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The general Leigh-Strassler deformation and Integrability

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Abstract: The success of the identification of the planar dilatation operator of $\mathcal{N}=4$ SYM with an integrable spin chain Hamiltonian has raised the question if this also is valid for a deformed theory. Several deformations of SYM have recently been under investigation in this context. In this work we consider the general Leigh-Strassler deformation. For the generic case the S-matrix techniques cannot be used to prove integrability. Instead we use R-matrix techniques to study integrability. Some new integrable points in the parameter space are found.

Keywords: AdS-CFT correspondence, Integrable field theories, bethe ansatz.

Contents

1. Introduction

In the last few years, several new discoveries have shed light on the AdS/CFT correspondence [1, 2, 3]. This correspondence maps strings moving in an $AdS_5 \times S^5$ background to an $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. The eigenvalues of the dilatation operator are mapped to the energies of closed string states [4]. A step in understanding this duality better was the discovery that the dilatation operator of the $\mathcal{N} = 4$ SYM is proportional to the Hamiltonian of an integrable spin chain [5, 6, 7].

Recently, progress has been made to extend the gauge/gravity-correspondence, in context of spin chains, towards more realistic models with less supersymmetry [8, 9, 10, 11]. For instance, if the background geometry for the string is $AdS_5 \times W$, where W is some compact manifold, the dual gauge theory should still be conformal. Other geometries, mainly orbifolds of $AdS_5 \times S^5$, corresponding to non-conformal theories have also been investigated [12, 13].

The success in using spin chains to study the duality beyond the BMN limit motivates studies of integrability of deformed correlators. One question that naturally arises in this context is whether integrability is related to supersymmetry, conformal invariance or have more geometrical reasons.

The Leigh-Strassler deformations [14] preserve $\mathcal{N}=1$ supersymmetry and conformal invariance, at least up to one loop. It is hence of great interest to investigate if there exist points in the parameter space where the dilatation operator is mapped to an integrable spin-chain Hamiltonian. This question has been under investigation in [15, 16, 17, 18]. In [16] this deformation was studied in a special case corresponding to a q-deformed (often called β-deformed) commutator. It was found that for the sector with three chiral fields the dilatation operator is integrable for q equals root of unity.

In reference [19], a way of generating supergravity duals to the β-deformed field theory was introduced, and in [8, 9, 20] agreement between the supergravity sigma model and the coherent state action coming from the spin chain describing the β-deformed dilatation operator was demonstrated. This way of creating supergravity duals was used in [21] to construct a three-parameter generalization of the β-deformed theory. The gauge theory dual to this supergravity solution was found in [21, 17] for $q=e^{i\gamma_j}$ with γ_j real, corresponding to certain phase deformations in the Lagrangian. This gauge theory is referred to as twisted SYM, from which the β-deformed theory is obtained when all the $\gamma_j=\beta$. The result is that the theory is integrable for any $q=e^{i\gamma_j}$ with γ_j real [17]. The general case with complex γ_j is not integrable [16, 18].

In the present work, the q-deformed analysis is extended to the more general Leigh-Strassler deformations with an extra complex parameter h, in order to find new integrable theories. A site dependent transformation is found which relates the γ_j-deformed case to a site dependent spin-chain Hamiltonian with nearest-neighbour interactions. In particular when all γ_j are equal, the transformation relates the q-deformed theory to the h-deformed theory, *i.e.* the theory only involving the parameter h. In particular, we find a new R-matrix, at least in the context of $\mathcal{N}=4$ SYM, for $q=0$ and $h=e^{i\theta}$ with θ real. We also find all R-matrices with a linear dependence on the spectral parameter which give the dilatation operator. A general ansatz for the R-matrix is given. Unfortunately, the most general solution is not found. However, we find a new hyperbolic R-matrix which corresponds to a basis-transformed Hamiltonian with only diagonal entries [18]. A reformulation of the general R-matrix shows that the structure of the equations obtained from the Yang-Baxter equations resemble the equations obtained in the eight vertex model. This gives a clear hint how to proceed.

In the dual supergravity theory, some attempts to construct backgrounds for non-zero h have been done [22, 23]. Apart from the five-flux there is also a three-flux. A step going beyond supergravity was taken in [24] where the BMN limit was considered. We hope our results will make it easier to find the supergravity dual of the general Leigh-Strassler deformed theory.

2. Marginal deformations of $\mathcal{N}$=4 supersymmetric Yang-Mills

To study marginal deformations of $\mathcal{N}=4$ SYM with $SU(N)$ gauge group, it is convenient to use $\mathcal{N}=1$ SYM superfields. The six real scalar fields of the $\mathcal{N}=4$ vector multiplet are combined into the lowest order terms of three complex $\mathcal{N}=1$ chiral superfields Φ_0, Φ_1

and Φ_2. It is well known that the $\mathcal{N}=1$ superpotential

$$W_{\mathcal{N}=1} = \frac{1}{3!} C^{IJK}_{abc} \Phi^a_I \Phi^b_J \Phi^c_K \, , \tag{2.1}$$

where C^{IJK}_{abc} is the coupling constant, describes a finite theory at one-loop if the following two conditions are fulfilled [25, 26]

$$3C_2(G) = \sum_I T(A_I), \qquad \text{and} \qquad C^{IKL}_{acd} \bar{C}^{bcd}_{JKL} = 2g^2 T(A_I) \delta_a{}^b \delta^I{}_J . \tag{2.2}$$

The constant $C_2(G)$ is the quadratic Casimir operator defined here[1] as $C_2(G) \cdot \mathbf{1} = \delta_{ab} T^a_A T^b_A$ where A is the adjoint representation of the group G which in the present context is the symmetry group $SU(N)$. The constant $T(M)$ is defined through $T(M)\delta^{ab} = \mathrm{Tr}(T^a_M T^b_M)$ for the representation M. The first condition of (2.2) implies that the β-function is zero. For an $SU(N)$ group with the superpotential (2.1) this is automatically fulfilled. The choice $C^{IJK}_{abc} = g\varepsilon^{IJK} f_{abc}$ therefore gives a superconformal $\mathcal{N}=1$ theory at one-loop. However, there are more general superpotentials satisfying the one-loop finiteness conditions. To explore marginal deformations of $\mathcal{N}=4$ SYM we consider the Leigh-Strassler superpotential [14]

$$W = \frac{1}{3!} \lambda \varepsilon^{IJK} \mathrm{Tr}\left[\left[\Phi_I, \Phi_J \right] \Phi_K \right] + \frac{1}{3!} h^{IJK} \mathrm{Tr}\left[\left\{ \Phi_I, \Phi_J \right\} \Phi_K \right] \, , \tag{2.3}$$

where h^{IJK} is totaly symmetric. The coupling constants can now be written as $C^{IJK}_{abc} = \lambda\varepsilon^{IJK} f_{abc} + h^{IJK} \mathrm{Tr}\left[\{T_a, T_b\} T_c\right]$. The non-zero couplings are chosen to be $h^{012} = \lambda(1-q)/(1+q)$ and $h^{III} = 2\lambda h/(1+q)$. In terms of the deformation parameters q and h the superpotential (2.3) becomes

$$W = \frac{2\lambda}{1+q} \mathrm{Tr}\left[\Phi_0\Phi_1\Phi_2 - q\Phi_1\Phi_0\Phi_2 + \frac{h}{3}\left(\Phi_0^3 + \Phi_1^3 + \Phi_2^3\right) \right] . \tag{2.4}$$

This deformed superpotential will be our main focus.

The presence of q and h in the superpotential (2.4) breaks the $SU(3)$ symmetry in the chiral sector. What is left of the symmetry is a $Z_3 \times Z_3$ symmetry. The first Z_3 permutes the Φ's and the second takes $\Phi_0 \to \omega\Phi_0$, $\Phi_1 \to \omega^2\Phi_1$ and $\Phi_2 \to \Phi_2$, where ω is a third root of unity.

The one-loop finiteness condition (2.2) is satisfied if

$$g^2 = \frac{\lambda^2}{(1+q)^2} \left[(1+q)^2 + \left((1-q)^2 + 2h^2\right) \left(\frac{N^2-4}{N^2} \right) \right] . \tag{2.5}$$

In the large-N limit, which we consider, the relation (2.5) becomes even more simple. The one-loop finiteness condition (2.2) also implies that the scalar field self-energy contribution from the fermion loop is the same as in the $\mathcal{N}=4$ scenario, due to the fact that the fermion loop has the contraction $C^{IKL}_{acd} \bar{C}^{bcd}_{JKL}$. The parameters in (2.5) span a space within which

[1] Our conventions are: T^a are the $SU(N)$ group generators, satisfying $T^a = T^{a\dagger}$. The normalization of T^a is given by $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$ from where it follows that $\mathrm{Tr}(T^a_A T^b_A) = N\delta^{ab}$ in the adjoint representation A.

there exists a manifold, or perhaps just isolated points, $\beta(g,\lambda,q,h)=0$ of superconformal theories to all loops [14]. In the limit $q\to 1$ and $h\to 0$ the $\mathcal{N}=4$ SYM is restored. Marginal deformations away from this fixed point will be explored in the following sections by means of integrable spin chains.

3. Dilatation operator

From the Leigh-Strassler deformation (2.4) of the $\mathcal{N}=4$ SYM theory it is possible to obtain the dilatation operator in the chiral sector. In this sector, the only contribution is coming from the F-term in the Lagrangian, under the assumption that the one-loop finiteness condition (2.2) is fulfilled. The scalar field part of the F-term can be expressed in terms of the superpotential as

$$\mathcal{L}_F = \left|\frac{\partial W}{\partial \Phi_0}\right|^2 + \left|\frac{\partial W}{\partial \Phi_1}\right|^2 + \left|\frac{\partial W}{\partial \Phi_2}\right|^2 . \tag{3.1}$$

Using ϕ_0, ϕ_1 and ϕ_2 to denote the complex component fields, the Lagrangian becomes (omitting the overall factor $2\lambda/(1+q)$ in (2.4))

$$\begin{aligned}\mathcal{L}_F &= \mathrm{Tr}\left[\phi_i\phi_{i+1}\bar\phi_{i+1}\bar\phi_i - q\phi_{i+1}\phi_i\bar\phi_{i+1}\bar\phi_i - q^*\phi_i\phi_{i+1}\bar\phi_i\bar\phi_{i+1}\right]\\ &+ \mathrm{Tr}\left[qq^*\phi_{i+1}\phi_i\bar\phi_i\bar\phi_{i+1} - qh^*\phi_{i+1}\phi_i\bar\phi_{i+2}\bar\phi_{i+2} - q^*h\phi_{i+2}\phi_{i+2}\bar\phi_i\bar\phi_{i+1}\right]\\ &+ \mathrm{Tr}\left[h\phi_{i+2}\phi_{i+2}\bar\phi_{i+1}\bar\phi_i + h^*\phi_i\phi_{i+1}\bar\phi_{i+2}\bar\phi_{i+2} + hh^*\phi_i\phi_i\bar\phi_i\bar\phi_i\right],\end{aligned} \tag{3.2}$$

where a summation over $i=0,1,2$ is implicitly understood and the indices of the fields ϕ_i are identified modulo three. In order to see how the dilatation operator acts on a general operator $O=\psi^{i_1\ldots i_L}\mathrm{Tr}\phi_{i_1}\ldots\phi_{i_L}$ to first loop order in the planar limit we calculate the Feynman graphs and regularize in accordance with [15, 16]. The vector space, spanning these operators, can be mapped to the vector space of a spin-1 chain (see [5] for details). We define the basis states $|0\rangle$, $|1\rangle$ and $|2\rangle$ for the spin chain which correspond to the fields ϕ_0, ϕ_1 and ϕ_2. By introducing the operators E_{ij}, which act on the basis states as $E_{ij}|k\rangle=\delta_{jk}|i\rangle$, the dilatation operator can be written as a spin-chain Hamiltonian with nearest-neighbour interactions, *i.e.* $\Delta=\sum_l H^{l,l+1}$ where

$$\begin{aligned}H^{l,l+1} &= E^l_{i,i}E^{l+1}_{i+1,i+1} - qE^l_{i+1,i}E^{l+1}_{i,i+1} - q^*E^l_{i,i+1}E^{l+1}_{i+1,i}\\ &+ qq^*E^l_{i+1,i+1}E^{l+1}_{i,i} - qh^*E^l_{i+1,i+2}E^{l+1}_{i,i+2} - q^*hE^l_{i+2,i+1}E^{l+1}_{i+2,i}\\ &+ hE^l_{i+2,i}E^{l+1}_{i+2,i+1} + h^*E^l_{i,i+2}E^{l+1}_{i+1,i+2} + hh^*E^l_{i,i}E^{l+1}_{i,i}\,.\end{aligned} \tag{3.3}$$

The direct product between the operators E_{ij} is suppressed. If we use the convention

$$|0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{3.4}$$

the Hamiltonian can be expressed as the matrix

$$H^{l,l+1} = \begin{pmatrix} h^*h & 0 & 0 & 0 & 0 & h & 0 & -q^*h & 0 \\ 0 & 1 & 0 & -q^* & 0 & 0 & 0 & 0 & h^* \\ 0 & 0 & q^*q & 0 & -qh^* & 0 & -q & 0 & 0 \\ 0 & -q & 0 & q^*q & 0 & 0 & 0 & 0 & -qh^* \\ 0 & 0 & -q^*h & 0 & h^*h & 0 & h & 0 & 0 \\ h^* & 0 & 0 & 0 & 0 & 1 & 0 & -q^* & 0 \\ 0 & 0 & -q^* & 0 & h^* & 0 & 1 & 0 & 0 \\ -qh^* & 0 & 0 & 0 & 0 & -q & 0 & q^*q & 0 \\ 0 & h & 0 & -q^*h & 0 & 0 & 0 & 0 & h^*h \end{pmatrix} . \qquad (3.5)$$

We will now search for special values of the parameters h and q for which the spin-chain Hamiltonian (3.3) is integrable. When h is absent, the analysis simplifies considerably, because the usual S-matrix techniques can be used [16, 18, 27]. The existence of a homogeneous eigenstate, an eigenstate of the form $\mid a\rangle \otimes \mid a\rangle \ldots \otimes \mid a\rangle$, is crucial for the S-matrix techniques to work. From this reference state, excitations can be defined. In this context, the state $|a\rangle$ is a pure state, that is, one of the states $|0\rangle$, $|1\rangle$ or $|2\rangle$.

When h is non-zero, the analysis become significantly harder. The only values for the parameters, for which it is possible to define a homogeneous eigenstate are $q = 1 + e^{i2\pi n/3}h$ or $q = -1$ and $h = e^{i2\pi n/3}$, where n is an arbitrary integer. In these cases the homogeneous eigenstates are

$$\mid a\rangle = \mid 0\rangle + e^{\frac{i2\pi m}{3}} \mid 1\rangle + e^{-\frac{i2\pi m}{3}} \mid 2\rangle, \qquad m \in Z\,. \qquad (3.6)$$

Clearly, the two Z_3 symmetries are manifest. For $q = 1 + he^{i2\pi n/3}$, the eigenvalues are zero, thus the corresponding states are protected. This case is related to the q-deformed Hamiltonian by a simple change of variables. We introduce a new basis

$$\begin{aligned} |0\rangle &= \frac{e^{\frac{i2\pi n}{3}}}{\sqrt{3}}(|\tilde{0}\rangle + |\tilde{1}\rangle + \tilde{2}\rangle)\,, \\ |1\rangle &= \frac{1}{\sqrt{3}}(|\widetilde{0}\rangle + e^{\frac{i2\pi}{3}}|\widetilde{1}\rangle + e^{-\frac{i2\pi}{3}}|\widetilde{2}\rangle)\,, \\ |2\rangle &= \frac{1}{\sqrt{3}}(|\widetilde{0}\rangle + e^{-\frac{i2\pi}{3}}|\widetilde{1}\rangle + e^{\frac{i2\pi}{3}}|\widetilde{2}\rangle), \end{aligned} \qquad (3.7)$$

where n is an integer. It will shortly be shown that the phase shift in $|0\rangle$ will imply that a phase $e^{\pm i2\pi/3}$ can be transformed away from h. The Hamiltonian expressed in the new basis (3.7) takes the same form as (3.3), but with new parameters $\tilde{q}$ and $\tilde{h}$ and an overall proportionality factor

$$e^{\frac{i2\pi}{3}} - qe^{-\frac{i2\pi}{3}} + he^{-\frac{i2\pi n}{3}}. \qquad (3.8)$$

The new parameters $\tilde{q}$ and $\tilde{h}$ can then be expressed in terms of the old parameters as

$$\tilde{q} = \frac{qe^{\frac{i2\pi}{3}} - e^{\frac{-i2\pi}{3}} - he^{-\frac{i2\pi n}{3}}}{e^{\frac{i2\pi}{3}} - qe^{\frac{-i2\pi}{3}} + he^{-\frac{i2\pi n}{3}}} \qquad (3.9)$$

$$\tilde{h} = \frac{1 - q + he^{-\frac{i2\pi n}{3}}}{e^{\frac{i2\pi}{3}} - qe^{\frac{-i2\pi}{3}} + he^{-\frac{i2\pi n}{3}}}. \qquad (3.10)$$

The case $q = he^{-i2\pi n/3} + 1$ corresponds to the q-deformed case, and if h's imaginary part comes from the phase $e^{i2\pi n/3}$, the remaining part is phase independent. This is in agreement with reference [28]. The integrable case q equals a phase will correspond to the case $q = he^{-i2\pi n/3} + 1$ with $h = \rho e^{i2\pi n/3}$ with ρ and q being real. It is also clear that the case $q = -1$ and $h = e^{i2\pi n/3}$ is related by the change of basis to a Hamiltonian of the form

$$H = \sum_i 3 \left[E_{22}^i E_{22}^{i+1} + E_{00}^i E_{00}^{i+1} + E_{11}^i E_{11}^{i+1} \right] . \tag{3.11}$$

This case looks perhaps trivial, but it is not. The different eigenvalues equal $3n$ with $n = 0, 1, 2, \ldots, L-2, L$. Note that the value $L-1$ is absent for this periodic spin chain[2]. The states have a large degree of degeneration.

For other values of q and h, a reference state does not have a precise meaning. Hence, we cannot adapt the S-matrix formalism. Instead, we will try to find an R-matrix, from which the Hamiltonian (3.3) is obtainable. The existence of an R-matrix $R(u)$, depending on the spectral parameter u, is sufficient to ensure integrability. All R-matrices necessarily have to satisfy the Yang-Baxter equation

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v). \tag{3.12}$$

The Hamiltonian can be obtained from the R-matrix through the following relation

$$\mathcal{P} \frac{d}{du} R(u)|_{u=u_0} = H \, , \tag{3.13}$$

where $\mathcal{P}$ is the permutation operator, with the additional requirement $R(u_0) = \mathcal{P}$ for some point $u = u_0$.

4. A first look for integrability

In this section, we will show how the transformation of basis (3.7) combined with a position dependent phase shift, sometimes called a twist, gives rise to new interesting cases of integrability. In [16], the q-deformed case was studied. It was shown that for q equals a root of unity, the phases can be transformed away into the boundary conditions. Furthermore, it was shown in [17] that the integrability properties do not get affected for any $q = e^{i\beta}$, where β is real. It was also established that a generalised SYM Lagrangian deformed with three phases γ_i, instead of just one variable, is integrable. The deformed theory is referred to as the twisted (or γ-deformed) SYM and the corresponding one-loop dilatation operator in the three scalar sector is

$$\begin{aligned} H^{l,l+1} = & \left[E_{00}^l E_{11}^{l+1} + E_{11}^l E_{22}^{l+1} + E_{22}^l E_{00}^{l+1} \right] \\ & - \left[e^{i\gamma_1} E_{10}^l E_{01}^{l+1} + e^{i\gamma_2} E_{21}^l E_{12}^{l+1} + e^{i\gamma_3} E_{02}^l E_{20}^{l+1} \right] \\ & - \left[e^{-i\gamma_1} E_{01}^l E_{10}^{l+1} + e^{-i\gamma_2} E_{12}^l E_{21}^{l+1} + e^{-i\gamma_3} E_{20}^l E_{02}^{l+1} \right] \\ & + \left[E_{11}^l E_{00}^{l+1} + E_{22}^l E_{11}^{l+1} + E_{00}^l E_{22}^{l+1} \right] . \end{aligned} \tag{4.1}$$

[2]Excitations can be created if two states of the same number are next to each other. For example the state $|112012\rangle$ has energy three and the next highest energy state is $|111112\rangle$ with energy 4×3. The state with the highest energy, equals to 6×3, is $|111111\rangle$.

A natural question to ask is if the phases can also be transformed away in a generic Hamiltonian of the form (3.3). If both q and h are present we can not, at least in any simple way, transform away the phase of the complex variables. However, when $q = re^{\pm 2\pi i/3}$ it is possible to do a position dependent coordinate transformation

$$|\tilde{0}\rangle_k = e^{i2\pi/3}|0\rangle_k\,, \qquad |\tilde{1}\rangle_k = e^{i2k\pi/3}|1\rangle_k\,, \qquad \text{and} \qquad |\tilde{2}\rangle_k = e^{-i2k\pi/3}|2\rangle_k\,, \tag{4.2}$$

as in [16][3] so that the phase of q is transformed away. Here, k refers to the site of the spin-chain state. This transformation changes the generators in the Hamiltonian as

$$\tilde{E}^l_{n,n+m} = e^{\frac{i2\pi ml}{3}} E^l_{n,n+m}\,. \tag{4.3}$$

This kind of transformation of basis generally results in twisted boundary conditions. Thus, the periodic boundary condition $|a\rangle_0 = |a\rangle_L$ for the original basis becomes in the new basis

$$|\tilde{0}\rangle_0 = |\tilde{0}\rangle_L\,, \qquad |\tilde{1}\rangle_0 = e^{\frac{i2\pi L}{3}}|\tilde{1}\rangle_L\,, \qquad \text{and} \qquad |\tilde{2}\rangle_0 = e^{\frac{-i2\pi L}{3}}|\tilde{2}\rangle_L\,, \tag{4.4}$$

where L is the length of the spin chain. A consequence is that the system is invariant under a rotation of q by introducing appropriate twisted boundary conditions (4.4). As an example, the q-deformed Hamiltonian with periodic boundary conditions with $q = he^{i2\pi n/3} + 1$ (see text above (3.11)), is equivalent to $qe^{i2\pi m/3} = he^{i2\pi n/3} + 1$ with twisted boundary conditions. Hence, the following cases are integrable

$$h = \rho e^{\frac{i2\pi n}{3}}\,, \quad q = (1+\rho)e^{\frac{i2\pi m}{3}} \qquad \text{and} \qquad q = -e^{\frac{i2\pi m}{3}}\,, \quad h = e^{\frac{i2\pi n}{3}}\,, \tag{4.5}$$

where ρ is real and can take both negative and positive values and n and m are arbitrary independent integers.

One can actually combine the twist transformation above with the shift of basis (3.7) in a non-trivial way. This combination will turn out to give a relation which maps the Hamiltonian with arbitrary q and vanishing h into the Hamiltonian with vanishing q and arbitrary h. The periodic boundary condition will, however, change for spin chains where the length is not a multiple of three.

In terms of matrices the transformation can be represented as follows. Let us represent the shift of basis (3.7) by the matrix T (with n set to zero)

$$T = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i2\pi/3} & e^{-i2\pi/3} \\ 1 & e^{-i2\pi/3} & e^{i2\pi/3} \end{pmatrix}\,, \tag{4.6}$$

and the transformation matrix related to the phase shift (4.2) by (but without the phase-shift in the zero state $|0\rangle$)

$$U_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i2\pi k/3} & 0 \\ 0 & 0 & e^{-i2\pi k/3} \end{pmatrix}\,. \tag{4.7}$$

[3]Note that the phase factor in $|0\rangle$ is not position dependent, it was only added in order to cancel the extra phase which would have appeared in front of the terms having h in them.

The transformation that takes the q-deformed to the h-deformed Hamiltonian is then

$$\widetilde{H} = T_1 H T_1^{-1}\,, \tag{4.8}$$

where

$$T_1 = (T \otimes T)(U_k \otimes U_{k+1})(T^{-1} \otimes T^{-1})\,. \tag{4.9}$$

Acting with this transformation on the Hamiltonian (3.3) we get the new Hamiltonian

$$\begin{aligned}\widetilde{H}^{l,l+1} &= q^* q E^l_{i,i} E^{l+1}_{i+1,i+1} - h q^* E^l_{i+1,i} E^{l+1}_{i,i+1} - h^* q E^l_{i,i+1} E^{l+1}_{i+1,i} \\ &+ h h^* E^l_{i+1,i+1} E^{l+1}_{i,i} + h E^l_{i+1,i+2} E^{l+1}_{i,i+2} + h^* E^l_{i+2,i+1} E^{l+1}_{i+2,i} \\ &- q^* E^l_{i+2,i} E^{l+1}_{i+2,i+1} - q E^l_{i,i+2} E^{l+1}_{i+1,i+2} + E^l_{i,i} E^{l+1}_{i,i}\,,\end{aligned} \tag{4.10}$$

Up to an overall factor, the transformation (4.8) change the couplings as

$$q \neq 0 \quad \text{and} \quad h = 0 \quad \Longleftrightarrow \quad \tilde{q} = 0 \quad \text{and} \quad \tilde{h} = -1/q \tag{4.11}$$

In terms of states, the map (4.8) generates the following change

$$|a\rangle_{1+3k} \to |a-1\rangle_{1+3k}\,, \quad |a\rangle_{2+3k} \to |a+1\rangle_{2+3k}\,, \quad |a\rangle_{3k} \to |a\rangle_{3k}\,, \tag{4.12}$$

where a takes the values 0, 1 or 2. Let us investigate how the transformation (4.10) affect the boundary conditions. From equation (4.12) we see that the original periodic boundary conditions $|a\rangle_0 = |a\rangle_L$ translate into

$$|0^{new}\rangle_0 = |2^{new}\rangle_L, \qquad |1^{new}\rangle_0 = |0^{new}\rangle_L \qquad \text{and} \qquad |2^{new}\rangle_0 = |1^{new}\rangle_L\,, \tag{4.13}$$

if the length L of the spin chain is one modulo three and the opposite, $|0^{new}\rangle_0 = |1^{new}\rangle_L$ *etc*, for the two modulo three case. If the length is a multiple of three the boundary conditions remain the same.

If we start from the Hamiltonian of the γ-deformed SYM (4.1), the transformation (4.8) leads to the Hamiltonian

$$\begin{aligned}H^{l,l+1} &= \left[E^l_{00}E^{l+1}_{11} + E^i_{11}E^{l+1}_{22} + E^i_{22}E^{l+1}_{00}\right] \\ &- \left[e^{-i\gamma_{3-l}}E^l_{20}E^{l+1}_{21} + e^{-i\gamma_{1-l}}E^l_{01}E^{l+1}_{02} + e^{-i\gamma_{2-l}}E^l_{12}E^{l+1}_{10}\right] \\ &- \left[e^{i\gamma_{3-l}}E^l_{02}E^{l+1}_{12} + e^{i\gamma_{1-l}}E^l_{10}E^{l+1}_{20} + e^{i\gamma_{2-l}}E^l_{21}E^{l+1}_{01}\right] \\ &+ \left[E^l_{00}E^{l+1}_{00} + E^l_{11}E^{l+1}_{11} + E^l_{22}E^{l+1}_{22}\right]\,.\end{aligned} \tag{4.14}$$

This Hamiltonian describes interactions which differ from systems we have previously encountered, since here the interactions are site dependent. This behavior shows up naturally in a non-commutative theory. In [17], it was discussed that the γ-deformed SYM corresponds to a form of non-commutative deformation of $\mathcal{N} = 4$ SYM.

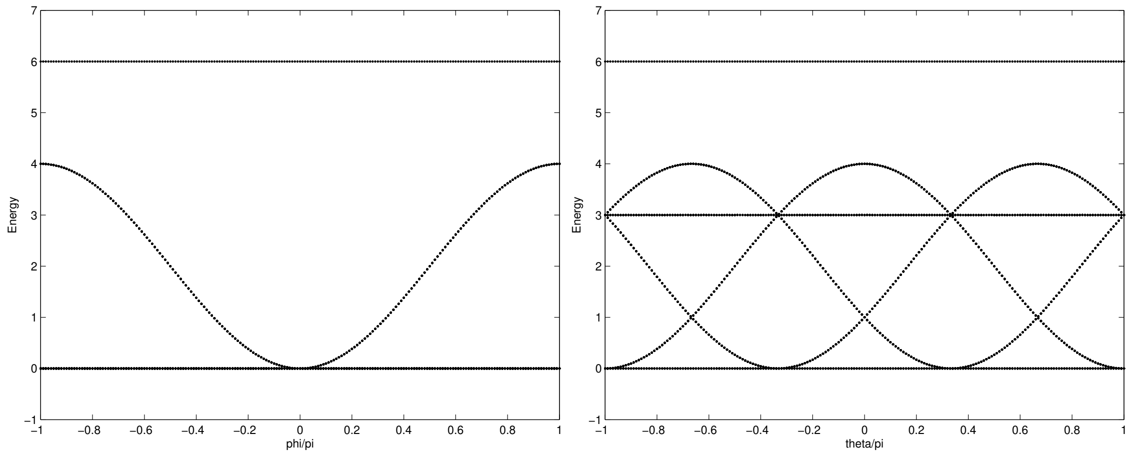

Figure 1: Spin chain with three sites. The left graph shows the energy spectrum as a function of the phase ϕ, when $q = e^{i\pi\phi}$ and $h = 0$. The right graph shows the spectrum as a function of the phase $\tilde{\theta}$, when $\tilde{h} = e^{i\tilde{\theta}}$ and $\tilde{q} = 0$.

If all the γ_i are equal, the Hamiltonian above will corresponds to our original Hamiltonian (3.3) with $q = 0$ and $h = e^{i\theta}$. The associated R-matrix is

$$\begin{aligned} R(u) = &\left[E^i_{01}E^{i+1}_{10} + E^i_{12}E^{i+1}_{21} + E^i_{20}E^{i+1}_{02}\right] \\ -&\ ue^{-i\theta}\left[E^i_{20}E^{i+1}_{21} + E^i_{01}E^{i+1}_{02} + E^i_{12}E^{i+1}_{10}\right] \\ -&\ ue^{i\theta}\left[E^i_{12}E^{i+1}_{02} + E^i_{20}E^{i+1}_{10} + E^i_{01}E^{i+1}_{21}\right] \\ +&\left[E^i_{00}E^{i+1}_{00} + E^i_{11}E^{i+1}_{11} + E^i_{22}E^{i+1}_{22}\right] \\ +&\ (1-u)\left[E^i_{10}E^{i+1}_{01} + E^i_{21}E^{i+1}_{12} + E^i_{02}E^{i+1}_{20}\right] . \end{aligned} \tag{4.15}$$

We have checked explicit that (4.15) satisfies the Yang-Baxter equation. This means that the theory is integrable!

In the rest of this section we will discuss the spectrum when the spin-chain Hamiltonian (3.3) is either q-deformed or h-deformed. Figure 1 shows the spectrum for a three-site spin-chain Hamiltonian. The left graph shows how the energy depends on the phase ϕ, with $q = e^{i\phi}$ and $h = 0$. The right graph shows instead how the eigenvalues vary as a function $\tilde{\theta}$, when $\tilde{h} = e^{i\tilde{\theta}}$ and $\tilde{q} = 0$. Figure 2 shows the same spectra for a four-site spin chain. All graphs contain energies which are the eigenvalues of several states. Highly degenerate states are generally a sign of integrability because they reflect a large number of symmetries in the theory.

Let us start by explaining the spectra in Figure 1. When h is zero there is only one sinus curve while when q is zero there are three sinus curves. The reason is the transformation (4.11), since it maps $q = e^{i\phi}$ and $h = 0$ into $\tilde{h} = e^{i\tilde{\theta}}$ and $\tilde{q} = 0$ with the relation of the phases $\tilde{\theta} = \pi - \phi + 2\pi n/3$. Therefore, for each value of q there exist several values of $\tilde{h}$ which differ by a phase $2\pi/3$. For $q = 0$, there is a state, independent of the phase, with energy three. This state is absent for $h = 0$. One example of such a state is $|000\rangle - |111\rangle$. The "inverse" transformation, see (4.12), of this state is $|120\rangle - |201\rangle$, which is zero due to

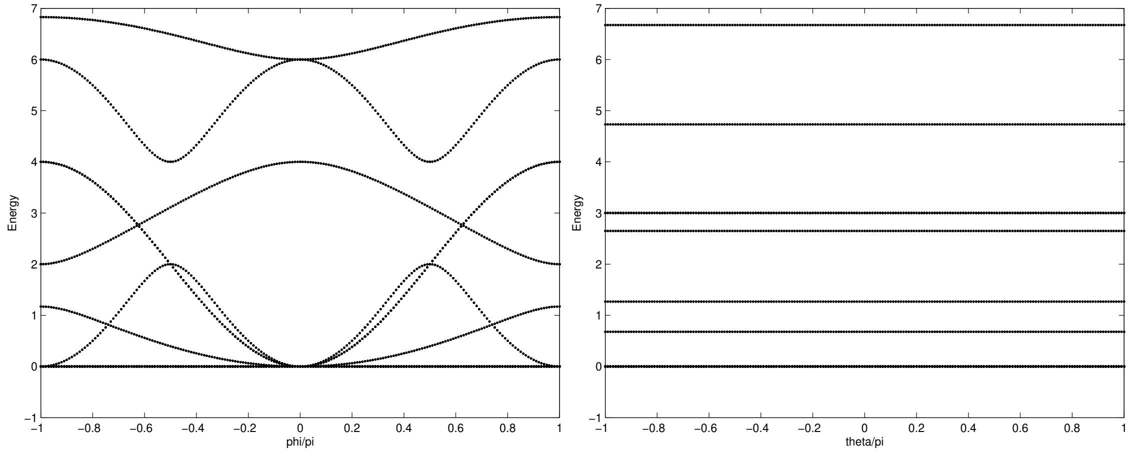

Figure 2: Spin chain with four sites. The left graph shows the energy spectrum as a function of the phase ϕ, when $q = e^{i\pi\phi}$ and $h = 0$. The right graph shows the spectrum as a function of the phase $\tilde{\theta}$, when $\tilde{h} = e^{i\tilde{\theta}}$ and $\tilde{q} = 0$.

periodicity.

The four-site spin chain (see Figure 2) differs substantially from the spin chain with three sites. The case $q = 0$ is completely phase-independent. The reason is the boundary conditions. Actually, spin chains with the number of sites differing from multiples of three will have spectra which do not depend on the phase. It will just coincide with the spectra for the case $q = e^{-i2\pi/3}$ and $h = 0$. Starting with the case q equal to a root of unity it is possible to make a transformation, changing the boundary conditions, such that the phase of q is removed [16]. The change in the boundary conditions is then

$$|0^o\rangle_0 = |0^o\rangle_L\,, \qquad |1^o\rangle_0 = e^{i\Phi}|1^o\rangle_L \qquad \text{and} \qquad |2^o\rangle_0 = e^{-i\Phi}|2^o\rangle_L\,, \tag{4.16}$$

where Φ is a phase factor, the exact form of which is not important for our purposes. The effect (4.16) has on the boundary conditions (4.13) is, when L is one modulo three,

$$|0^{new}\rangle_0 = |2^{new}\rangle_L\,, \qquad |1^{new}\rangle_0 = e^{i\Phi}|0^{new}\rangle_L \qquad \text{and} \qquad |2^{new}\rangle_0 = e^{-i\Phi}|1^{new}\rangle_L\,. \tag{4.17}$$

If we make the shift $|1^{new}\rangle \to e^{i\Phi}|1^{new}\rangle$ we see that this corresponds to the boundary conditions (4.13). The same procedure can be made when L is two modulo three. This means that any q equal to root of unity[4] can be mapped to any $\tilde{h}$ with the phase $\tilde{\theta} = \pi + 2\pi p/n + 2\pi m/3$. All values of h will then give the same energy spectrum due to the fact that p,n and m are arbitrary integer numbers, so the possible values of $\tilde{\theta}$ will in principle fill up the real axis. This implies that the energy must be the same for all values of $\tilde{\theta}$. For $q = e^{-i2\pi/3}$ and $h = 0$ there is a direct map (see (3.9)) to the case $q = 0$ and $h = -e^{2\pi m/3}$ which does not change the boundary conditions. The energy spectra for these two cases must be the same. Consequently, the spectra for "all" points coincide with the spectrum of $q = e^{-i2\pi/3}$ and $h = 0$. The fact that the shape of the eigenvalue distribution

[4] $q = e^{i\phi}$ is a root of unity iff $n\phi = 0 \bmod 2\pi$ for n an integer. The phase is then $\phi = 2\pi p/n$ where p is an integer.

changes drastically depending on how many sites there are suggests that a well-defined large L-limit does not exist. However, it might still be possible to find a well-defined large L-limit if only L-multiples of three is considered.

5. R-matrix

We will now try to make a general ansatz for an R-matrix which has the possibility to give rise to our Hamiltonian (3.3). A linear ansatz will turn out to lead to the cases we found in the previous section. To find a new solution the ansatz need to be more complicated, for instance consisting of hyper-elliptic functions. We are interested in an R-Matrix of the following form

$$\begin{aligned} R(u) &= aE_{i,i}\otimes E_{i,i} + bE_{i,i}\otimes E_{i+1,i+1} + \bar{b}E_{i+1,i+1}\otimes E_{i,i} \\ &+ cE_{i,i+1}\otimes E_{i+1,i} + \bar{c}E_{i+1,i}\otimes E_{i,i+1} \\ &+ dE_{i+1,i+2}\otimes E_{i,i+2} + \bar{d}E_{i+2,i+1}\otimes E_{i+2,i} \\ &+ eE_{i+2,i}\otimes E_{i+2,i+1} + \bar{e}E_{i,i+2}\otimes E_{i+1,i+2}, \end{aligned} \tag{5.1}$$

where the coefficients are functions of a spectral parameter u.

Written on matrix form the R-matrix is

$$R = \begin{pmatrix} a & 0 & 0 & 0 & 0 & e & 0 & \bar{d} & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 & e \\ 0 & 0 & \bar{b} & 0 & d & 0 & \bar{c} & 0 & 0 \\ 0 & \bar{c} & 0 & \bar{b} & 0 & 0 & 0 & 0 & d \\ 0 & 0 & \bar{d} & 0 & a & 0 & e & 0 & 0 \\ \bar{e} & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ 0 & 0 & c & 0 & \bar{e} & 0 & b & 0 & 0 \\ d & 0 & 0 & 0 & 0 & \bar{c} & 0 & \bar{b} & 0 \\ 0 & e & 0 & \bar{d} & 0 & 0 & 0 & 0 & a \end{pmatrix}. \tag{5.2}$$

A natural first step to look for a R-matrix solution is to make a linear ansatz which will give the Hamiltonian (3.3) as in (3.13).

The Hamiltonian can also be defined through the permuted $\mathfrak{R}$-matrix

$$\mathfrak{R} \equiv \mathcal{P}R, \tag{5.3}$$

where $\mathcal{P}$ is the 9×9 permutation matrix.

If $R(u)|_{u=u_0} = \mathcal{P}$ or $\mathfrak{R}(u)|_{u=u_0} = \mathcal{P}$, the Hamiltonian is obtained as

$$H = \mathcal{P}\frac{d}{du}R(u)|_{u=u_0} \quad \text{or} \quad H = \mathcal{P}\frac{d}{du}\mathfrak{R}(u)|_{u=u_0}. \tag{5.4}$$

The linear ansatz below has the property that it gives the Hamiltonian (3.3) in accordance with the first formula in (5.4)

$$\begin{aligned} a(u) &= (h^*h-k)u+\alpha\,, & b(u) &= -qu\,, & \bar{d}(u) &= -q^*hu\,, \\ c(u) &= (q^*q-k)u+\alpha\,, & \bar{b}(u) &= -q^*u\,, & e(u) &= hu\,, \\ \bar{c}(u) &= (1-k)u+\alpha\,, & d(u) &= h^*u\,, & \bar{e}(u) &= -qh^*u\,, \end{aligned} \tag{5.5}$$

with k and α being free parameters, the Yang-Baxter equations turn out to be independent of α while they demand k to be $k = \frac{1}{2}(1 + h^*h + q^*q)$. Inserting the linear ansatz in the Yang-Baxter equation we find that the equation is satisfied either if

$$q = e^{i\phi} \text{ and } h = 0 \quad \text{or} \quad q = 0 \text{ and } h = e^{i\theta} \,, \tag{5.6}$$

where ϕ and θ can be any phase, or if the following equations holds

$$e^{i3\phi} r = \left(1 + \rho e^{i3\theta}\right) \,, \tag{5.7}$$

$$e^{-i3\phi} r = \left(1 + \rho e^{-i3\theta}\right) \,, \tag{5.8}$$

$$r = \pm(1 \pm \rho) \,, \tag{5.9}$$

where we used the notation that $q = re^{i\phi}$ and $h = \rho e^{i\theta}$ and let r and ρ be any real numbers. Here we immediately see that the relations between the real parts of q and h are given by the last equation, hence we only need to consider which angles are not in contradiction to that. The solution is

$$q = re^{i2\pi n/3} \,, \qquad h = (1 + r)e^{i2\pi m/3} \,, \tag{5.10}$$

where we once again let r take any real number. Now we would like to see whether there exist solutions if an ansatz is made with the permuted version of the R-matrix ansatz (5.5). We obtain

$$\begin{aligned}
a(u) &= (h^*h - k)u + \alpha \,, & c(\mu) &= -q^*u \,, & e(u) &= hu \,, \\
b(u) &= (1 - k)u + \alpha \,, & \bar{c}(u) &= -qu \,, & \bar{d}(u) &= -q^*hu \,, \\
\bar{b}(u) &= (q^*q - k)u + \alpha \,, & d(u) &= -qh^*u \,, & \bar{e}(u) &= h^*u \,.
\end{aligned} \tag{5.11}$$

The conditions from the Yang-Baxter equation read

$$q^* = -q^2 \,, \qquad h^* = h^2 \,, \tag{5.12}$$

with no restriction on k and α. The only solution to this is

$$q = -e^{2\pi n/3} \,, \qquad h = e^{2\pi m/3} \,, \tag{5.13}$$

(or $q = 0$ and $h = 0$). This is the other type of solution we expected from the last section. The one corresponding to $q = -1$ and $h = e^{i2\pi m/3}$ and that one but with twisted boundary conditions. Hence a R-matrix with a linear dependence on the spectral parameter u can not give us more integrable cases than already found. We need a more general R-matrix solution to find new interesting cases.

5.1 Symmetries revealed

In order to address the problem of finding the most general solution for the R-matrix (5.2) it is an advantage to make use of the symmetries. We choose the representation

$$R = \sum_{i=1}^{3} (\omega_i T_i \otimes S_i + \bar{\omega}_i S_i \otimes T_i + \gamma_i E_i \otimes E_{2i}) \,. \tag{5.14}$$

All indices in this section are modulo three if not otherwise stated. The generators S_i, T_i and E_i are

$$
\begin{aligned}
S_1 &= \begin{pmatrix} 0 & 0 & 1 \\ e^{-\frac{i2\pi}{3}} & 0 & 0 \\ 0 & e^{\frac{i2\pi}{3}} & 0 \end{pmatrix}, \quad
S_2 = \begin{pmatrix} 0 & 0 & 1 \\ e^{\frac{i2\pi}{3}} & 0 & 0 \\ 0 & e^{-\frac{i2\pi}{3}} & 0 \end{pmatrix}, \quad
S_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
T_1 &= \begin{pmatrix} 0 & e^{\frac{i2\pi}{3}} & 0 \\ 0 & 0 & e^{-\frac{i2\pi}{3}} \\ 1 & 0 & 0 \end{pmatrix}, \quad
T_2 = \begin{pmatrix} 0 & e^{-\frac{i2\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{i2\pi}{3}} \\ 1 & 0 & 0 \end{pmatrix}, \quad
T_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\
E_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{i2\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{i2\pi}{3}} \end{pmatrix}, \quad
E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{i2\pi}{3}} & 0 \\ 0 & 0 & e^{-\frac{i2\pi}{3}} \end{pmatrix}, \quad
E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
\tag{5.15}
$$

How the functions in the R-Matrix (5.2) are expressed in terms of the functions ω_i, $\bar{\omega}_i$ and γ_i can be found in Appendix (A.1). The generators are related by

$$
\begin{array}{lll}
S_k S_l = e^{-i\frac{2\pi(l-k)}{3}} T_{2k-l} & S_k T_l = E_{k-l} & S_k E_l = e^{i\frac{2\pi l}{3}} S_{k+l} \\
T_k S_l = e^{-\frac{i2\pi(l-k)}{3}} E_{l-k} & T_k T_l = e^{-i\frac{2\pi(l-k)}{3}} S_{2k-l} & T_k E_l = T_{k-l} \\
E_k S_l = S_{k+l} & E_k T_l = e^{i\frac{2\pi}{3}k} T_{l-k} & E_k E_l = E_{k+l}
\end{array}
\tag{5.16}
$$

Using these relations it is straightforward to obtain the Yang-Baxter equations which can be found in Appendix (A.2). A nice feature of these equations is that all of them, except the fourth, the fifth and the sixth, can be generated from the first equation through the cyclic permutations $\omega_{n+1} \to \bar{\omega}_{n+1} \to \gamma_3$ and $\gamma_2 \to \omega_n \to \bar{\omega}_n \to \gamma_1 \to \omega_{n+2} \to \bar{\omega}_{n+2}$. The remaining three equations are related to each other by the same cyclic permutation. The structure of the equations (A.2) is similar to the Yang-Baxter equations in the eight vertex model [29, 30]

$$
\omega_n \omega_l' \omega_j'' - \omega_l \omega_n' \omega_k'' + \omega_j \omega_k' \omega_n'' - \omega_k \omega_j' \omega_l'' = 0 \, , \tag{5.17}
$$

for all cyclic permutations (j, k, l, n) of (1,2,3,4). These equations can neatly be represented by writing the elements in rectangular objects

$$
\begin{array}{|c|c|c|c|}
\hline
\omega_{\mathbf{n}} & \omega_l & \omega_{\mathbf{j}} & \omega_k \\
\hline
\omega_l & \omega_{\mathbf{n}} & \omega_k & \omega_{\mathbf{j}} \\
\hline
\omega_{\mathbf{j}} & \omega_k & \omega_{\mathbf{n}} & \omega_l \\
\hline
\omega_k & \omega_{\mathbf{j}} & \omega_l & \omega_{\mathbf{n}} \\
\hline
\omega_{\mathbf{n}} & \omega_l & \omega_{\mathbf{j}} & \omega_k \\
\hline
\end{array} \, . \tag{5.18}
$$

Note the beautiful toroidal pattern. The object above should be interpreted as follows. The first three rows represent the equation (5.17) with the first column representing the first term in (5.17)

$$
\begin{array}{|c|}
\hline
\omega_{\mathbf{n}} \\
\hline
\omega_l \\
\hline
\omega_{\mathbf{j}} \\
\hline
\end{array} = \omega_n \omega_l' \omega_j'' \, , \tag{5.19}
$$

and the next column is equal to the second term in (5.17)

$$\begin{array}{|c|}\hline \omega_l \\ \hline \omega_{\mathbf{n}} \\ \hline \omega_k \\ \hline \end{array} = -\omega_l \omega_n' \omega_k'' \,. \tag{5.20}$$

The next three rows represent another equation of eight vertex model

$$\begin{array}{|c|c|c|c|}\hline \omega_l & \omega_{\mathbf{n}} & \omega_k & \omega_{\mathbf{j}} \\ \hline \omega_{\mathbf{j}} & \omega_k & \omega_{\mathbf{n}} & \omega_l \\ \hline \omega_k & \omega_{\mathbf{j}} & \omega_l & \omega_{\mathbf{n}} \\ \hline \end{array} = \omega_l \omega_j' \omega_k'' - \omega_n \omega_k' \omega_j'' + \omega_k \omega_n' \omega_l'' - \omega_j \omega_l' \omega_n'' = 0 \,. \tag{5.21}$$

Our equations can also be represented in terms of similar rectangular objects, with the same toroidal pattern

$$\begin{array}{|c|c|c|c|c|c|}\hline \boldsymbol{\omega_2} & \omega_1 & \boldsymbol{\bar{\omega}_2} & \gamma_1 & \boldsymbol{\gamma_3} & \bar{\omega}_3 \\ \hline \omega_1 & \boldsymbol{\omega_2} & \gamma_1 & \boldsymbol{\bar{\omega}_2} & \bar{\omega}_3 & \boldsymbol{\gamma_3} \\ \hline \boldsymbol{\gamma_3} & \gamma_1 & \boldsymbol{\omega_2} & \bar{\omega}_3 & \boldsymbol{\bar{\omega}_2} & \omega_1 \\ \hline \gamma_1 & \boldsymbol{\gamma_3} & \bar{\omega}_3 & \boldsymbol{\omega_2} & \omega_1 & \boldsymbol{\bar{\omega}_2} \\ \hline \boldsymbol{\bar{\omega}_2} & \bar{\omega}_3 & \boldsymbol{\gamma_3} & \omega_1 & \boldsymbol{\omega_2} & \gamma_1 \\ \hline \end{array} \,. \tag{5.22}$$

The first three rows give the second equation in (A.2) with $n = 3$. The next three rows are the seventh equation in (A.2) with $n = 1$. This suggests that the system of equations (A.2) should have a nice solution, just like the eight vertex model. The first row determines the rest of the entries, thus all equations can be represented with just the upper row. Hence, all the 36 equations can be represented by the following rows

$$\begin{array}{|c|c|c|c|c|c|}\hline \omega_{n+1} & \omega_n & \bar{\omega}_{n+1} & \gamma_1 & \gamma_3 & \bar{\omega}_{n+2} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline \omega_{n+1} & \omega_n & \bar{\omega}_n & \gamma_2 & \gamma_1 & \bar{\omega}_{n+1} \\ \hline \end{array}$$
$$\begin{array}{|c|c|c|c|c|c|}\hline \omega_n & \omega_{n+1} & \bar{\omega}_n & \gamma_2 & \gamma_3 & \bar{\omega}_{n+2} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline \omega_2 & \gamma_2 & \omega_1 & \gamma_1 & \omega_3 & \gamma_3 \\ \hline \end{array} \tag{5.23}$$
$$\begin{array}{|c|c|c|c|c|c|}\hline \bar{\omega}_2 & \omega_2 & \bar{\omega}_1 & \omega_1 & \bar{\omega}_3 & \omega_3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline \gamma_1 & \bar{\omega}_2 & \gamma_2 & \bar{\omega}_1 & \gamma_3 & \bar{\omega}_3 \\ \hline \end{array} \,.$$

The solution to the eight vertex model is a product of theta functions. The cyclicity and periodicity properties of the eight vertex model is mirrored into the rectangular object. Due to the combination of addition theorems for theta functions and the intrinsic properties of the equations, the rectangular objects make it easy to see if an ansatz solves all the equations. We believe that the addition theorems for theta functions generating the solution of the eight vertex model should be possible to generalize to any even sized rectangular object. It would then be interesting to see if those equations are related to an R-matrix of arbitrary dimension.

5.2 A hyperbolic solution

If the following ansatz $\omega_i = e^{uQ_i}$, $\bar{\omega}_i = e^{u\bar{Q}_i}$ and $\gamma_i = e^{uK_i}$, where we let Q_i, $\bar{Q}_i$ and K_i be arbitrary constants is made, it leads us to the following solution

$$\begin{aligned} \omega_1 &= e^{Q_1 u} \,, \quad & \bar{\omega}_1 &= e^{Q_2 u} \,, \quad & \gamma_1 &= e^{Q_2 u} \,, \\ \omega_2 &= e^{Q_2 u} \,, \quad & \bar{\omega}_2 &= e^{Q_1 u} \,, \quad & \gamma_2 &= e^{Q_1 u} \,, \\ \omega_3 &= e^{Q_3 u} \,, \quad & \bar{\omega}_3 &= e^{Q_3 u} \,, \quad & \gamma_3 &= e^{Q_3 u} \,., \end{aligned} \tag{5.24}$$

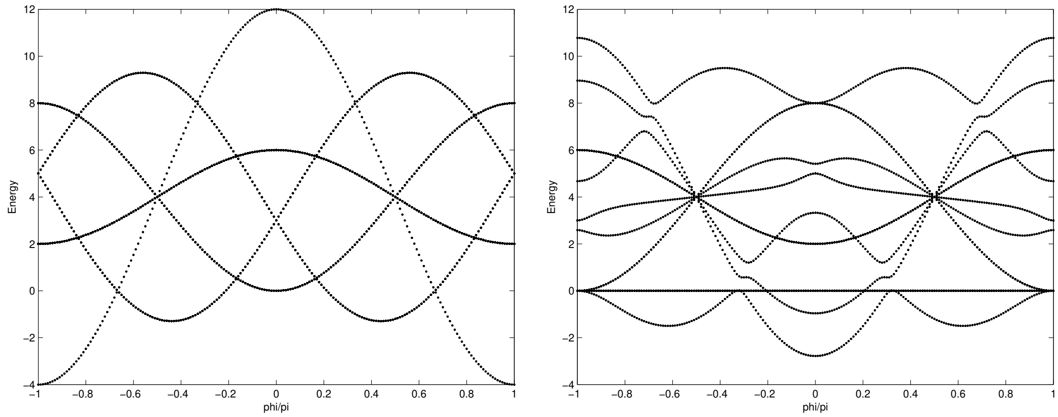

Figure 3: The eigenvalue dependence on the phase for the the Hamiltonian (5.25).

The following Hamiltonian is obtained from the above R-matrix solution

$$
\begin{aligned}
H^{l,l+1} &= E^l_{i,i} \otimes E^{l+1}_{i+1,i+1} + e^{i\phi} E^l_{i+1,i} \otimes E^{l+1}_{i,i+1} + e^{-i\phi} E^l_{i,i+1} \otimes E^{l+1}_{i+1,i} \\
&+ E^l_{i+1,i+1} \otimes E^{l+1}_{i,i} + e^{-i\phi} E^l_{i+1,i+2} \otimes E^{l+1}_{i,i+2} + e^{i\phi} E^l_{i+2,i+1} \otimes E^{l+1}_{i+2,i} \\
&+ e^{-i\phi} E^l_{i+2,i} \otimes E^{l+1}_{i+2,i+1} + e^{i\phi} E^l_{i,i+2} \otimes E^{l+1}_{i+1,i+2} + E^l_{i,i} \otimes E^{l+1}_{i,i}.
\end{aligned} \tag{5.25}
$$

where $e^{i\phi} = (Q_2 e^{i2\pi/3} + Q_1 e^{-i2\pi/3})/(Q_1^2 + Q_2^2 - Q_1 Q_2)$ (we put Q_3 to zero because it does not give us any more information). Here we also made use of the fact that the Hamiltonian obtained from the procedure (3.13) can be rescaled plus that something proportional to the identity matrix can be added. Actually this Hamiltonian can be related with the transformation (3.9) to a completely diagonal Hamiltonian, such that it is included in the integrable models mentioned in [18]. In figure 3 the graph to the left shows how the energy eigenvalues of the Hamiltonian (5.25) depends on the phase ϕ. The graph to the right shows the eigenvalues, of the Hamiltonian if we change the sign in front of the second and third term in (5.25), depending on the phase ϕ. The graph to the right looks very amusing. It looks very similar to the graph to the left if that is turned upside down and deformed in a considerable symmetrical way.

6. Broken $\mathbf{Z_3} \times \mathbf{Z_3}$ symmetry

Relaxing the one-loop finiteness condition (2.2), by choosing $h^{000} = h^{222} = 0$ and $h^{111} \propto h$ in the superpotential (2.3) breaks the $Z_3 \times Z_3$ symmetry. The superpotential is

$$
W \propto \mathrm{Tr}\left[\Phi_0 \Phi_1 \Phi_2 - q \Phi_1 \Phi_0 \Phi_2 + \frac{h}{3} \Phi_1^3\right], \tag{6.1}
$$

where an overall factor is excluded. This superpotential is actually easier to study since the dilatation operator has homogeneous vacua $|0\rangle|0\rangle \dots |0\rangle$ and $|2\rangle|2\rangle \dots |2\rangle$. The mixing-matrix for the anomalous dimensions has the form of a spin-chain Hamiltonian arising from R-matrices found by Fateev-Zamolodchikov (or XXZ) [31] and the Izergin-Korepin [32]. This type of models were considered in [33] even though the authors never completely classified them. They have a $U(1)$-symmetry which can be used to get rid of the phase in the complex variable h.

In this setting, there is no longer a cancelation between the fermion loop and the scalar self-energy. The additional contribution to the Hamiltonian is of the form (B.5) (see Appendix B for details). The spin chain obtained from the superpotential 6.1 is, with $q=-1$,

$$H = \left(\begin{array}{ccc|ccc|ccc}
0 & & & & & & & & \\
 & 1-\frac{h^*h}{2} & & 1 & & & & & \\
 & & 1 & & h^* & & 1 & & \\
\hline
 & 1 & & 1-\frac{h^*h}{2} & & & & & \\
 & & h & & 0 & & h & & \\
 & & & & & 1-\frac{h^*h}{2} & & 1 & \\
\hline
 & & 1 & & h^* & & 1 & & \\
 & & & & & 1 & & 1-\frac{h^*h}{2} & \\
 & & & & & & & & 0
\end{array}\right). \tag{6.2}$$

The term $h^*h/2$, is the fermion loop contribution from the self-energy. We will show that for the special values $q=-1$ and $h=e^{i\phi}\sqrt{2}$, this Hamiltonian can be obtained from the spin-1 XXZ R-matrix. The phase of h is redundant, the energy does not depend on it, and can be phased away through the transformation $|\tilde{1}\rangle = e^{-i\phi/2}|1\rangle$. The R-matrix for the XXZ-model is [31]

$$R(u) = \left(\begin{array}{ccc|ccc|ccc}
s & & & & & & & & \\
 & t & & r & & & & & \\
 & & T & & a^* & & R & & \\
\hline
 & r & & t & & & & & \\
 & & a & & \sigma & & a & & \\
 & & & & & t & & r & \\
\hline
 & & R & & a^* & & T & & \\
 & & & & & r & & t & \\
 & & & & & & & & s
\end{array}\right) \qquad
\begin{array}{l}
s = 1 \\
t = \epsilon J \sinh(u) \\
r = J \sinh 2\eta \\
a = e^{i\phi} J \frac{\sinh u \sinh 2\eta}{\sinh(u+\eta)} \\
R = J \frac{\sinh \eta \sinh 2\eta}{\sinh(u+\eta)} \\
T = J \frac{\sinh u \sinh(u-\eta)}{\sinh(u+\eta)} \\
\sigma = \epsilon t + R \\
J = \frac{1}{\sinh(u+2\eta)}
\end{array} \tag{6.3}$$

where $\epsilon = \pm 1$. The ϵ in t in (6.3) is added after checking that the R-matrix still satisfies Yang-Baxter equation. If we put $u=0$, the R-matrix becomes the permutation matrix. Thus, a Hamiltonian can be obtained from the R-matrix by the usual procedure $H = PR'|_{u=0}$. Performing the derivatives at the point $u=0$ gives

$$\begin{aligned}
&s' = 0\,, \qquad t' = \epsilon\frac{1}{\sinh 2\eta}\,, \qquad r' = -\frac{\cosh 2\eta}{\sinh 2\eta}\,, \qquad a' = e^{i\phi}\frac{1}{\sinh\eta}\,, \\
&R' = -\frac{\cosh\eta}{\sinh\eta} - \frac{\cosh 2\eta}{\sinh 2\eta}\,, \qquad T' = -\frac{1}{\sinh 2\eta}\,, \qquad \sigma' = \epsilon t' + R'\,.
\end{aligned} \tag{6.4}$$

Multiplying all parameters with $\sinh 2\eta$, the new variables, evaluated at $\eta = \pi/4$, leads to

$$\tilde{s}' = 0\,, \quad \tilde{t}' = -1\,, \quad \tilde{r}' = 0\,, \quad \tilde{a}' = e^{i\phi}\sqrt{2}\,, \quad \tilde{R}' = -1\,, \quad \tilde{T}' = -1\,, \quad \tilde{\sigma}' = 0\,. \tag{6.5}$$

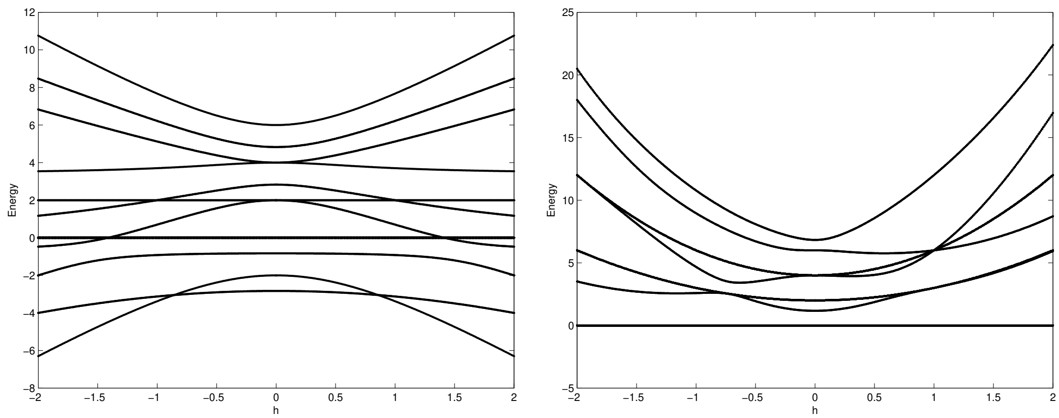

Figure 4: To the left is the spectra for the case $h^{000} = h^{222} = 0$ and $h^{111} = h$ depending on h when $q = -1$, and to the right is the spectra for the case h^{III} all equal to h (up to a constant factor) depending on h when $q = -1$

with the corresponding Hamiltonian

$$H = \left(\begin{array}{ccc|ccc|ccc}
0 & & & & & & & & \\
 & 0 & & \pm 1 & & & & & \\
 & & -1 & & e^{-i\phi}\sqrt{2} & & -1 & & \\
\hline
 & \pm 1 & & 0 & & & & & \\
 & & e^{i\phi}\sqrt{2} & & 0 & & e^{i\phi}\sqrt{2} & & \\
 & & & & & 0 & & \pm 1 & \\
\hline
 & & -1 & & e^{-i\phi}\sqrt{2} & & -1 & & \\
 & & & & & \pm 1 & & 0 & \\
 & & & & & & & & 0
\end{array}\right) \tag{6.6}$$

If we make the choice $\epsilon = -1$, this is the spin chain Hamiltonian with deformation $h = e^{i\phi}\sqrt{2}$ and $q = -1$! Looking at the left graph of Figure 4 of a four-site spin chain we see that two lines cross at this point. This might, however, just be a coincidence. A special feature with $q = -1$ is that there is a Z_2-symmetry due to the invariance under exchange of the fields Φ_0 and Φ_2.

The right graph shows the same spectrum, but with all couplings h^{III} equal to h, up to a constant factor. The point $h = 1 - \sqrt{3}$ is special, since at this point the transformation (3.9) is "self-dual" , which means here that $\tilde{q} = q$ and $\tilde{h} = h$.

7. Conclusions

We have studied the dilation operator, corresponding to the general Leigh-Strassler deformation with h non-zero of $\mathcal{N} = 4$ SYM, in order to find new integrable points in the parameter-space of couplings. In particular we have found a relationship between the γ-deformed SYM and a site dependent spin-chain Hamiltonian. When all parameters γ_i are equal, this relates an entirely q-deformed to an entirely h-deformed superpotential. For $q = 0$ and the $h = e^{i\theta}$, where θ is real, we have found a new R-matrix (see 4.15).

We found a way of representing a general ansatz for the R-matrix, with the right form to give the dilatation operator, which makes the structure of the Yang-Baxter equations clear. The equations can be represented in terms of rectangular objects, which reveals that the underlying structure is a generalized version of the structure of the eight-vertex model. We presented all values of the parameters q and h for which the spin-chain Hamiltonian can be obtained from R-matrices with a linear dependence on the spectral parameter. Most of them were related to the q-deformed case through a simple shift of basis with a real phase β, or a shift with a twist with the phase $\pm 2\pi/3$, which reflects the Z_3-symmetry.

We also found a new hyperbolic R-matrix (5.25) which, through a simple change of basis, gives a Hamiltonian with only diagonal terms which was included in the cases studied in [18]. We had a brief look at a case with broken $Z_3 \times Z_3$ symmetry and found that the matrix of anomalous dimensions can for some special values of the parameters be obtained from the Fateev-Zamolodchikov R-matrix.

We conjecture that the Yang-Baxter equations found for the general R-matrix have a solution which is a generalized version of the solution to the eight-vertex model. If this solution exists, it is plausible that there will exist more points in the parameter space for which the dilatation operator is integrable. To find a general solution to these equations would be of interest in its own right. From a mathematical point of view, it is then interesting to generalize the solution to an R-matrix of arbitrary dimension.

The found relationship between the q- and the h-deformed superpotential should be visible in the dual string theory, and should also give a clue of what that string theory looks like. Another way to approach the problem, as mentioned in [9], is to first find a coherent state spin chain and from that reconstruct the dual geometry. The coherent state spin chain [9] is valid for small β, *i.e.* $q \approx 1$. We believe that making use of the basis transformation (3.9) makes it possible to create a coherent state spin chain for $q \approx 1$ and small h acting with the transformation (3.9) on a q close to one gives a new q close to one and a new small h. We also hope that due to the relation between vanishing h and vanishing q it is possible to write a coherent sigma model for both q and h close to one. It would then be very interesting to find the dual geometry, which corresponds to a further away deformation of the $\mathcal{N} = 4$ SYM.

One other thing of interest is to extend the analysis to other sectors of the theory and to higher loop order. In the β-deformed case it is possible to argue that the integrability holds to higher loop order [8], because the dilatation operator is related with a unitary transformation to the case of the usual $\mathcal{N} = 4$ SYM. In the same way can we argue about the h-deformed case, even though we have to consider the induced effects of the spin chain periodicity.

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A. Yang-Baxter equations for the general case

The functions in the R-Matrix (5.2) are expressed in terms of the functions ω_i, $\bar{\omega}_i$ and γ_i as

$$
\begin{aligned}
a(u) &= \gamma_1(u) + \gamma_2(u) + \gamma_3(u)\,, \\
b(u) &= \gamma_1(u)e^{i2\pi/3} + \gamma_2(u)e^{-i2\pi/3} + \gamma_3(u)\,, \\
\bar{b}(u) &= \gamma_2(u)e^{i2\pi/3} + \gamma_1(u)e^{-i2\pi/3} + \gamma_3(u) \\
c(u) &= \omega_1(u) + \omega_2(u) + \omega_3(u)\,, \\
c(u) &= \bar{\omega}_1(u) + \bar{\omega}_2(u) + \bar{\omega}_3(u)\,, \\
d(u) &= \omega_2(u)e^{i2\pi/3} + \omega_1(u)e^{-i2\pi/3} + \omega_3(u)\,, \\
\bar{d}(u) &= \bar{\omega}_1(u)e^{i2\pi/3} + \bar{\omega}_2(u)e^{-i2\pi/3} + \bar{\omega}_3(u)\,, \\
e(u) &= \omega_1(u)e^{i2\pi/3} + \omega_2(u)e^{-i2\pi/3} + \omega_3(u)\,, \\
\bar{e}(u) &= \bar{\omega}_2(u)e^{i2\pi/3} + \bar{\omega}_1(u)e^{-i2\pi/3} + \bar{\omega}_3(u)\,, .
\end{aligned}
\tag{A.1}
$$

Yang-Baxter equations from the R-matrix ansatz (5.14) read

$$
\begin{aligned}
\omega_{n+1}\omega'_{n+2}\gamma''_3 - \omega_{n+2}\omega'_{n+1}\gamma''_2 + \gamma_3\bar{\omega}'_n\bar{\omega}''_{n+1} - \gamma_2\bar{\omega}'_{n+1}\bar{\omega}''_n + \bar{\omega}_{n+1}\gamma'_2\omega''_{n+1} - \bar{\omega}_n\gamma'_3\omega''_{n+2} &= 0\,, \\
\omega_{n+1}\omega'_{n+2}\gamma''_1 - \omega_{n+2}\omega'_{n+1}\gamma''_3 + \gamma_1\bar{\omega}'_{n+2}\bar{\omega}''_n - \gamma_3\bar{\omega}_n\bar{\omega}''_{n+2} + \bar{\omega}_n\gamma'_3\omega''_{n+1} - \bar{\omega}_{n+2}\gamma'_1\omega''_{n+2} &= 0\,, \\
\omega_{n+1}\omega'_{n+2}\gamma''_2 - \omega_{n+2}\omega'_{n+1}\gamma''_1 + \gamma_2\bar{\omega}'_{n+1}\bar{\omega}''_{n+2} - \gamma_1\bar{\omega}_{n+2}\bar{\omega}''_{n+1} + \bar{\omega}_{n+2}\gamma'_1\omega''_{n+1} - \bar{\omega}_{n+1}\gamma'_2\omega''_{n+2} &= 0\,, \\
\omega_1\bar{\omega}'_{n+1}\omega''_2 - \bar{\omega}_1\omega'_{2n+1}\bar{\omega}''_3 + \omega_2\bar{\omega}'_{n+2}\omega''_0 - \bar{\omega}_2\omega'_{2n-1}\bar{\omega}''_1 + \omega_0\bar{\omega}'_n\omega''_1 - \bar{\omega}_0\omega'_{2n}\bar{\omega}''_2 &= 0\,, \\
\gamma_2\omega'_{n+1}\gamma''_1 + \gamma_3\omega'_{n-1}\gamma''_2 + \gamma_1\omega'_n\gamma''_3 - \omega_1\gamma'_n\omega''_2 - \omega_2\gamma'_{n+1}\omega''_3 - \omega_3\gamma'_{n-1}\omega''_1 &= 0\,, \\
\gamma_1\bar{\omega}'_{n-1}\gamma''_2 + \gamma_2\bar{\omega}'_{n+1}\gamma''_3 + \gamma_3\bar{\omega}'_n\gamma''_1 - \bar{\omega}_1\gamma'_{n-1}\bar{\omega}''_2 - \bar{\omega}_2\gamma'_{n+1}\bar{\omega}''_3 - \bar{\omega}_3\gamma'_n\bar{\omega}''_1 &= 0\,, \\
\bar{\omega}_{n+1}\bar{\omega}'_{n+2}\omega''_{n+1} - \bar{\omega}_{n+2}\bar{\omega}'_{n+1}\omega''_n - \omega_n\gamma'_3\gamma''_1 + \omega_{n+1}\gamma'_1\gamma''_3 - \gamma_1\omega'_{n+1}\bar{\omega}''_{n+2} + \gamma_3\omega'_n\bar{\omega}''_{n+1} &= 0\,, \\
\bar{\omega}_{n+2}\bar{\omega}'_{n+1}\omega''_{n+2} - \bar{\omega}_{n+1}\bar{\omega}'_{n+2}\omega''_n - \omega_n\gamma'_3\gamma''_2 + \omega_{n+2}\gamma'_2\gamma''_3 - \gamma_2\omega'_{n+2}\bar{\omega}''_{n+1} + \gamma_3\omega'_n\bar{\omega}''_{n+2} &= 0\,, \\
\bar{\omega}_{n+1}\bar{\omega}'_{n+2}\omega''_{n+2} - \bar{\omega}_{n+2}\bar{\omega}'_{n+1}\omega''_{n+1} - \omega_{n+1}\gamma'_1\gamma''_2 + \omega_{n+2}\gamma'_2\gamma''_1 + \gamma_1\omega'_{n+1}\bar{\omega}''_{n+1} - \gamma_2\omega'_{n+2}\bar{\omega}''_{n+2} &= 0\,, \\
\omega_{n+2}\omega'_n\bar{\omega}''_{n+1} - \omega_n\omega'_{n+2}\bar{\omega}''_n + \bar{\omega}_{n+1}\gamma'_3\gamma''_1 - \bar{\omega}_n\gamma'_1\gamma''_3 + \gamma_1\bar{\omega}'_n\omega''_{n+2} - \gamma_3\bar{\omega}'_{n+1}\omega''_n &= 0\,, \\
\omega_n\omega'_{n+1}\bar{\omega}''_n - \omega_{n+1}\omega'_n\bar{\omega}''_{n+2} - \bar{\omega}_{n+2}\gamma'_3\gamma''_2 + \bar{\omega}_n\gamma'_2\gamma''_3 - \gamma_2\bar{\omega}'_n\omega''_{n+1} + \gamma_3\bar{\omega}'_{n+2}\omega''_n &= 0\,, \\
\omega_{n+2}\omega'_{n+1}\bar{\omega}''_{n+1} - \omega_{n+1}\omega'_{n+2}\bar{\omega}''_{n+2} + \bar{\omega}_{n+1}\gamma'_2\gamma''_1 - \bar{\omega}_{n+2}\gamma'_1\gamma''_2 + \gamma_1\bar{\omega}'_{n+2}\omega''_{n+2} - \gamma_2\bar{\omega}'_{n+1}\omega''_{n+1} &= 0\,,
\end{aligned}
\tag{A.2}
$$

Here, we have defined $\omega = \omega(u-v)$, $\omega' = \omega(u)$ and $\omega'' = \omega(v)$.

B. Self-energy with broken $\mathbf{Z_3} \times \mathbf{Z_3}$ symmetry

We will follow the prescription of [34] to compute the contribution to the Hamiltonian from the superpotential (6.1), when conformal invariance is broken. The additional terms are coming from the self-energy fermion loop.

The scalar self-energy of the vertices is, in $\mathcal{N}=4$ SYM,

$$
\frac{g_{YM}^2(L+1)}{8\pi^2} N : \mathrm{Tr}\left(\bar{\phi}_i\phi_i\right) :, \tag{B.1}
$$

where $L = \log x^{-2} - (1/\epsilon + \gamma + \log\pi + 2)$. The scalar-vector contribution to this is $-\frac{g_{YM}^2(L+1)}{8\pi^2}$, and the fermion loop contribution is $\frac{g_{YM}^2(L+1)}{4\pi^2}$. Half of the fermion contribution comes from the superpotential; this is the part which will be altered by the extra h-dependent part of the superpotential. Hence, the additional term to the new spin chain, besides the F-term scalar part, is

$$\frac{h^* h}{1+q^* q}\frac{g^2(L+1)}{8\pi^2} N : \mathrm{Tr}\left(\bar\phi_1 \phi_1\right) : . \tag{B.2}$$

Then, we will have an effective scalar interaction which just comes from the F-term (since we have the same cancelation as in the $\mathcal{N}=4$ SYM)[34]

$$\pm\sqrt{\frac{2}{(1+q^*q)}}\frac{g_{YM}^2 L}{16\pi^2} : V_F : , \tag{B.3}$$

where

$$\begin{aligned} V_F = &\left(\mathrm{Tr}\left[\phi_i\phi_{i+1}\bar\phi_{i+1}\bar\phi_i - q\phi_{i+1}\phi_i\bar\phi_{i+1}\bar\phi_i - q^*\phi_i\phi_{i+1}\bar\phi_i\bar\phi_{i+1}\right]\right. \\ &+ \mathrm{Tr}\left[qq^*\phi_{i+1}\phi_i\bar\phi_i\bar\phi_{i+1} - qh^*\phi_0\phi_2\bar\phi_1\bar\phi_1 - q^*h\phi_1\phi_1\bar\phi_2\bar\phi_0\right] \\ &+ \left.\mathrm{Tr}\left[h\phi_1\phi_1\bar\phi_0\bar\phi_2 + h^*\phi_2\phi_0\bar\phi_1\bar\phi_1 + hh^*\phi_1\phi_1\bar\phi_1\bar\phi_1\right]\right) . \end{aligned} \tag{B.4}$$

The plus-minus sign in (B.3) depends on which sign we choose for the superpotential. Since all terms are multiplied by the same divergent factor we can set $L = \log x^{-2}$, just as in the case of $\mathcal{N}=4$. The contribution from the self-energy to the dilatation operator is

$$\frac{h^*h}{1+q^*q}\left(E_{11}\otimes I + I\otimes E_{11}\right), \tag{B.5}$$

and the F-term scalar interaction contribute with

$$\begin{aligned} &\pm\sqrt{\frac{2}{(1+q^*q)}}\left(E^l_{i,i}E^{l+1}_{i+1,i+1} - qE^l_{i+1,i}E^{l+1}_{i,i+1} - q^*E^l_{i,i+1}E^{l+1}_{i+1,i}\right. \\ &+ qq^*E^l_{i+1,i+1}E^{l+1}_{i,i} - qh^*E^l_{i+1,i+2}E^{l+1}_{i,i+2} - q^*hE^l_{1,0}E^{l+1}_{1,2} \\ &+ \left. hE^l_{1,2}E^{l+1}_{1,0} + h^*E^l_{2,1}E^{l+1}_{0,1} + hh^*E^l_{1,1}E^{l+1}_{1,1}\right) . \end{aligned} \tag{B.6}$$

We will now consider the case when $q=-1$. The total dilatation operator simplifies to

$$H = \left(\begin{array}{ccc|ccc|ccc} 0 & & & & & & & & \\ & 1-\frac{h^*h}{2} & & 1 & & & & & \\ & & 1 & & -qh^* & & 1 & & \\ \hline & 1 & & 1-\frac{h^*h}{2} & & & & & \\ & & h & & 0 & & h & & \\ & & & & & 1-\frac{h^*h}{2} & & 1 & \\ \hline & & 1 & & h^* & & 1 & & \\ & & & & & 1 & & 1-\frac{h^*h}{2} & \\ & & & & & & & & 0 \end{array}\right) . \tag{B.7}$$

Here we have chosen a relative minus sign between the contribution from the fermion loop and the scalar interaction term.

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