^2_{12}\ +\ 32\,\bar{A}(\chi_{12})\,L^r_6\,\chi^2_{12}\ -\ 1/4\,\bar{A}(\chi_{12})^2\,\chi_{12}\ +\ \bar{A}(\chi_{12})\bar{B}(\chi_1,\chi_2,0)\ \left[1/9\,\chi_1\chi_2+4/9\,\chi^2_{12}\right]\\
&+\ 4/9\,\bar{A}(\chi_{12})\bar{B}(\chi_1,\chi_2,0,k)\,\chi_{12}\ -\ 2\,\bar{A}(\chi_{12},\varepsilon)\,\pi_{16}\,\chi^2_{12}\ +\ 32\,\bar{A}(\chi_{34})\,L^r_1\,\chi_{12}\chi_{34}\ +\ 8\,\bar{A}(\chi_{34})\,L^r_2\,\chi_{12}\chi_{34}\\
&-\ 32\,\bar{A}(\chi_{34})\,L^r_4\,\chi_{12}\chi_{34}\ +\ 32\,\bar{A}(\chi_{34})\,L^r_6\,\chi_{12}\chi_{34}\ -\ \bar{A}(\chi_{34})\bar{B}(\chi_p,\chi_\pi,0)\ [2/9\,R^\pi_{12}\,\chi_p\chi_\pi+1/9\,R^\pi_{12}\,\chi_q\chi_\pi\\
&+\ 1/6\,R^\pi_{12}\,\chi_1\chi_2]\ +\ 1/2\,\bar{A}(\chi_{34})\bar{B}(\chi_\pi,\chi_\pi,0)\,R^\pi_{12}\,\chi_\pi\chi_{12}\ +\ \bar{A}(\chi_{34})\bar{B}(\chi_1,\chi_2,0)\ [1/3\,\chi_{12}\chi_{34}+1/9\,R^\pi_{12}\,\chi_1\chi_2\\
&+\ 1/3\,R^\pi_{12}\,\chi_{12}\chi_{34}-2/9\,R^\pi_{12}\,\chi^2_{12}]\ +\ \bar{A}(\chi_{34})\bar{B}(\chi_1,\chi_2,0,k)\ [1/6\,\chi_{34}+1/9\,R^\pi_{12}\,\chi_{12}+1/6\,R^\pi_{12}\,\chi_{34}]\\
&-\ 1/2\,\bar{A}(\chi_{34},\varepsilon)\,\pi_{16}\,\chi_{12}\chi_{34}\ -\ 1/4\,\bar{A}(\chi_{1s})\bar{A}(\chi_{2s})\,\chi_{12}\ -\ \bar{B}(\chi_p,\chi_p,0)\,\pi_{16}\ \left[1/24\,R^d_p\,\chi_q\chi_{12}+1/8\,R^d_p\,\chi_{12}\chi_{34}\right]\\
&-\ 2\,\bar{B}(\chi_p,\chi_p,0)\,L^r_0\,R^d_p\,\chi_p\chi_{12}\ -\ 5\,\bar{B}(\chi_p,\chi_p,0)\,L^r_3\,R^d_p\,\chi_p\chi_{12}\ +\ 8\,\bar{B}(\chi_p,\chi_p,0)\,L^r_4\,R^p_{q\pi}\,\chi_p\chi_{12}\chi_{34}\\
&+\ \bar{B}(\chi_p,\chi_p,0)\,L^r_5\ \left[4\,R^p_{q\pi}\,\chi^3_p+2\,R^d_p\,\chi_1\chi_2\right]\ -\ 16\,\bar{B}(\chi_p,\chi_p,0)\,L^r_6\,R^p_{q\pi}\,\chi_p\chi_{12}\chi_{34}\\
&-\ \bar{B}(\chi_p,\chi_p,0)\,L^r_8\ \left[8\,R^p_{q\pi}\,\chi^3_p+4\,R^d_p\,\chi_1\chi_2\right]\ +\ \bar{B}(\chi_p,\chi_p,0)^2\ \left[9/32\,R^p_{q\pi}R^d_p\,\chi_p\chi_{12}-1/32\,R^p_{q\pi}R^d_p\,\chi_q\chi_{12}\right]\\
&+\ \bar{B}(\chi_p,\chi_p,0)\bar{B}(\chi_p,\chi_\pi,0)\ \left[1/6\,R^\pi_{12}R^d_p\,\chi^2_p+1/12\,R^\pi_{12}R^d_p\,\chi_1\chi_2\right]\ +\ 1/6\,\bar{B}(\chi_p,\chi_p,0)\bar{B}(\chi_1,\chi_2,0)\,R^q_{p\pi}R^d_p\,\chi_p\chi_{12}\\
&+\ 1/12\,\bar{B}(\chi_p,\chi_p,0)\bar{B}(\chi_1,\chi_2,0,k)\,R^q_{p\pi}R^d_p\,\chi_p\ +\ 1/8\,\bar{B}(\chi_p,\chi_p,0,\varepsilon)\,\pi_{16}\ \left[R^d_p\,\chi_p\chi_{12}+R^d_p\,\chi_{12}\chi_{34}\right]\\
&+\ \bar{B}(\chi_p,\chi_\pi,0)\,L^r_7\ \left[16/3\,R^d_pR^z_{\pi34q}\,\chi_p+32/3\,R^d_pR^z_{\pi34q}\,\chi_{12}\right]\ +\ \bar{B}(\chi_p,\chi_\pi,0)\,L^r_8\ \left[8/3\,R^d_pR^z_{\pi34q}\,\chi_p\right.\\
&\left.+\ 16/3\,R^d_pR^z_{\pi34q}\,\chi_{12}\right]\ +\ 8\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L^r_4\,R^\pi_{12}\,\chi^2_\pi\chi_{12}\ +\ 4\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L^r_5\,R^\pi_{12}\,\chi^2_\pi\chi_{12}\\
&-\ 16\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L^r_6\,R^\pi_{12}\,\chi^2_\pi\chi_{12}\ -\ 16\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L^r_7\,R^\pi_{12}\,(R^z_{\pi4})^2\chi_{12}\ -\ 4\,\bar{B}(\chi_\pi,\chi_\pi,0)\,L^r_8\ \left[R^\pi_{12}\,\chi_{12}\chi^2_3\right.\\
&\left.+\ R^\pi_{12}\,\chi_{12}\chi^2_4\right]\ -\ 1/16\,\bar{B}(\chi_1,\chi_1,0)\bar{B}(\chi_2,\chi_2,0)\,R^d_1R^d_2\,\chi_{12}\ +\ 32/3\,\bar{B}(\chi_1,\chi_2,0)\,L^r_7\,R^d_1R^d_2\,\chi_{12}\\
&+\ 16/3\,\bar{B}(\chi_1,\chi_2,0)\,L^r_8\,R^d_1R^d_2\,\chi_{12}\ +\ 16/3\,\bar{B}(\chi_1,\chi_2,0,k)\,L^r_7\,R^d_1R^d_2\ +\ 8/3\,\bar{B}(\chi_1,\chi_2,0,k)\,L^r_8\,R^d_1R^d_2\\
&+\ H^F(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\ \left[-1/4\,\chi_p\chi_{12}-3/8\,\chi^2_{12}-1/8\,(R^p_{q\pi})^2\chi^2_{12}+3/8\,R^p_{q\pi}R^c_p\,\chi^2_{12}-3/32\,(R^c_p)^2\chi^2_{12}\right]\\
&+\ H^F(1,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\ \left[-5/8\,R^p_{q\pi}\,\chi_q\chi_{12}+1/4\,R^p_{q\pi}\,\chi_{ps}\chi_{12}+3/16\,R^p_{s\pi}\,\chi_q\chi_{12}+7/16\,R^p_{s\pi}\,\chi_{12}\chi_s\right.\\
&\left.-\ 1/8\,R^c_p\,\chi_{ps}\chi_{12}\right]\ +\ 1/4\,H^F(1,\chi_{p3},\chi_{q4},\chi_{34},\chi_{12})\,\chi_{12}\chi_{34}\ +\ H^F(1,\chi_\pi,\chi_p,\chi_{12},\chi_{12})\ \left[1/2\,R^p_{q\pi}R^\pi_{12}\,\chi^2_{12}\right.\\
&\left.-\ 1/8\,R^p_{q\pi}R^v_{\pi12}\,\chi^2_{12}+1/4\,R^\pi_{pp}R^c_p\,\chi^2_{12}+1/16\,R^c_pR^v_{\pi12}\,\chi^2_{12}\right]\ +\ H^F(1,\chi_\pi,\chi_\pi,\chi_{12},\chi_{12})\ \left[1/2\,(R^\pi_{12})^2\chi^2_{12}\right.\\
&\left.+\ 1/2\,R^\pi_{12}R^v_{\pi12}\,\chi^2_{12}+5/32\,(R^v_{\pi12})^2\,\chi^2_{12}\right]\ +\ H^F(1,\chi_\pi,\chi_{1s},\chi_{2s},\chi_{12})\ \left[-3/16\,R^\pi_{ps}\,\chi_p\chi_{12}-5/16\,R^\pi_{ps}\,\chi_q\chi_{12}\right.\\
&\left.+\ 1/2\,R^\pi_{12}\,\chi^2_{12}-1/16\,R^v_{\pi ps}\,\chi_\pi\chi_{12}-1/16\,R^v_{\pi12}\,\chi_{12}\chi_s\right]\ +\ H^F(1,\chi_1,\chi_{12},\chi_2,\chi_{12})\ \left[-1/8\,R^p_{q\pi}R^c_q\,\chi^2_{12}\right.\\
&\left.+\ 3/4\,R^1_{2\pi}R^2_{1\pi}\,\chi^2_{12}+1/16\,R^c_1R^c_2\,\chi^2_{12}\right]\ +\ 1/4\,H^F(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\,\chi^2_{12}\\
&-\ H^F(2,\chi_p,\chi_p,\chi_{12},\chi_{12})\ \left[1/8\,R^p_{q\pi}R^d_p\,\chi^2_{12}-5/16\,R^c_pR^d_p\,\chi^2_{12}\right]\ +\ H^F(2,\chi_p,\chi_\pi,\chi_{12},\chi_{12})\ \left[1/4\,R^\pi_{pp}R^d_p\,\chi^2_{12}\right.\\
&\left.+\ 1/16\,R^d_pR^v_{\pi12}\,\chi^2_{12}\right]\ -\ H^F(2,\chi_p,\chi_{12},\chi_q,\chi_{12})\ \left[1/8\,R^q_{p\pi}R^d_p\,\chi^2_{12}-1/16\,R^c_qR^d_p\,\chi^2_{12}\right]\\
&-\ 1/8\,H^F(2,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\,R^d_p\,\chi_{ps}\chi_{12}\ +\ 5/32\,H^F(5,\chi_p,\chi_p,\chi_{12},\chi_{12})\,(R^d_p)^2\chi^2_{12}\\
&+\ 1/16\,H^F(5,\chi_1,\chi_2,\chi_{12},\chi_{12})\,R^d_1R^d_2\,\chi^2_{12}\ +\ H^F_1(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\ \left[2\,\chi^2_{12}+(R^p_{q\pi})^2\chi^2_{12}-R^p_{q\pi}R^c_p\,\chi^2_{12}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ 1/2\,(R^c_p)^2\chi_{12}^2\big] \;+\; H^F_1(1,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\,R^p_{q\pi}R^z_{sqp}\,\chi_{12}^2 \;-\; H^F_1(1,\chi_{ps},\chi_{qs},\chi_p,\chi_{12})\,R^p_{s\pi}\,\chi_{12}^2 \\
&- H^F_1(1,\chi_{ps},\chi_{qs},\chi_\pi,\chi_{12})\,R^\pi_{12}R^z_{sp\pi}\,\chi_{12}^2 \;+\; 1/2\,H^F_1(1,\chi_{12},\chi_p,\chi_\pi,\chi_{12})\,\big[R^p_{q\pi}R^v_{\pi 12}\,\chi_{12}^2 - R^\pi_{pp}R^z_{qp\pi}R^c_p\,\chi_{12}^2\big] \\
&- H^F_1(1,\chi_{12},\chi_\pi,\chi_\pi,\chi_{12})\,\big[1/2\,R^\pi_{12}R^v_{\pi 12}\,\chi_{12}^2 + 1/4\,(R^v_{\pi 12})^2\chi_{12}^2\big] \;+\; H^F_1(1,\chi_{12},\chi_1,\chi_2,\chi_{12})\,\big[1/2\,R^p_{q\pi}R^c_q\,\chi_{12}^2 \\
&- R^1_{2\pi}R^2_{1\pi}\,\chi_{12}^2\big] \;+\; 1/2\,H^F_1(3,\chi_{12},\chi_p,\chi_p,\chi_{12})\,\big[R^p_{q\pi}R^d_p\,\chi_{12}^2 - R^c_pR^d_p\,\chi_{12}^2\big] \\
&+ 1/2\,H^F_1(3,\chi_{12},\chi_p,\chi_q,\chi_{12})\,R^q_{p\pi}R^d_p\,\chi_{12}^2 \;+\; 1/2\,H^F_1(3,\chi_{12},\chi_p,\chi_\pi,\chi_{12})\,R^\pi_{12}R^z_{pq\pi}R^d_p\,\chi_{12}^2 \\
&- 1/4\,H^F_1(7,\chi_{12},\chi_p,\chi_p,\chi_{12})\,(R^d_p)^2\chi_{12}^2 \;-\; 3/4\,H^F_{21}(1,\chi_p,\chi_p,\chi_{12},\chi_{12})\,\chi_{12}^2 \\
&- 3/8\,H^F_{21}(1,\chi_p,\chi_{1s},\chi_{2s},\chi_{12})\,R^p_{q\pi}R^z_{sqp}\,\chi_{12}^2 \;+\; 3/8\,H^F_{21}(1,\chi_{ps},\chi_{qs},\chi_p,\chi_{12})\,\big[R^p_{q\pi}\,\chi_{12}^2 + R^p_{s\pi}\,\chi_{12}^2 \\
&- R^c_p\,\chi_{12}^2\big] \;+\; 3/8\,H^F_{21}(1,\chi_{ps},\chi_{qs},\chi_q,\chi_{12})\,R^q_{p\pi}R^z_{spq}\,\chi_{12}^2 \;+\; 3/8\,H^F_{21}(1,\chi_{ps},\chi_{qs},\chi_\pi,\chi_{12})\,R^\pi_{12}R^z_{pq\pi}R^z_{sp\pi}\,\chi_{12}^2 \\
&- 3/8\,H^F_{21}(1,\chi_\pi,\chi_{1s},\chi_{2s},\chi_{12})\,R^\pi_{12}R^z_{s1\pi}R^z_{s2\pi}\,\chi_{12}^2 \;+\; H^F_{21}(1,\chi_{12},\chi_p,\chi_p,\chi_{12})\,\big[3/8\,\chi_{12}^2 + 3/8\,(R^p_{q\pi})^2\chi_{12}^2 \\
&- 3/8\,R^p_{q\pi}R^c_p\,\chi_{12}^2 + 3/32\,(R^c_p)^2\chi_{12}^2\big] \;-\; H^F_{21}(1,\chi_{12},\chi_p,\chi_\pi,\chi_{12})\,\big[3/8\,R^p_{q\pi}R^v_{\pi 12}\,\chi_{12}^2 - 3/16\,R^c_pR^v_{\pi 12}\,\chi_{12}^2\big] \\
&+ 3/32\,H^F_{21}(1,\chi_{12},\chi_\pi,\chi_\pi,\chi_{12})\,(R^v_{\pi 12})^2\,\chi_{12}^2 \;+\; H^F_{21}(1,\chi_{12},\chi_1,\chi_2,\chi_{12})\,\big[-3/8\,R^p_{q\pi}R^c_q\,\chi_{12}^2 + 3/4\,R^1_{2\pi}R^2_{1\pi}\,\chi_{12}^2 \\
&+ 3/16\,R^c_1R^c_2\,\chi_{12}^2\big] \;+\; 3/4\,H^F_{21}(1,\chi_{12},\chi_{12},\chi_{12},\chi_{12})\,\chi_{12}^2 \;+\; 3/4\,H^F_{21}(1,\chi_{34},\chi_{p3},\chi_{q4},\chi_{12})\,\chi_{12}^2 \\
&- 3/8\,H^F_{21}(3,\chi_{ps},\chi_p,\chi_{qs},\chi_{12})\,R^d_p\,\chi_{12}^2 \;-\; 3/8\,H^F_{21}(3,\chi_{12},\chi_p,\chi_p,\chi_{12})\,\big[R^p_{q\pi}R^d_p\,\chi_{12}^2 - 1/2\,R^c_pR^d_p\,\chi_{12}^2\big] \\
&- H^F_{21}(3,\chi_{12},\chi_p,\chi_q,\chi_{12})\,\big[3/8\,R^q_{p\pi}R^d_p\,\chi_{12}^2 - 3/16\,R^c_qR^d_p\,\chi_{12}^2\big] \;+\; 3/16\,H^F_{21}(3,\chi_{12},\chi_p,\chi_\pi,\chi_{12})\,R^d_pR^v_{\pi 12}\,\chi_{12}^2 \\
&+ 3/32\,H^F_{21}(7,\chi_{12},\chi_p,\chi_p,\chi_{12})\,(R^d_p)^2\chi_{12}^2 \;+\; 3/16\,H^F_{21}(7,\chi_{12},\chi_1,\chi_2,\chi_{12})\,R^d_1R^d_2\,\chi_{12}^2 . \qquad (33)
\end{aligned}
$$

V. NUMERICAL ANALYSIS AND CONCLUSIONS

The NNLO results given in the preceding section depend on a number of $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ LEC:s. Eventually, their proper values, from which the physical LEC:s l_i^r and c_i^r can be obtained, should be determined by a fit of the NNLO expressions to lattice QCD data. For the present purpose, in order to have some reasonable estimates of the LEC:s, we simply use the values obtained from experiment in Ref. [15], which used three-flavor continuum χPT to NNLO.

The combination of LEC:s used in this paper corresponds to 'fit 10' of Ref. [15], which had $F_0 = 87.7$ MeV and $\mu = 770$ MeV. The parameters which were not determined by that fit, namely L_0^r, L_4^r and L_6^r, have been set to zero at the abovementioned scale μ, as have all the K_i^r. Also, L_{11}^r has been treated according to the condition $L_{11}^r = -l_4^r/4 = -(2L_4^r + L_5^r)$. It should be noted that L_0^r cannot be determined from experiment, but is obtainable from partially quenched simulations. Some recent results on L_4^r and L_6^r may be found in Ref. [16]. If we temporarily reintroduce the superscripts $(2pq)$, the choice of LEC:s for the numerical analysis is

$$
\begin{aligned}
F &= F_0\Big|_{\text{fit 10}}, \\
L^{r(2pq)}_{i=1\ldots10} &= L^r_{i=1\ldots10}\Big|_{\text{fit 10}}, \\
L^{r(2pq)}_{11} &= -\,(2L_4^r + L_5^r)\Big|_{\text{fit 10}}, \\
L^{r(2pq)}_{0} &= L^{r(2pq)}_{12} = K_i^{r(2pq)} = 0\,. \qquad (34)
\end{aligned}
$$

In order to set the scale for the lowest order masses, for a quark mass of 100 MeV we have roughly $\chi \sim 0.01$ GeV^2 and for 500 MeV we have $\chi \sim 0.25$ GeV^2. A crude estimate of the range of validity of PQχPT is for all $\chi_i, \chi_{ij} \lesssim 0.3$ GeV^2.

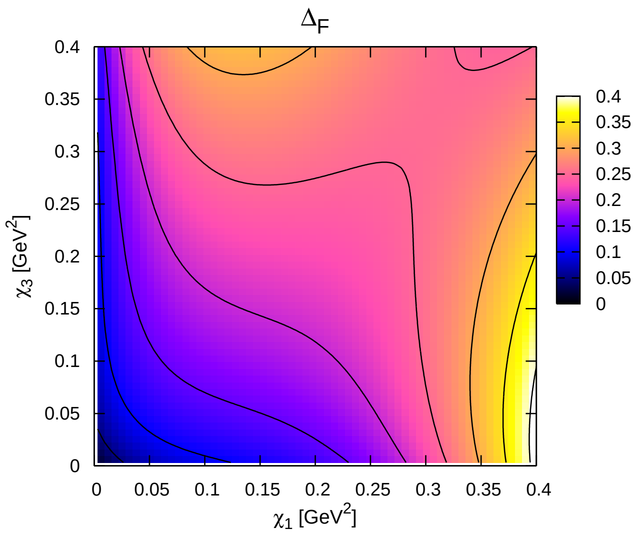

FIG. 3: The NNLO modification Δ_F of the decay constant for $d_{\rm val} = 1$ and $d_{\rm sea} = 1$ as a function of the valence quark mass parameter χ_1 and the sea-quark mass parameter χ_3. The function plotted represents the sum of the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ contributions.

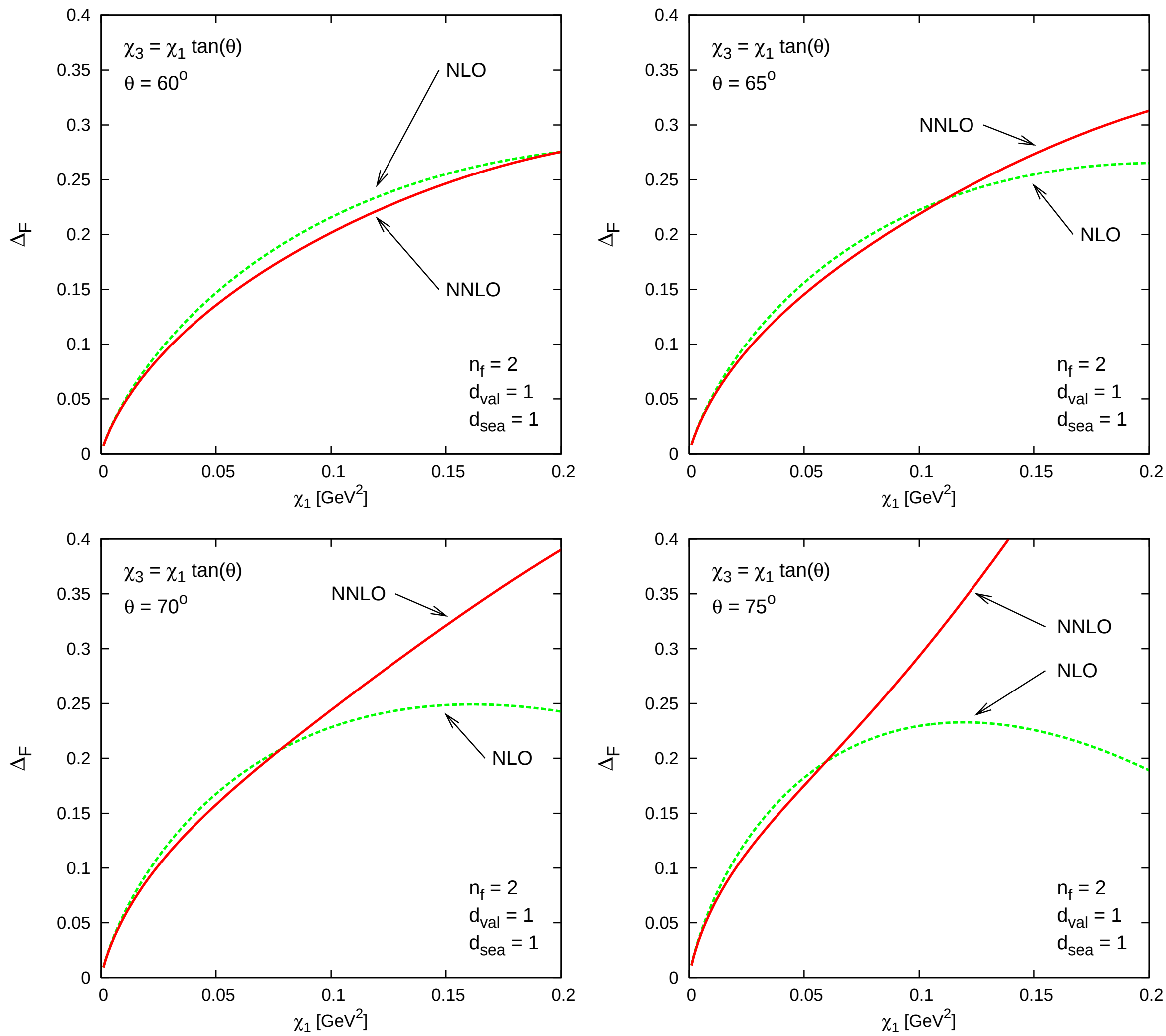

FIG. 4: Comparison of the NLO and NNLO corrections Δ_F to the decay constant for $d_{\mathrm{val}} = 1$ and $d_{\mathrm{sea}} = 1$. The curves labeled 'NLO' represent the $\mathcal{O}(p^4)$ contribution to the decay constant, and those labeled 'NNLO' represent the sum of the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ contributions. The dependence on the sea-quark mass χ_3 has been parameterized as $\chi_3 = \chi_1 \tan(\theta)$. Present Lattice QCD simulations are performed in the region where $\theta > 45°$.

A. Numerical Results for Decay Constants

The modification of the decay constant, when corrections up to NNLO are taken into account, is parameterized in terms of the relative change Δ_F, which is defined as

$$F_{\mathrm{phys}} \;=\; F\,(1 + \Delta_F)\,, \qquad (35)$$

so that Δ_F is expected to disappear in the chiral limit, when both sea and valence quark masses are set to zero. That this condition is satisfied can be seen directly from Fig. 3. However, if the sea-quark mass is set to zero for a constant valence quark mass, the decay constant approaches a constant value. This behavior is expected and may be attributed to the unphysical quenched chiral logarithms and similar effects.

The numerical results for the pseudoscalar meson decay constants are shown in Figs. 3 and 4. For simplicity, we have considered the case with $d_{\mathrm{val}} = 1$ and $d_{\mathrm{sea}} = 1$, i.e. the valence quark masses and sea-quark masses are degenerate, but different from each other. Thus the results presented correspond to Eq. (16) for NLO, and Eqs. (17) and (18) for NNLO.

The convergence of the chiral expansion for the decay constant is investigated in Fig. 4 for different combinations of sea and valence quark masses. We have chosen the sea-quark mass larger than the valence quark mass in our examples since this is the most likely regime for lattice QCD simulations. Overall, Fig. 3 shows that the

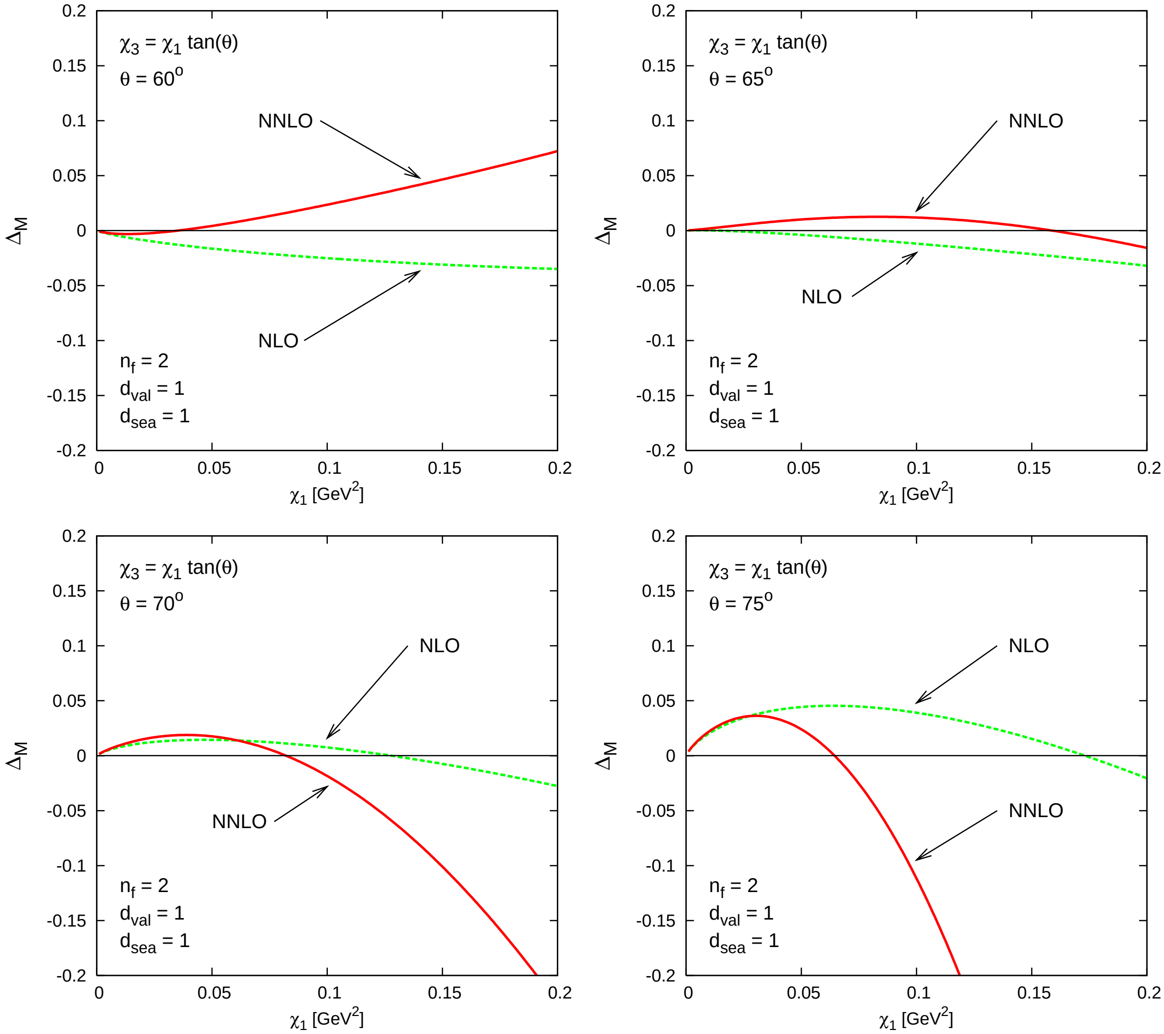

FIG. 5: Comparison of the NLO and NNLO corrections Δ_M to the meson mass for $d_{\rm val} = 1$ and $d_{\rm sea} = 1$. The curves labeled 'NLO' represent the $\mathcal{O}(p^4)$ contribution to the meson mass, and those labeled 'NNLO' represent the sum of the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ contributions. The dependence on the sea-quark mass χ_3 has been parameterized as $\chi_3 = \chi_1 \tan(\theta)$. Present Lattice QCD simulations are performed in the region where $\theta > 45°$.

NNLO corrections to the decay constant are significant at least for our choice of LEC:s. Nevertheless, the comparison between the NLO and NNLO curves in Fig. 4 also shows that there exists a rather large range of input quark masses for which the convergence of the chiral expansion is reasonable. The cases with less degenaracies in the input quark masses could also be considered, but such plots are difficult to present in an instructive way since these functions depend on up to four different quark masses. One possible way to proceed is to look at various special cases where the ratio between the sea or valence quark masses is fixed. To illustrate the effects of nondegenerate sea and valence quarks, such plots are given in the next subsection for the meson mass.

B. Numerical Results for Meson Masses

The numerical results for the pseudoscalar meson masses are shown in Figs. 5 through 8. Part of the presentation is similar to that of the decay constant results in the preceding section, i.e. results are given for $d_{\rm val} = 1$ and $d_{\rm sea} = 1$. These results correspond to Eq. (26) for NLO, and Eqs. (27) and (28) for NNLO. In addition some results are also presented for $d_{\rm val} = 1$, $d_{\rm sea} = 2$ in Fig. 7 and for $d_{\rm val} = 2$, $d_{\rm sea} = 1$ in Fig. 8. The modification of the meson mass when NNLO corrections are taken into account is parameterized in terms of the quantity Δ_M, which is defined as

$$M^2_{\rm phys} = M^2_0 \, (1 + \Delta_M)\,, \qquad (36)$$

so that Δ_M represents a relative deviation from the lowest order result M_0^2. With this definition, Δ_M is expected to disappear in the chiral limit, when both sea and valence quark masses approach zero at a constant ratio.

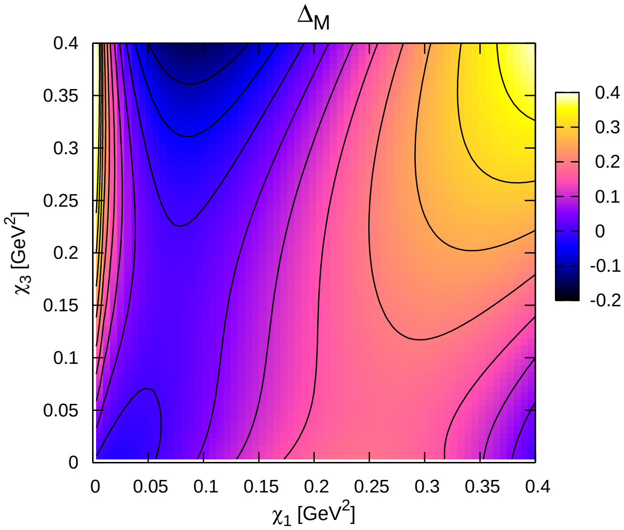

FIG. 6: The NNLO modification Δ_M of the meson mass for $d_{\rm val} = 1$ and $d_{\rm sea} = 1$ as a function of the valence quark mass parameter χ_1 and the sea-quark mass parameter χ_3. The function plotted represents the sum of the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ contributions. Note the divergence of the result near $\chi_1 = 0$, which vanishes for $\chi_3 = 0$.

In general, the appearance of Fig. 6 indicates that the NNLO contribution to the meson mass is more pronounced than for the decay constant, which is due to the fact that the NLO contribution to the meson mass vanishes for certain values of the input quark masses. Another noteworthy feature of Fig. 6 is the logarithmic divergence for small values of the valence quark mass χ_1. This feature, which is well-known at NLO and persists at NNLO, is due to the unphysical quenched chiral logarithms. This divergence disappears if the sea and valence quark masses are sent to zero in constant ratio, which is also confirmed by Figs. 5 through 8.

There is no *a priori* reason to believe that the convergence of the meson mass should be intrinsically worse than that of the decay constant. Nevertheless, the combined NLO + NNLO result in Fig. 6 vanishes in several places, with the NLO and NNLO contributions canceling each other. The decay constants showed a more uniform convergence for the masses and input LEC:s used here. In this respect the curves shown in Fig. 5 are less instructive as to the convergence of the chiral expansion than those shown in Fig. 4. Admittedly, however, the inclusion of NNLO effects into the pseudoscalar meson mass

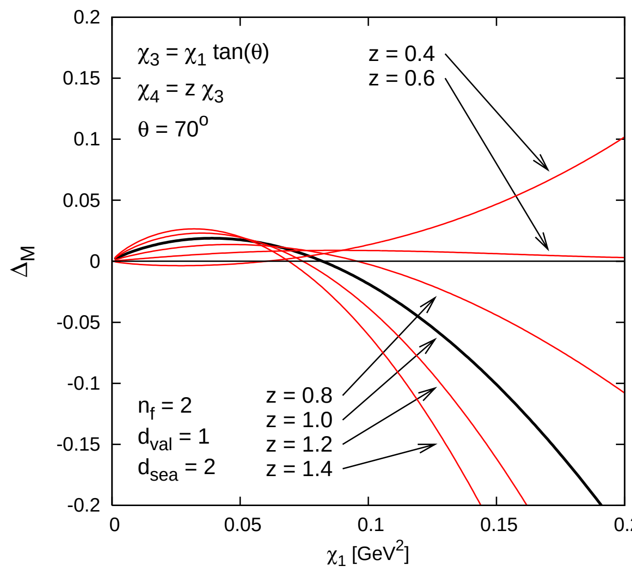

FIG. 7: The modification Δ_M of the NNLO meson mass for $d_{\rm val} = 1$ and $d_{\rm sea} = 2$. The sea quark mass dependence has been parameterized as $\chi_4 = z\chi_3$ and the relation between the sea and valence quark masses as $\chi_3 = \chi_1 \tan(\theta)$.

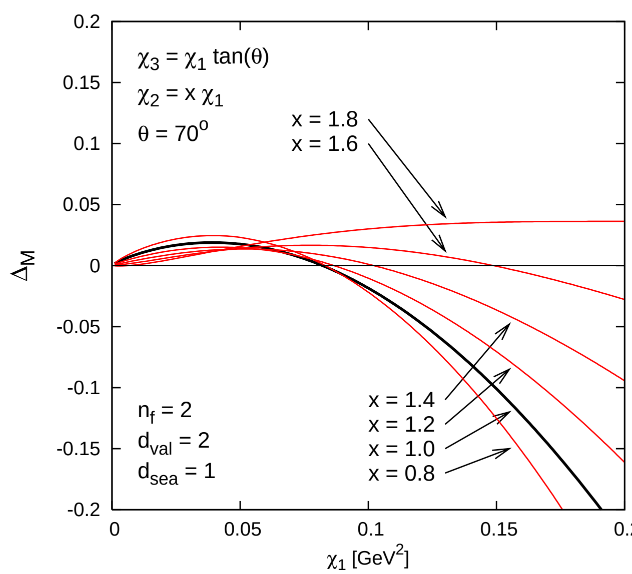

FIG. 8: The modification Δ_M of the NNLO meson mass for $d_{\rm val} = 2$ and $d_{\rm sea} = 1$. The valence quark mass dependence has been parameterized as $\chi_2 = x\chi_1$ and the relation between the sea and valence quark masses as $\chi_3 = \chi_1 \tan(\theta)$.

in PQχPT is a major effect which changes features of the NLO calculation.

It is also noteworthy that the convergence of the chiral expansion for the quantities considered apparently depends rather strongly on the values of the LEC:s. For example, the obvious trial choice of setting all the L_i^r to zero leads, incidentally, to a much larger NNLO contribution which converges less well also for the decay constants. Setting several of the K_i^r to nonzero values can also change this picture quite dramatically. Thus a definite statement about convergence can only be made when some estimates are available for the K_i^r.

C. Determination of LEC:s from Lattice QCD Simulations

In the previous sections, a sample of numerical results have been presented in order to illustrate the effects of the NNLO contributions when the LEC:s are known. Since the intended area of application of the NNLO formulas given in this paper is the determination of LEC:s from PQQCD Lattice simulations, some attention should also be devoted to finding efficient methods for this task. Again, it is understood that the $L_i^{r(2pq)}$ and $K_i^{r(2pq)}$ are simply denoted by L_i^r and K_i^r.

At NLO in PQχPT, the expressions for the pseudoscalar meson masses and decay constants can be cleanly separated into a tree-level contribution which is a function of the L_i^r and a remaining part, independent of the L_i^r, which involves the chiral logarithms. Thus at NLO it is possible to develop various optimal strategies for the purpose of obtaining estimates of the L_i^r by fits to Lattice QCD data [4]. At NLO, the dependence on the L_i^r is proportional to $2L_4^r\chi_{34} + L_5^r\chi_{12}$ for the decay constant and $(4L_6^r - 2L_4^r)\chi_{34} + (2L_8^r - L_5^r)\chi_{12}$ for the masses, from which an overall factor χ_{12} has been removed. The LEC:s can thus be determined from the 1+1 case by using a constant sea quark mass and varying the valence quark mass. The more complicated cases with nondegenerate quarks, i.e. $\chi_1 \neq \chi_2$ or $\chi_3 \neq \chi_4$ are not needed at NLO.

A similar analysis which involves the determination of LEC:s from a fit to Lattice QCD simulations using NNLO PQχPT is more challenging, not only because of the complexity of the NNLO expressions, but mainly since the chiral logarithms now involve the L_i^r. Furthermore the coefficient functions of the L_i^r at NNLO have a more complicated dependence on both sea and valence quark masses. On the other hand, the situation for the K_i^r at NNLO is similar to that of the L_i^r at NLO, although the much larger number of K_i^r parameters, along with the appearance of contributions bilinear in the L_i^r certainly complicates the analysis.

In general, at $\mathcal{O}(p^6)$ the expressions for the pseudoscalar meson masses and decay constants depend on 12 K_i^r parameters, 5 for the decay constants and an additional 7 for the masses. The dependence of the decay constant on the K_i^r is proportional to

$$\begin{aligned}&16\,\chi_{12}^2 K_{19}^r \;-\; 8\,\chi_1\chi_2\,(K_{19}^r - K_{23}^r) \;+\; 16\,\chi_{12}\chi_{34}\,K_{20}^r\\ &-\;16\,\chi_3\chi_4\,K_{21}^r \;+\; 32\,\chi_{34}^2\,(K_{21}^r + K_{22}^r)\,,\end{aligned} \tag{37}$$

from which all 5 appearing constants, $K_{19\ldots23}^r$ can be determined but at least one case with nondegenerate sea quark masses needs to be calculated. From the 1+1 case, three combinations can be determined. Also, to separate K_{19}^r from K_{23}^r a case with nondegenerate valence quark masses, $d_{\rm val} = 2$, is needed, otherwise only the combination $K_{19}^r + K_{23}^r$ can be obtained. Likewise nondegenerate sea quark masses, $d_{\rm sea} = 2$, are needed to disentangle K_{21}^r from K_{22}^r, simulations with degenerate sea quark masses allow only the determination of $K_{21}^r + 8K_{22}^r$. Similarly, the dependence of the meson masses on the K_i^r is

$$\begin{aligned}&-\;\chi_{12}^2\,(32\,K_{17}^r + 32\,K_{19}^r - 96\,K_{25}^r - 32\,K_{39}^r)\\ &+\;\chi_1\chi_2\,(16\,K_{19}^r - 16\,K_{23}^r - 48\,K_{25}^r)\\ &-\;\chi_{12}\chi_{34}\,(64\,K_{18}^r + 32\,K_{20}^r - 64\,K_{26}^r - 64\,K_{40}^r)\\ &+\;\chi_3\chi_4\,(32\,K_{21}^r - 32\,K_{26}^r)\\ &-\;\chi_{34}^2\,(64\,K_{21}^r + 64\,K_{22}^r - 64\,K_{26}^r - 192\,K_{27}^r)\,,\end{aligned} \tag{38}$$

from which an overall factor of χ_{12} has again been removed. From the 1+1 case we can obtain 3 combinations, one more from a case with $d_{\rm val} = 2$ and one more from a case with $d_{\rm sea} = 2$. The particular combinations can be easily read off from Eq. (38). Thus, in order to obtain independent information on as many of the K_i^r:s as possible, simulations with nondegenerate quark masses are clearly preferrable. On the other hand, a possible strategy for the determination of the L_i^r from NNLO fits is to use only one distinct sea and valence quark mass, since then the number of distinct combinations of K_i^r parameters is minimal, while the possibility of distinguishing all of the $L_{4\ldots6}^r$ and L_8^r is retained.

At NNLO, additional L_i^r are present in the expressions for the decay constant and the meson mass, namely L_7^r and the $L_{0\ldots3}^r$. Of these, L_7^r can be determined from the diagonal mesons [4] by consideration of the coefficient of the double pole. This calculation is not yet performed up to NNLO. The other four L_i^r are those relevant at NLO for meson-meson scattering. Their values have to be used as input but could in principle be separated from the K_i^r since they multiply nonanalytic dependences on the quark masses. A detailed study of this is beyond the scope of the present work.

The present analysis shows that while the number of free parameters in a fit of the NNLO expressions to Lattice QCD data is large, the possibility of producing a large amount of simulation data with different combinations of sea and valence quark masses should compensate for this. Furthermore, it should be kept in mind that many of the in principle free parameters L_i^r are already quite accurately known from previous work.

D. Conclusions

In summary, we have calculated the pseudoscalar meson masses and decay constants to NNLO in Partially Quenched Chiral Perturbation Theory for two flavors of sea-quarks. Explicit analytical formulas of reasonable length and complexity have been given, along with a numerical analysis which demonstrates the numerical implementation of the results. Although the NNLO results appear to be of a reasonable magnitude, definite statements concerning the convergence of the chiral expansion have to be postponed, as the magnitude of the results depends rather sensitively on a number of largely unknown LEC:s.

An implementation of the present results will be made available by the authors in the near future [17]. The availability of these results makes it possible to perform a NNLO chiral extrapolation from partially quenched lattice data. However, usage of the present results requires Lattice QCD data on the pseudoscalar meson masses and decay constants which has been extrapolated to the continuum limit and to infinite volume at different values of the quark masses. Although such data are not yet available, this situation will undoubtedly change in the near future. The ultimate goal of such fits to lattice data is thus to provide reliable estimates of the (unphysical) low-energy constants of PQχPT, from which the values of the physical ones can then be extracted.

Acknowledgments

The program FORM 3.0 has been used extensively in these calculations [18]. This work is supported by the European Union TMR network, Contract No. HPRN-CT-2002-00311 (EURIDICE). TL wishes to thank Eduardo de Rafael, Marc Knecht, Laurent Lellouch and Jürg Gasser for instructive discussions, and F. Farchioni for correspondence. TL also thanks the Mikael Björnberg memorial foundation for a travel grant.

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