Preprint typeset in JHEP style - HYPER VERSION

LU-TP 07-09

On the Dipole Swing and the Search for Frame Independence in the Dipole Model

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Abstract: Small-x evolution in QCD is conveniently described by Mueller's dipole model which, however, does not include saturation effects in a way consistent with boost invariance. In this paper we first show that the recently studied zero and one dimensional toy models exhibiting saturation and explicit boost invariance can be interpreted in terms positive definite $k \to k+1$ dipole vertices. Such $k \to k+1$ vertices can in the full model be generated by combining the usual dipole splitting with $k-1$ simultaneous dipole swings. We show that, for a system consisting of N dipoles, one needs to combine the dipole splitting with at most $N-1$ simultaneous swings in order to generate all colour correlations induced by the multiple dipole interactions.

Keywords: Dipole Model, Small-x evolution, Saturation.

Contents

1. Introduction

The small-x region in QCD can be described by the well known, linear, BFKL equation which predicts a power like growth in x for the gluon density. Such a fast growth is problematic since it breaks the unitarity bound at high energies. As the gluon density becomes large, non-linear effects cannot be ignored and it was early suggested that effects from parton saturation should tame the growth of the gluon density, in accordance with unitarity [1].

Starting the evolution from a colour singlet quark-antiquark pair, a colour dipole, Mueller [2, 3] formulated a dipole model in transverse coordinate space which reproduces the BFKL equation to leading order. The transverse coordinate formulation also allows one to go beyond the BFKL equation since it is here easier to take into account multiple interactions. This is so because the transverse coordinates of the partons are frozen during the evolution, and it is therefore rather easy to sum the multiple scattering series in an eikonal approximation. This was exploited by Mueller, who was thus able to obtain a unitarised formula for the scattering amplitude.

Within the dipole formalism, Balitsky [4] derived an infinite hierarchy of equations for the dipole scattering amplitudes. Kovchegov [5] derived a closed equation for the amplitude using a mean field approximation, and this equation is referred to as the Balitsky-Kovchegov (BK) equation. The same hierarchy of equations also follows from the JIMWLK equation [6–9] which is the master equation of the Color Glass Condensate (CGC) formalism [10] and describes the non-linear evolution of dense hadronic matter in the presence of saturation effects.

In Mueller's model the multiple dipole interactions correspond to multiple pomeron exchange, and in the Lorentz frame where the collision is studied these multiple interactions lead to the formation of pomeron loops. However, these loops cannot be formed during the evolution of the dipole cascade since this evolution is linear. Thus only those loops which are cut in the specific Lorentz frame used for the calculation are accounted for, while none of the remaining loops is included. This implies that the model is not frame independent. To minimize the error, the optimal frame to use is the one where the colliding dipole cascades are of the same density, since multiple scatterings then become important at rapidities where one may still neglect saturation effects in the evolving dipole cascades.

In order to obtain a frame independent formalism it is necessary to include saturation effects also in the evolution of each dipole cascade. There have been various attempts to include such saturation effects in a consistent manner, but no explicitely frame independent formalism has yet been presented.

To gain insight and possible hints towards a solution, a simple 1+0 dimensional (rapidity constitutes the only dimension) toy model in which transverse coordinates are neglected was constructed in [3,11]. This model has been further studied in [12,13], and the resulting frame independent evolution can be interpreted as a coherent emission of new "dipoles" from the multi-"dipole" state.

In [14] we developed a dipole cascade model for DIS and pp collisions based on Mueller's model but also including effects of energy-momentum conservation. This model was further extended in [15] to include saturation effects in the cascade evolution through the so called dipole swing mechanism [15, 16], which gives an additional $2 \to 2$ transition during the evolution. Monte Carlo (MC) simulations show that the evolution is almost frame independent, and the model results are in good agreement with inclusive and diffractive data from HERA and the Tevatron [17].

In this paper we will first show that the explicitely frame independent evolution in the toy model mentioned above, and also in its 1+1 dimensional generalization in [18], can be given a probabilistic interpretation in terms of positive definite $k \to k+1$ dipole vertices. Such $k \to k+1$ transitions can in the full model be generated by combining the dipole splitting with the dipole swing.

In case each individual dipole is restricted to single scattering only, we show that one needs to combine the dipole splitting with at most one swing at a time in order to generate the necessary colour correlations. As remarked in [12], the toy model evolutions mentioned above describe the multiple scatterings of individual dipoles, and we will here show that one can in the real model generate the correlations induced by the dipole scatterings by combining a splitting with several simultaneous swings. For a system consisting of N spatially uncorrelated dipoles, it is easy to see that one needs to combine the dipole splitting with at most $N-1$ simultaneous swings. In a process where one splitting is combined with $k-1$ swings, k dipoles are replaced by $k+1$ dipoles, thus giving a $k \to k+1$ transition.

However, starting the evolution from a single $q\bar{q}$ pair, one obtains dipoles which are connected in chains, and in this case not all swings are allowed. It is here important to keep track of the correct topology of the dipole state. While this is never a problem in the original formulation which only contains the $1 \to 2$ splitting, it is here very important to avoid the formation of unphysical states. Although it has been checked for a large number of cases, a formal proof that it is always enough with $N-1$ swings is not avaliable, and the result is a conjecture.

The paper is organized as follows. In the next section we shortly review Mueller's dipole model and the question of frame independence in the evolution. In section 3 we will review the toy models in zero and one transverse dimensions formulated in [11, 18], and show how these can be interpreted in terms of the $k \to k+1$ transitions mentioned above. Then in section 4 we consider the evolution in the full model and argue that also in this case the correct evolution can be formulated in terms of $k \to k+1$ vertices. In section 5 we go on to study the colour topology of the evolution, and we show how one can generate the needed colour correlations using the dipole swing. Finally, in section 6, we present our conclusions.

2. Approaches Towards a Frame Independent Formalism

In Mueller's model [2, 3] for onium-onium scattering a colour dipole formed by a colour charge at transverse coordinate x and an anti-charge at y can split into two dipoles by emitting a soft gluon at position z with the following probability

$$\frac{d\mathcal{P}}{dY} = \frac{\bar{\alpha}}{2\pi} d^2z \frac{(x-y)^2}{(x-z)^2(z-y)^2} \equiv d^2z \mathcal{M}(x,y,z), \tag{2.1}$$

$$\bar{\alpha} \equiv \frac{\alpha_s N_c}{\pi} \text{ and } Y \equiv \ln \frac{1}{\mathrm{x}_{Bj}}. \tag{2.2}$$

We refer to $\mathcal{M}$ as the dipole kernel and to Y as the rapidity, which here acts as the time variable in which the evolution proceeds. A dipole cascade is then formed when each dipole splits repeatedly. When two such cascades collide, a right-moving dipole (x_i, y_i) and a left-moving dipole (x_j, y_j) interact with a probability

$$f_{ij} = f(x_i, y_i | x_j, y_j) = \frac{\alpha_s^2}{8} \left[\log\left(\frac{(x_i - y_j)^2 (y_i - x_j)^2}{(x_i - x_j)^2 (y_i - y_j)^2} \right) \right]^2. \tag{2.3}$$

All dipole interactions are assumed to be independent, and the S-matrix element is given by $S = \exp(-\sum_{ij} f_{ij})$.

In this formalism, saturation effects occur only due to multiple scatterings while the evolution of the dipole cascade satisfies the usual BFKL equation. This implies that the formalism is not frame independent and in order to obtain a frame independent formalism such saturation effects must be properly included in the cascade evolution.

Different approaches have been proposed to obtain this. It was noted that the CGC formalism is not complete in the sense that it does not contain any gluon splittings, or gluon number fluctuations. This problem comes from the fact that the Balitsky-JIMWLK (B-JIMWLK) equations[1], which can schematically be written as

$$\partial_Y \langle T^k \rangle = \mathcal{M} \otimes \{\langle T^k \rangle - \langle T^{k+1} \rangle\}, \tag{2.4}$$

only couples the $k-$dipole scattering amplitude T^k to the $(k+n)$-dipole amplitudes with $n = 0, 1, \ldots$. From the view of target[2] evolution, this means that one includes all gluon merging diagrams, while the gluon splitting diagrams are absent. Gluon splittings are equivalent to dipole splittings, and the dipole model has been used to add fluctuations into the formalism. The modified B-JIMWLK equations then read

$$\partial_Y \langle T^k \rangle = \mathcal{M} \otimes \{\langle T^k \rangle - \langle T^{k+1} \rangle\} + \mathscr{K} \otimes \langle T^{k-1} \rangle, \tag{2.5}$$

where $\mathscr{K}$ is a kernel representing the fluctuations in the target.

Viewed in the opposite direction, the fluctuation effects in the target correspond to saturation effects in the projectile evolution. Given the form of the modified B-JIMWLK equations above, it may seem natural to include a $2 \to 1$ vertex in the dipole evolution. Such an interpretation has, however, the drawback that it cannot be interpreted as a classical evolution process since the $2 \to 1$ vertex is not positive definite, as was shown by Iancu et al. [19]. In [16] Kozlov et al. calculated directly the $4 \to 2$ gluon merging vertex within the dipole language and were led to the conclusion that one should include a $2 \to 4$ dipole vertex which is composed of a splitting and a swing.

In [15] we included the $2 \to 2$ dipole "swing", in addition to the $1 \to 2$ dipole splitting, in the evolution. If we have two dipoles (x_i, y_i) and (x_j, y_j), the swing will replace them by (x_i, y_j) and (x_j, y_i). The dipole swing can be interpreted in two ways. First, as a way to approximate colour quadrupoles as two independent dipoles formed by the closest charge-anticharge pairs, in which case the swing is naturally suppressed by N_c^2. Secondly, we may view it as the result of a gluon exchange between the dipoles, which results in a change in the colour flow. In this case the swing would be proportional to α_s^2, which again, compared to $\bar{\alpha}$, is formally suppressed by N_c^2.

The dipole swing is related to the pomeron interactions studied by Bartels and Ryskin [20, 21]. Here a pomeron is interpreted as two gluons in a colour singlet state. In a four gluon system with two singlet pairs, gluon exchange can give a transformation (a "switch"

[1] Throughout this paper, we will only consider the large-N_c version of these equations.

[2] In the CGC approach, it is usually assumed that the target is a dense hadron while the projectile is an elementary colour dipole.

or "swing") (12) + (34) $\to$ (13) + (24), where a parenthesis denotes two gluons in a singlet state.

We note that the swing is here not a vertex in the same sense as the splitting process since, unlike the splitting, the swing is not proportional to dY but rather happens instantaneously. In the MC implementation we assign a "colour" to each dipole and two dipoles are allowed to swing if their colour indices match[3]. The swing is then determined by the weight

$$P(\text{swing}) \propto \frac{(x_1 - y_1)^2 (x_2 - y_2)^2}{(x_1 - y_2)^2 (x_2 - y_1)^2}. \tag{2.6}$$

Here the two initial dipoles are determined by the coordinates (x_1, y_1) and (x_2, y_2), and the new by (x_1, y_2) and (x_2, y_1). This form favours the formation of small dipoles. It also preserves one of the results in Mueller's original formulation, namely that the total weight for a dipole chain is given by the product $\prod_i \frac{1}{r_i^2}$, where r_i is the size of dipole i and the product runs over all "remaining" dipoles in the cascade.

In our formalism the total number of dipoles does not decrease. For each event, many of the dipoles will not interact and these have to be considered as being virtual. In this case saturation effects do not have to decrease the total number of dipoles but rather only the number of interacting, or "real", dipoles. The dipole swing has this property since it is more likely that two dipoles are replaced by two smaller dipoles, as can easily be seen from (2.6), and smaller dipoles have smaller interaction probabilities. Thus the number of interacting dipoles will actually decrease, and in pomeron language this means that the swing generates pomeron mergings.

An essential feature of our formalism is that the dipoles in the cascade form connected chains[4], rather than a collection of uncorrelated dipoles, as in a reaction-diffusion type of formalism. A dipole chain cannot end in a gluon, and it is not possible to remove a dipole without reconnecting its neighbors. A generic $2 \to 1$ vertex is therefore not possible in this formalism[5], and the dipole swing gives the simplest process from which one can form closed chains during the evolution.

3. The Toy Models

3.1 The 1+0 dimensional toy model

In this section we will review the toy model which was studied in detail in [12] (see also [13]). This model was first presented by Mueller [3] and it is interesting since, besides having some structural aspects in common with the dipole model, it offers analytical solutions which have been very hard to obtain for the full model.

[3]The number of effective colours is N_c^2 which is the number of possible colour configurations for a given colour-anticolour pair.

[4]In Mueller's original formulation this is not relevant since there the dipoles evolve truly independently. However, in our implementation of energy-momentum conservation, neighboring dipoles affect each other and it is then relevant that the cascade is formulated as a dipole chain.

[5]The only allowed merging process is in case two neighboring dipoles merge.

The model is defined such that at any rapidity Y the system is specified only by the number of dipoles. The probability to find the system in the n-dipole state at time Y is denoted by $P_n(Y)$. Here transverse coordinates are completely neglected and Y defines the only coordinate in the model.

If we let $\boldsymbol{\mathcal{H}}$ denote the Hamiltonian of the system we have

$$\mathcal{H}_{nm} = \langle m|\boldsymbol{\mathcal{H}}|n\rangle = \mathcal{R}(n)(\delta_{m,n+1} - \delta_{m,n}). \tag{3.1}$$

$\mathcal{R}(n)$ is so far an unspecified function, which determines the splitting rate of the n dipole state. The probability $P_n(Y)$ evolves according to

$$\partial_Y P_n = \mathcal{H}_{mn} P_m. \tag{3.2}$$

With S_{nm} we denote the S-matrix for the scattering of two dipole states of n and m dipoles respectively. If we assume that each dipole scatters independently with a probability τ, we have $S_{nm} = (1-\tau)^{nm}$. We here assume that $\tau << 1$ and if this is not the case, one should replace $1-\tau$ by $\exp(-\tau)$. The physical $S-$matrix, $\mathcal{S}$, is obtained by taking an average over all possible events. Using matrix notation we have

$$\mathcal{S}(Y_1+Y_2) = \boldsymbol{p}^T(Y_1)\boldsymbol{S}\boldsymbol{q}(Y_2). \tag{3.3}$$

Here $\boldsymbol{p}^T(Y_1) = (P_1(Y_1), P_2(Y_1), \dots)$ is the row vector of the configuration probabilities for the right moving onium (evolved up to Y_1) while $\boldsymbol{q}$ denotes the column vector of configuration probabilities for the left moving onium (evolved up to Y_2).

In (3.3) we have anticipated that $\mathcal{S}$ depends only on the total rapidity interval Y_1+Y_2, which defines boost invariance. This implies that we have $(\partial_{Y_1} - \partial_{Y_2})\mathcal{S} = 0$, and requiring this in (3.3) one obtains

$$\boldsymbol{p}^T(Y_1)\boldsymbol{\mathcal{H}}\boldsymbol{S}\boldsymbol{q}(Y_2) - \boldsymbol{p}^T(Y_1)\boldsymbol{S}\boldsymbol{\mathcal{H}}^T\boldsymbol{q}(Y_2) = 0. \tag{3.4}$$

A sufficient condition for a solution is to require that

$$\boldsymbol{\mathcal{H}}\boldsymbol{S} = \boldsymbol{S}\boldsymbol{\mathcal{H}}^T \tag{3.5}$$

which means that $\boldsymbol{\mathcal{H}S}$ is symmetric (since $\boldsymbol{S}$ is symmetric). It is now easily seen that the condition (3.5) requires $\mathcal{R}(n)$ in (3.1) to be given by

$$\mathcal{R}(n) = c\,(1-(1-\tau)^n) \tag{3.6}$$

where c is a constant, $c = \mathcal{R}(1)/\tau$. By rescaling Y, we might as well assume c to equal 1.

3.2 Stochastic evolution with $k \to k+1$ vertices

We note that (3.6) can be rewritten as

$$\mathcal{R}(n) = c\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\tau^k. \tag{3.7}$$

This suggests that we can interpret the evolution in terms of $k \to k+1$ transitions with weights $(-1)^k \tau^k$. However, the alternating signs implies that one cannot interpret these vertices in a probabilistic formulation.

We will now show that one can nevertheless interpret the evolution in terms of positive definite $k \to k+1$ vertices. Thus the evolution can still be formulated as a stochastic process, but we will also see that the probabilistic interpretation of the evolution implies that it cannot be reduced into a formalism which describes a system of incoherent particles.

Assume we have a system of n particles X, which we also refer to as dipoles, satisfying the following rules. Each isolated X can emit a new X with a probability per unit time (rapidity) given by τ, *i.e.* we have a reaction $X \to X + X$ which occurs with probability $\tau > 0$. In addition to this, k isolated dipoles can undergo a transition $kX \to (k+1)X$ with probability τ^k.

The evolution of the probabilities $P_n(Y)$ for these X then satisfies

$$\partial_Y P_n(Y) = -\sum_{k=1}^{n} \mathcal{R}^{(n)}_{k\to k+1} P_n(Y) + \sum_{k=1}^{n-1} \mathcal{R}^{(n-1)}_{k\to k+1} P_{n-1}(Y) \tag{3.8}$$

where $\mathcal{R}^{(n)}_{k\to k+1}$ is the splitting rate for the process where k dipoles are replaced by $k+1$ dipoles in a state containing n dipoles.

Now, the splitting rate $\mathcal{R}^{(n)}_{k\to k+1}$ is not simply given by $\binom{n}{k}\tau^k$ as one could expect naively, but it is instead given by

$$\mathcal{R}^{(n)}_{k\to k+1} = \binom{n}{k} \tau^k (1-\tau)^{n-k}, \tag{3.9}$$

since for each $k \to k+1$ we must also multiply with the probability that no more than k dipoles were involved in the emission of the new dipole. Obviously, k must run from 1 to n, and summing all contributions we obtain the total splitting rate as

$$\sum_{k=1}^{n} \mathcal{R}^{(n)}_{k\to k+1} = \sum_{k=1}^{n} \binom{n}{k} \tau^k (1-\tau)^{n-k} = (1-(1-\tau)^n) = \mathcal{R}(n) \tag{3.10}$$

where $\mathcal{R}(n)$ was defined in (3.6). The positive definite transition rates in (3.9) thus give a boost invariant evolution as before. Note also that in this case the $k \to k+1$ transitions are not universal since they depend on n, unlike the rates in (3.7). The $k \to k+1$ splitting vertex therefore not only depends on the state of the k emitters, but it does also depend on the rest of the dipoles in the cascade. We will in the forthcoming sections see that the dipole swings give a very similar evolution in the full model.

We also note that the $k \to k+1$ transitions can be made manifest by writing down the Hamiltonian

$$\boldsymbol{\mathcal{H}} = \sum_{k=1}^{\infty} \frac{\tau^k}{k!} (\boldsymbol{N}^{-1/2}\boldsymbol{a}^\dagger - \mathbf{1})(\boldsymbol{a}^\dagger)^k \boldsymbol{a}^k \prod_{l=k+1}^{\infty} (1-\tau) \sum_{m=l}^{\infty} |m\rangle\langle m|, \tag{3.11}$$

where $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$ are dipole creation and annihilation operators respectively, and $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$.

3.3 Evolution equations

In this section we will first show that the evolution equations for the scattering amplitudes derived in [12] are described *exactly* by the $k \to k+1$ transitions in (3.9). We will then go on to point out that there is a fundamental structural reason for the fact that the attempts to interpret the full model evolution given in (2.5) in a probabilistic manner have run into problems. We will see that it is not possible to interpret this equation using a probabilistic $2 \to 1$ vertex even in the toy model. However, we note that it is not necessary to include a $2 \to 1$ vertex to obtain saturation. In fact any $2 \to n$ vertex where only one of the n dipoles interact also corresponds to a $2 \to 1$ transition, and this will be discussed more later.

First we write the S-matrix given in (3.3) as

$$\mathcal{S}(Y_1+Y_2) = \boldsymbol{p}^T(Y_1)\boldsymbol{s_q}(Y_2), \quad \boldsymbol{s_q} \equiv \boldsymbol{Sq} \tag{3.12}$$

where the nth component of the vector $\boldsymbol{s_q}(Y_2)$ is the S-matrix element of a projectile, evolved up to Y_1, made up from n dipoles scattering against a generic target, which is evolved up to Y_2. It is then easy to see that $(\boldsymbol{s_q})_n \equiv \langle s^n \rangle$ satisfies the following evolution equation [12]

$$\partial_Y \langle s^n \rangle = \mathcal{R}(n)\{\langle s^{n+1}\rangle - \langle s^n \rangle\} \tag{3.13}$$

Using the relation $s = 1-t$, where t denotes the scattering amplitude, one can similarly derive the equations obeyed by $\langle t^n \rangle$. Since τ is assumed to be small, one can expand $\mathcal{R}(n)$ in each equation and drop contributions which are negligible in all regimes (dilute and dense systems). Doing this, the authors in [12] arrived at the following evolution equations for $\langle t\rangle$, $\langle t^2\rangle$ and $\langle t^3\rangle$,

$$\begin{aligned}
\partial_Y\langle t\rangle &= \langle t\rangle - \langle t^2\rangle, & (3.14)\\
\partial_Y\langle t^2\rangle &= 2(\langle t^2\rangle - \langle t^3\rangle) + \tau\langle t(1-t)^2\rangle, & (3.15)\\
\partial_Y\langle t^3\rangle &= 3(\langle t^3\rangle - \langle t^4\rangle) + 3\tau\langle t^2(1-t)^2\rangle + \tau^2\langle t(1-t)^3\rangle. & (3.16)
\end{aligned}$$

If one neglects all terms proportional to τ, then it is seen that the resulting hierarchy corresponds to the large N_c version of the B-JIMWLK hierarchy.

Let us now see how equations (3.14)-(3.16) arise from the transition rates $\mathcal{R}^{(n)}_{k\to k+1}$. The evolution of the S-matrix elements are given by

$$\begin{aligned}
\partial_Y\langle s\rangle &= \mathcal{R}^{(1)}_{1\to 2}\{\langle s^2\rangle - \langle s\rangle\}, & (3.17)\\
\partial_Y\langle s^2\rangle &= (\mathcal{R}^{(2)}_{1\to 2} + \mathcal{R}^{(2)}_{2\to 3})\{\langle s^3\rangle - \langle s^2\rangle\}, & (3.18)\\
\partial_Y\langle s^3\rangle &= (\mathcal{R}^{(3)}_{1\to 2} + \mathcal{R}^{(3)}_{2\to 3} + \mathcal{R}^{(3)}_{3\to 4})\{\langle s^4\rangle - \langle s^3\rangle\}. & (3.19)
\end{aligned}$$

It is then straightforward to obtain the following equations for the scattering amplitudes

$$\begin{aligned}
\partial_Y\langle t\rangle &= \mathcal{R}^{(1)}_{1\to 2}\{\langle t\rangle - \langle t^2\rangle\} = \langle t\rangle - \langle t^2\rangle, & (3.20)\\
\partial_Y\langle t^2\rangle &= 2\mathcal{R}^{(1)}_{1\to 2}(\langle t\rangle - \langle t^2\rangle) - \mathcal{R}^{(2)}_{1\to 2}\langle t(1-t)^2\rangle - \mathcal{R}^{(2)}_{2\to 3}\langle t(1-t)^2\rangle & \\
&= 2(\langle t^2\rangle - \langle t^3\rangle) + 2\tau\langle t(1-t)^2\rangle - \tau\langle t(1-t)^2\rangle. & (3.21)
\end{aligned}$$

We indeed see that the first equation is equal to (3.14) and that the second equation is equal to (3.15). It is also straightforward to see that the equation for $\langle t^3\rangle$ agrees with (3.16).

Next, we comment on the structure of the equation given in (2.5). Assume we wish to view the process as a stochastic evolution with a $1 \to 2$ splitting vertex, $f_{1\to 2}$, and a $2 \to 1$ merging vertex, $k_{2\to 1}$. We here assume the total splitting rate to be the incoherent sum of the individual splitting rates. For the evolution of the 2−dipole state we have

$$\partial_Y \langle s^2\rangle = f^{(2)}_{1\to 2}\{\langle s^3\rangle - \langle s^2\rangle\} + k^{(2)}_{2\to 1}\{\langle s\rangle - \langle s^2\rangle\} \tag{3.22}$$

which gives

$$\begin{aligned}\partial_Y \langle t^2\rangle &= 2f^{(1)}_{1\to 2}\{\langle t\rangle - \langle t^2\rangle\} + k^{(2)}_{2\to 1}\langle t(1-t)\rangle - f^{(2)}_{1\to 2}\langle t(1-t)^2\rangle \\ &= (2f^{(1)}_{1\to 2} - f^{(2)}_{1\to 2} + k^{(2)}_{2\to 1})\langle t\rangle + (-2f^{(1)}_{1\to 2} + 2f^{(2)}_{1\to 2} - k^{(2)}_{2\to 1})\langle t^2\rangle - f^{(2)}_{1\to 2}\langle t^3\rangle \\ &= k^{(2)}_{2\to 1}\langle t\rangle + (f^{(2)}_{1\to 2} - k^{(2)}_{2\to 1})\langle t^2\rangle - f^{(2)}_{1\to 2}\langle t^3\rangle .\end{aligned} \tag{3.23}$$

We thus see that the $2 \to 1$ contribution not only generates the "fluctuation" term, $k^{(2)}_{2\to 1}\langle t\rangle$, but it does also modify the $\langle t^2\rangle$ term. If this term is to be unaffected by the additional vertex, as in (2.5), then we have to set $k^{(2)}_{2\to 1} = 0$. This is actually very similar to what happens in the full model. In that case the integral over the proposed $2 \to 1$ vertex has to be zero, which implies that the vertex cannot be positive definite, as was noted in [19]. We can also try to add another vertex such that the total contribution to the $\langle t^2\rangle$ term cancels. Assume for example the existence of an additional $2 \to 0$ vertex $g_{2\to 0}$. We then get

$$\partial_Y \langle t^2\rangle = (k^{(2)}_{2\to 1} + 2g^{(2)}_{2\to 0})\langle t\rangle + (f^{(2)}_{1\to 2} - k^{(2)}_{2\to 1} - g^{(2)}_{2\to 0})\langle t^2\rangle - f^{(2)}_{1\to 2}\langle t^3\rangle \tag{3.24}$$

from which we conclude that

$$k^{(2)}_{2\to 1} + g^{(2)}_{2\to 0} = 0 \tag{3.25}$$

which means that either $k^{(2)}_{2\to 1}$ or $g^{(2)}_{2\to 0}$ has to be negative. Thus we conclude that this approach has big problems, as one must choose $k^{(2)}_{2\to 1}$ either to be 0, or it must be negative, which means that one cannot obtain a probabilistic formulation.

3.4 The 1+1 dimensional toy model

A somewhat more complicated 1+1 dimensional model is presented in [18]. The structure of this model is very similar to the 1+0 dimensional model, but the difference is that this time a dipole state is not only specified by the total number of dipoles, but it also depends on the distribution of these dipoles along some additional transverse axis.

We denote the position of a dipole along this axis with x_i, and the generic n-dipole state is denoted $|\{x_i\}\rangle = |x_1, \ldots, x_n\rangle$. The assumption in [18] is that the dipole state evolves *only* by the addition of a single new dipole at some position x_{n+1}. In that case the frame independence equation in (3.5) can easily be solved, and the simplest solution for the total splitting rate $\mathcal{R}_i(\{n\})$ is given by [18]

$$\mathcal{R}(\{x_i\} \to \{x_i\} + x_{n+1}) = \frac{1 - \prod_{i=1}^n (1 - \tau(x_i|x_{n+1}))}{\tau}. \tag{3.26}$$

Here τ is a constant which can, by a redefinition of Y, set to be equal to 1.

We now show that the evolution can once again be formulated as a probabilistic process in terms of coherent $k \to k+1$ transitions as in sec 3.2. In this case we assume we have a system of dipoles, X_i, which live on a one dimensional spatial axis. We assume this axis to represent the position of the "point-like" dipoles. This axis is assumed to be continuous, so that the index i actually represents a continuous label x_i. An isolated dipole X_i can then emit another dipole X_j at position x_j with a probability $\tau_{ij} = \tau(x_i|x_j)$. However, in the presence of more than one X_i, the new X_j can also be emitted coherently from several dipoles with a probability given by the product of the individual emission probabilities. For a system of n dipoles located at positions $x_1, \ldots, x_n$, the total $k \to k+1$ splitting rates $\mathcal{R}^{(n)}_{k\to k+1}(\{x_i\} \to \{x_i\} + x_{n+1})$ are then given by

$$\mathcal{R}^{(n)}_{k\to k+1}(\{x_i\} \to \{x_i\} + x_{n+1}) = \frac{1}{k!} \sum_{i_1 \neq \cdots \neq i_k}^{n} \tau_{i_1,n+1}\tau_{i_2,n+1}\ldots\tau_{i_k,n+1} \prod_{m\neq 1,\ldots,k}^{n} (1-\tau_{i_m,n+1}), \tag{3.27}$$

and their sum satisfies

$$\sum_{k=1}^{n} \mathcal{R}^{(n)}_{k\to k+1}(\{x_i\} \to \{x_i\} + x_{n+1}) = 1 - \prod_{i=1}^{n}(1-\tau_{i,n+1}). \tag{3.28}$$

which is equal to (3.26). Note that once again the positive definite splitting rates do not only depend on the state of the emitting dipoles, but also on the state of the other dipoles in the cascade. This is unavoidable if one wants to obtain a probabilistic evolution.

The $k \to k+1$ splitting rates in (3.27) are very similar in structure to the processes generated by the dipole swing, to be discussed in the forthcoming sections. Anticipating the discussion there, we can interpret (3.27) as a process where the newly produced dipoles swing multiply with the rest of the dipoles in the cascade (a concrete example of this is shown in fig 5). Note that in the toy models both the splitting and the scatterings of the dipoles are determined by the quantities τ_{ij}. If we would assume that these also determine the swing probability, then the $k \to k+1$ splitting rates in (3.27) would describe processes where the newly produced dipole i swings with $k-1$ dipoles from the cascade, and the factor $\prod_j(1-\tau_{ij})$ could then be interpreted as the probability that i swings with no more than $k-1$ dipoles.

One difficulty is, however, that the swing in its form in the full model cannot really provide saturation in the toy models since the dipoles have no size here [6]. This follows from the fact that both toy models have trivial topologies, in the sense that the dipole state is assumed to evolve only by the addition of a new dipole without changing the emitting state. In the toy models saturation occurs because k dipoles emit a single dipole with the same strength as a single dipole. In our implementation of the dipole swing however,

[6] In [18], the spatial axis was interpreted as being related to the dipole size while we feel a more close analogy is to interpret it as a spatial coordinate where dipoles of some fixed size live. Irrespective of the interpretation, however, direct comparison with the full model is made difficult by the assumption that the toy model state only evolves by the addition of a single dipole.

saturation occurs because the swing decreases the sizes of the dipoles, and smaller dipoles have a smaller probability to interact.

4. Evolution in the full model

We will here argue that the evolution in the full model also can be formulated as a probabilistic process in terms of $k \to k+1$ transitions.

The S-matrix is in the full model given by

$$\begin{aligned} S_Y &= \sum_{N,M} P_N(Y_0) P_M(Y-Y_0) \prod_{i=1}^{N} \prod_{j=1}^{M} \Big(1 - f(x_i, y_i | x_j, y_j) \Big). \\ &= \Big\langle \prod_{i=1}^{N} \prod_{j=1}^{M} (1 - f_{ij}) \Big\rangle. \end{aligned} \tag{4.1}$$

where f_{ij} is given by (2.3).

Let us consider the evolution initiated by a pair of oppositely moving $q\bar{q}$ pairs. We then consider generic dipole states $\mathscr{A}_N$, containing $N-1$ gluons, which at each rapidity step can evolve into states $\mathscr{A}_{N+1}$ containing N gluons. The splitting rate is denoted $\mathcal{R}(\mathscr{A}_N \to \mathscr{A}_{N+1})$.

The scattering between the states $\mathscr{A}_N$ and $\mathscr{B}_M$ is then frame independent if

$$\begin{aligned} &\sum_{\mathscr{A}_{N+1}} \int_z \mathcal{R}(\mathscr{A}_N \to \mathscr{A}_{N+1}) \Big\{ 1 - \prod_{j \in \mathscr{B}_M} \{ 1 - (\sum_{i \in \mathscr{A}_{N+1} \setminus \mathscr{A}_N} - \sum_{i \in \mathscr{A}_N \setminus \mathscr{A}_{N+1}}) f(i|j) \} \Big\} = \\ &\sum_{\mathscr{B}_{M+1}} \int_z \mathcal{R}(\mathscr{B}_M \to \mathscr{B}_{M+1}) \Big\{ 1 - \prod_{i \in \mathscr{A}_N} \{ 1 - (\sum_{j \in \mathscr{B}_{M+1} \setminus \mathscr{B}_M} - \sum_{j \in \mathscr{B}_M \setminus \mathscr{B}_{M+1}}) f(i|j) \} \Big\}. \end{aligned} \tag{4.2}$$

The notations in this equation are as follows. The integral $\int_z$ denotes the integration over the transverse position of the Nth emitted gluon. In $\prod_{j \in \mathscr{B}_M}$, the index j runs over all dipoles in the state $\mathscr{B}_M$. The set denoted by $\mathscr{A}_{N+1} \setminus \mathscr{A}_N$ consists of those new dipoles produced in the last step of the evolution. Similarly, $\mathscr{A}_N \setminus \mathscr{A}_{N+1}$ denotes the set of all dipoles which are present in $\mathscr{A}_N$, but not in $\mathscr{A}_{N+1}$, *i.e.* those dipoles which were removed from the cascade in the last step of the evolution. Finally, $f(i|j)$ stands for the scattering amplitude between the dipoles i and j, *i.e.* the expression f_{ij} in (2.3) and (4.1). The sum $\sum_{\mathscr{A}_{N+1}}$ is over all N gluon states which can be reached from $\mathscr{A}_N$ in one step.

In the toy model analogy, the difference $\sum_{i \in \mathscr{A}_{N+1} \setminus \mathscr{A}_N} - \sum_{i \in \mathscr{A}_N \setminus \mathscr{A}_{N+1}}$ consists of only the newly produced dipole, i, since all others dipoles are assumed to be unaffected by the evolution. In that case eq (4.2) reduces to

$$\begin{aligned} &\sum_i \mathcal{R}(\mathscr{A}_N \to \mathscr{A}_N + i) \Big\{ 1 - \prod_{j \in \mathscr{B}_M} (1 - f(i|j)) \Big\} \\ &= \sum_i \mathcal{R}(\mathscr{B}_M \to \mathscr{B}_M + i) \Big\{ 1 - \prod_{j \in \mathscr{A}_M} (1 - f(i|j)) \Big\}. \end{aligned} \tag{4.3}$$

The most simple solution is given by

$$\mathcal{R}(\mathscr{A}_N \to \mathscr{A}_N + i) = 1 - \prod_{j \in \mathscr{A}_N} (1 - f(i|j)), \tag{4.4}$$

which we recognize from (3.26). We will now see that the evolution in the full model can be formulated probabilistically in terms of $k \to k+1$ vertices as in the toy models. At first we will formulate the evolution as in eq (3.7) which implies that the $2 \to 3$ transition will appear to have negative sign. However, we know from above how to treat these signs, and thus give the evolution a probabilistic interpretation. We will also see how one can interpret these vertices in terms of the dipole swing.

To this end, we consider first the situation where $N = 2$ and $M = 1$. The state $\mathscr{A}_2$ must consist of two connected dipoles since we know that a single, isolated dipole (the state $\mathscr{A}_1$) evolves by a dipole splitting (assuming the evolution is initiated by a single $q\bar{q}$ pair). We then denote the two dipoles in $\mathscr{A}_2$ with a and b, and the single dipole in the state $\mathscr{B}_1$ is denoted by l. We can then write (4.2) as

$$\sum_{\mathscr{A}_3} \int_z \mathcal{R}(a, b \to \mathscr{A}_3)(\sum_{i \in \mathscr{A}_3 \setminus \mathscr{A}_2} - \sum_{i \in \mathscr{A}_2 \setminus \mathscr{A}_3})f(i|l) = \int_z \mathcal{R}(l \to (l_1, z) + (z, l_2)) \times$$
$$\times \Big(\mathscr{F}(l, a, z) + \mathscr{F}(l, b, z) - \mathscr{F}(l, a, z)\mathscr{F}(l, b, z) \Big) \tag{4.5}$$

where

$$\mathscr{F}(l, a, z) = f(l_1, z|a) + f(z, l_2|a) - f(l|a). \tag{4.6}$$

Here l_1 and l_2 denote the transverse positions of the partons of dipole l. From studying the case $N = M = 1$ (the scattering between two elementary dipoles), we know that each isolated dipole evolves by a $1 \to 2$ splitting. We therefore write $\sum_{\mathscr{A}_3} \mathcal{R}(a, b \to \mathscr{A}_3)$ as

$$\sum_{\mathscr{A}_3} \mathcal{R}(a, b \to \mathscr{A}_3) = \mathcal{R}^{(1)}(a \to (a_1, z) + (z, a_2)) + \mathcal{R}^{(1)}(b \to (b_1, z) + (z, b_2))$$
$$+ \sum_{\mathscr{A}_3^{(2)}} \mathcal{R}^{(2)}(a, b \to \mathscr{A}_3^{(2)}). \tag{4.7}$$

Here $\mathscr{A}_3^{(2)}$ denotes the set of all 2 gluon states which can be reached from $\mathscr{A}_2 = \{a, b\}$ via the vertex $\mathcal{R}^{(2)}$. Now, we know from [11] that the incoherent contributions, $\mathcal{R}^{(1)}$, above are equal to the first order contributions (in $f(i|j)$) in (4.5). Thus we are left with the equation

$$\sum_{\mathscr{A}_3^{(2)}} \int_z \mathcal{R}^{(2)}(a, b \to \mathscr{A}_3^{(2)})(\sum_{i \in \mathscr{A}_3^{(2)} \setminus \mathscr{A}_2} - \sum_{i \in \mathscr{A}_1 \setminus \mathscr{A}_3^{(2)}})f(i|l) =$$
$$= - \int_z \mathcal{R}(l \to (l_1, z) + (z, l_2)) \Big(\mathscr{F}(l, a, z) \cdot \mathscr{F}(l, b, z) \Big). \tag{4.8}$$

In the toy model, where the dipole state evolves by the addition of a single dipole i only, we know that $\mathcal{R}^{(2)}(a, b \to \mathscr{A}_3^{(2)}) = -f(i|a)f(i|b)$ (in case we use the formulation in (3.7)).

Indeed in that case we see that both sides in (4.8) equals $-\sum_i f(i|a)f(i|b)f(i|l)$. It is then clear that we must have a $2 \to 3$ transition.

By similarly studying the case where $N = 3$ and $M = 1$, we would conclude that we need an additional $3 \to 4$ dipole vertex and so on. We are thus led to a picture where the dipole state evolves by $k \to k + 1$ transitions. If these transitions can be generated by combining the dipole splitting with the dipole swing, as we will argue in the next sections, we furthermore obtain a probabilistic interpretation of the evolution, as was discussed in the end of sec 3.4.

Before we going on, we also note that care has to be taken to the fact that the frame independence equation in (4.2) may contain divergences. In the original dipole model, these divergences arise from the dipole splitting kernel in (2.1), but the frame independence equation is still finite since the expressions in the brackets in (4.2) vanish at these singular points. This is both due to the topology of the dipole splitting and also to the colour transparency of small dipoles. Any new vertex to be introduced into the model must retain this property since otherwise equation (4.2) would not make any sense.

5. The Colour Topology of the Evolution

5.1 Colour Flow

Although we are not able to explicitely write down the splitting rate $\mathcal{R}^{(3)}(a, b \to \mathscr{A}_3^{(2)})$ in the full model, we will in this section argue that the correct topology of the evolution is the one induced by the dipole swing.

We first write eq (4.2) in Mueller's original formulation,

$$\sum_{i=1}^{N}\int_z \mathcal{M}(i|z)\Big\{1-\prod_{j=1}^{M}(1-(\sum_{k\in new}-\sum_{k\in old})f(k|j))\Big\} =$$
$$=\sum_{i=1}^{M}\int_z \mathcal{M}(i|z)\Big\{1-\prod_{j=1}^{N}(1-(\sum_{k\in new}-\sum_{k\in old})f(k|j))\Big\}. \tag{5.1}$$

Here $\mathcal{M}(i|z)$ is the usual dipole kernel in (2.1) for a dipole i emitting a gluon at position z, and for simplicity we denote with $\sum_{k\in new(old)}$ the sum over the dipoles produced (destroyed) in the last step.

In case the newly produced dipoles only scatter against one target dipole, eq (5.1) linearizes, and in that case the equality is known to hold [11]. Note that this case does not restrict us to one pomeron exchange, it is only the dipoles produced in the last step which should scatter against a single dipole. There may be still be several scatterings between dipoles produced earlier, with the restriction that each dipole only scatters once. In such events, the colour topology may be described by open chains stretching between the target and the projectile. An example is shown in fig 1. Here, the simple $1 \to 2$ splitting is sufficient for producing all possible colour configurations.

In case the newly produced dipoles scatter off two or more target dipoles, however, more complicated topologies are formed, and the simple $1 \to 2$ splitting is not sufficient

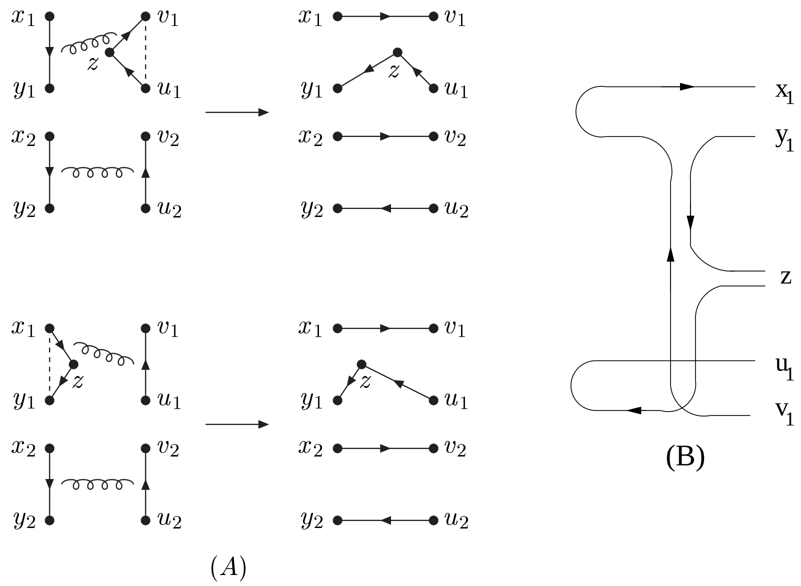

Figure 1: A process where the newly produced dipoles exchange only a single gluon with the target. In this case the colour correlations can be generated by the dipole splitting only. Here the position of z is integrated over and is therefore not fixed, and we only show one possible colour flow. Note that each interaction implies a change in the colour flow, which goes from colour to anticolour as indicated by the arrows. It is not important whether there are also other interactions or not. In fig (A) we assume an additional interaction between (x_2, y_2) and (u_2, v_2). In fig (B) we show the same colour flow in the corresponding Feynman diagram.

anymore. Let us first consider the case where each dipole is restricted to single scattering only. In this case, the newly produced two dipoles can at most scatter off two target dipoles. An example is shown in fig 2 where three different colour configurations are formed (there are three more configurations which can be formed by reversing the colour flow in each line). Here two right-moving dipoles (x_1, y_1) and (x_2, y_2), are connected to a single left-moving dipole (u, v). With the restriction that each dipole scatters only once, the dipole (u, v) can obviously not be connected to more than two oppositely moving dipoles. In this case one can produce all colour configurations by combining the dipole splitting with only one swing. This is in agreement with the findings in [16]. In fig 3 we show how the three configurations in fig 2 can be generated from a dipole split and one swing, by first forming the configuration in fig 3(A). Figure 3 can be compared to eq (4.8) where the right hand side of that equation describes the evolution and the scattering of the dipole (u, v) (l in (4.8)). The vertex $\mathcal{R}^{(2)}$ on the left hand side would then correspond to the diagrams showing the evolution of (x_1, y_1) and (x_2, y_2) into the three dipoles $(x_1, z), (z, y_2)$ and (x_2, y_1).

We conclude that in an approximation where multiple scatterings are allowed, but with the restriction that *each dipole scatters only once*, the maximal correlation induced

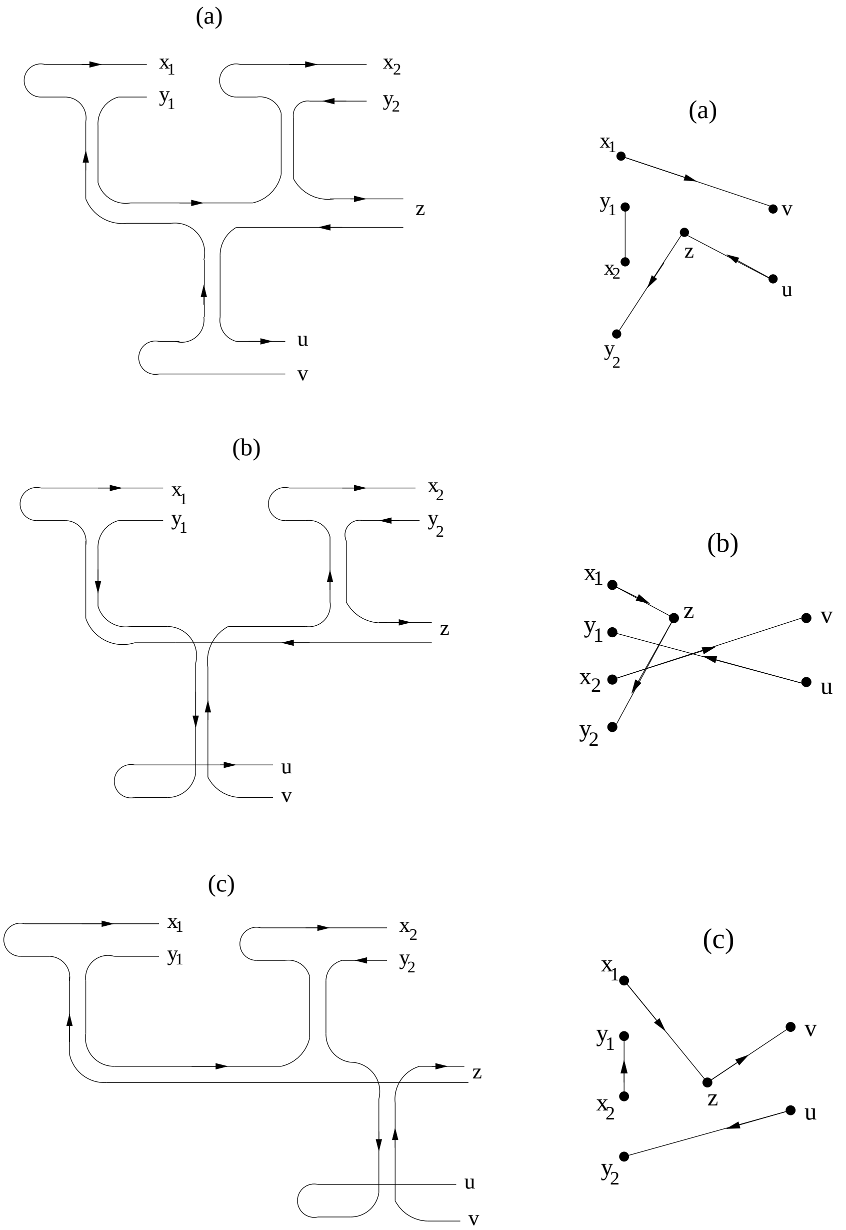

Figure 2: The colour flow in a process where two right-moving dipoles (x_1, y_1) and (x_2, y_2) are linked to a single left-moving dipole (u, v). Three different configurations, $(a), (b)$ and (c), are shown.

between the dipoles is that between a pair, and such a correlation can be generated by a simple swing.

Actually, in this approximation, explicit frame independence in zero transverse dimensions can be achieved by including a $2 \to 1$ vertex in addition to the usual $1 \to 2$ splitting. This reflects the fact that the maximal correlation induced is that between a pair of dipoles. Note also that, in the situation described above, it is always only one out of the three dipoles produced via the combination of the dipole splitting and the swing which

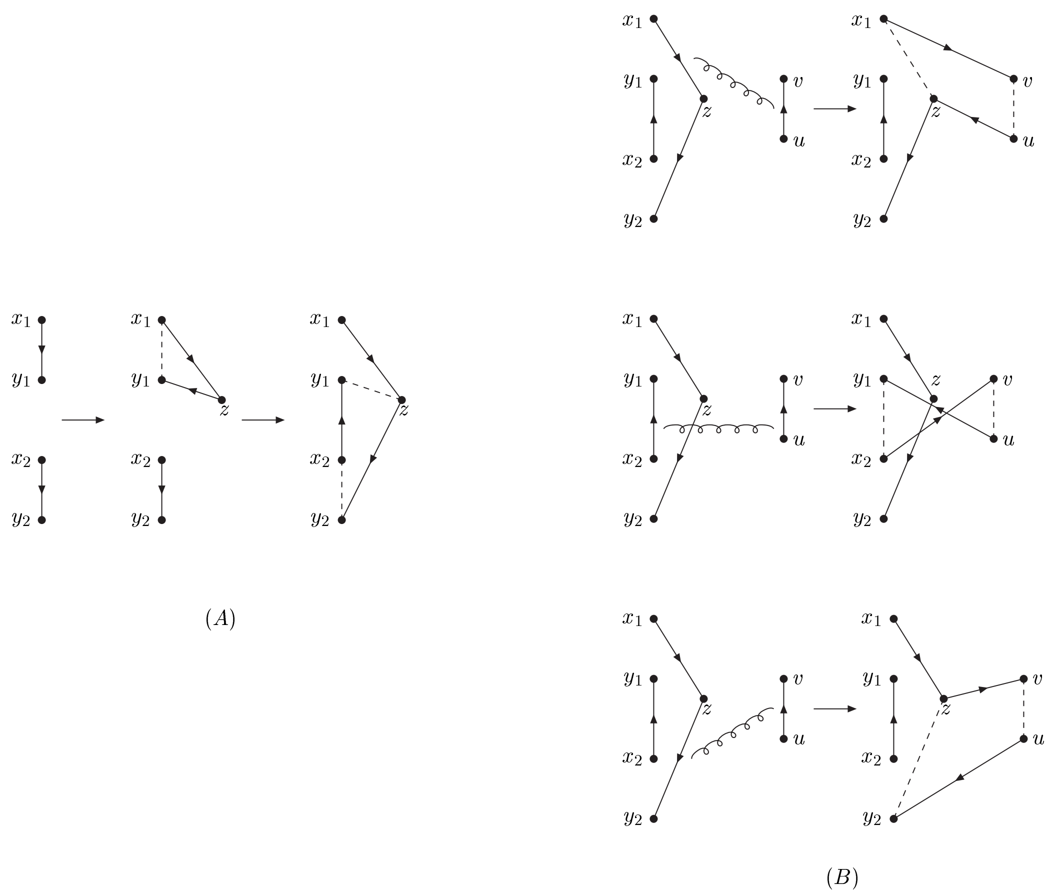

Figure 3: The three configurations marked by $(a), (b)$ and (c) in fig 2 can all be generated when the dipole (u, v) interacts with one of three dipoles from the configuration shown in fig (A). This is illustrated in fig (B). As illustrated, the configuration in fig (A) can in turn be generated via the dipole swing.

interacts with the target. In that sense the swing corresponds to a $2 \to 1$ transition. We can thus obtain an effective $2 \to 1$ transition without actually decreasing the number of dipoles.

If a single dipole can *scatter multiply*, one swing will not be enough. In this case one dipole can for example split into two new dipoles, and these two dipoles can then interact with more than two target dipoles, inducing higher order correlations. Before going on, we note that there is an ambiguity in the statement that one dipole scatters multiply. Since each scattering implies a recoupling of the colour flow, a dipole which interacts is replaced by a new dipole. What we rather mean here is that the partons of the dipole can exchange multiple gluons.

Consider the diagrams shown in fig 4. Here a single left-moving dipole (u, v) is linked to three right-moving dipoles $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), as shown in the figure. The two colour configurations shown in the figure can then be generated as illustrated in fig 5: First one generates the configuration $\{(x_3, y_2), (x_2, y_1), (x_1, z), (z, y_3)\}$ by combining a dipole splitting with two swings as shown in fig 5(A) (this is obviously not the only process

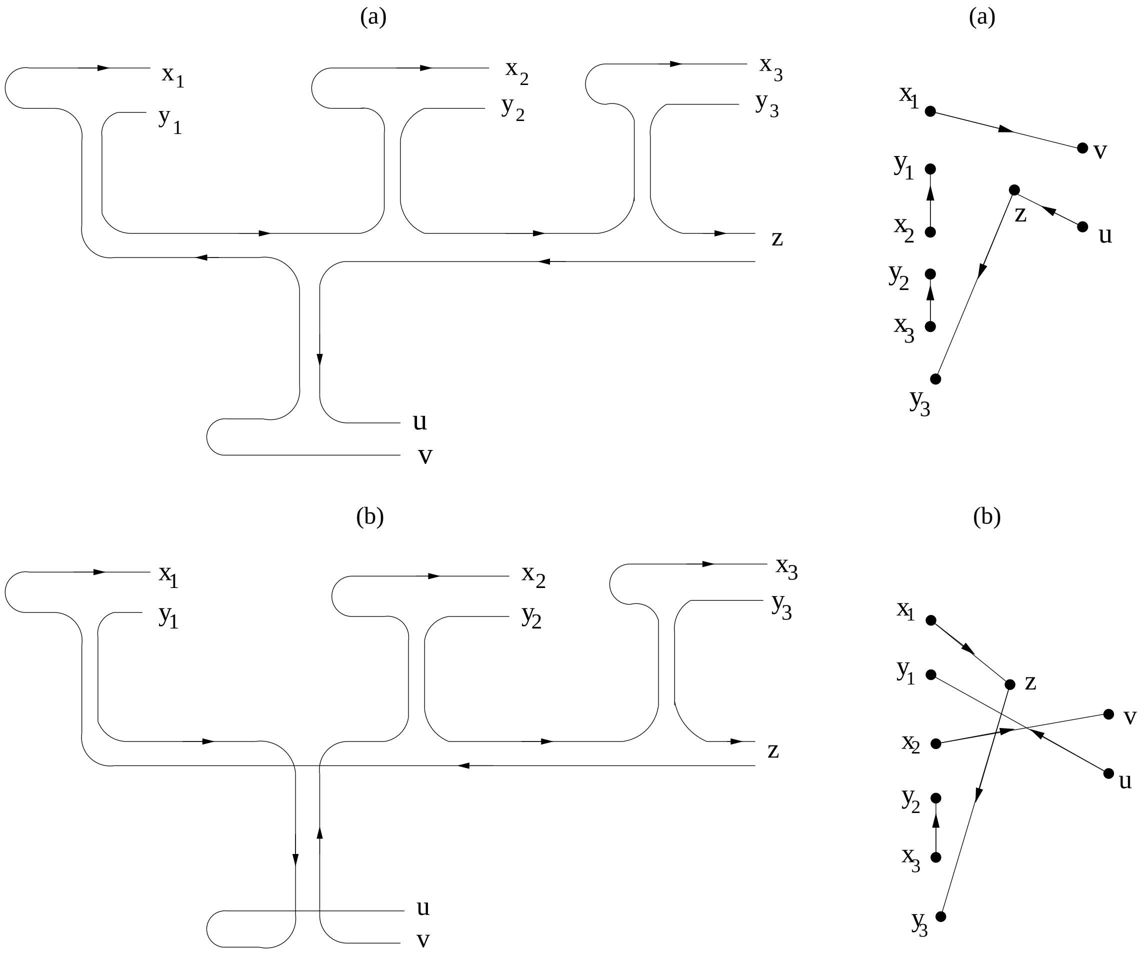

Figure 4: The colour flow in a process where three right-moving dipoles (x_1, y_1), (x_2, y_2) and (x_3, y_3) interact with a single left-moving dipole (u, v) after the emission of a gluon located at z. For simplicity, we only show two, marked (a) and (b), out of the four possible configurations.

from which this final configuration can be generated). One of the four dipoles in this state can then collide with the dipole (u, v). If for example (u, v) collides with (x_1, z), the configuration marked by (a) in fig 4 is produced.

The same process can also be viewed as an evolution of the dipole (u, v), which then splits into (u, z) and (z, v), and fig 5 shows also how the two configurations in fig 4 can be generated when the dipoles (u, z) and (z, v) interact with $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) exchanging now 3 gluons (fig 5(C)). Thus at least one of the dipoles (u, z) and (z, v) must scatter multiply in this case, since it would otherwise be impossible to generate the necessary colour correlations. As remarked in [12], the evolution equations in sec 3.3 actually describe such events where a newly produced dipole scatters multiply. We can also compare the processes in fig 5 to eq (4.2) where one side of the equation describes the multiple scatterings of the dipoles (u, z) and (v, z), while the other side describes how the 3-dipole system evolves into a 4-dipole system which then exchanges a single gluon with

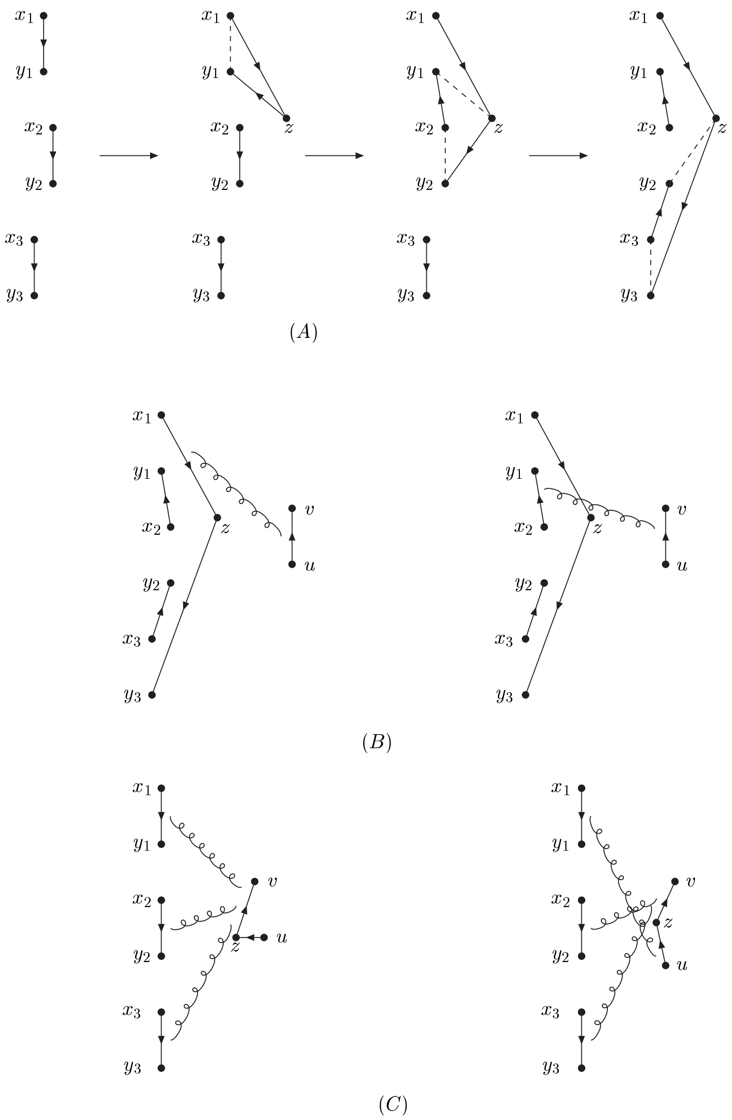

Figure 5: The configurations marked by (a) and (b) in figure 4 can be generated when the dipole (u, v) interacts with different dipoles from the configuration in fig (A), as illustrated in fig B. There are two more configurations which can be obtained when (u, v) interacts with the other two dipoles in fig (A). In fig (C) we show how the same configurations can be generated when the evolution is instead put into the dipole (u, v).

(u, v).

5.2 Generating arbitrary correlations using at most $N - 1$ swings

We might then expect that the correlation induced by the scattering between a single left-moving dipole and k right-moving dipoles can be generated by a splitting followed by $k - 1$ swings in the right-moving system. Note that such a process gives a $k \to k + 1$ transition. In the example above this was possible since we were able to form the configuration $\{(x_3, y_2), (x_2, y_1), (x_1, z), (z, y_3)\}$ by combining one splitting with two swings. This will always be possible if, given an arbitrary set of N dipoles we always can generate all possible $(N+1)$-dipole states, by combining a splitting with at most $N-1$ dipole swings. We will below argue that this is indeed the case. In case we have N spatially disconnected dipoles, the proof is easy. However, starting from a single $q\bar{q}$ pair, the evolution does not generate spatially disconnected dipoles, and in this case the result is a conjecture.

5.2.1 Spatially disconnected dipoles

As a warm up, we first show the statement in case we have N spatially disconnected dipoles $\{(x_i, y_i)\}_{i=1}^N$. We then wish to evolve this state into some arbitrary $N+1$ dipole state,

$$\prod_i (x_i, y_i) \to (x_k, z)(z, y_j) \prod_{i \neq k, p(i) \neq j} (x_i, y_{p(i)}), \tag{5.2}$$

using at most $N-1$ dipole swings. Here $p(i)$ is a permutation of $i = 1, \ldots, N$. We first start by emitting gluon z from the dipole (x_k, y_k),

$$\prod_i (x_i, y_i) \to (x_k, z)(z, y_k) \prod_{i \neq k} (x_i, y_i). \tag{5.3}$$

The result then follows if we can show that, for an arbitrary permutation $p(i)$, we can with $N-1$ swings always make the transformation

$$\prod_{i=1}^{N} (x_i, y_i) \to \prod_{i=1}^{N} (x_i, y_{p(i)}). \tag{5.4}$$

To this end, we perform the following swings in the indicated order,

1 $(x_N, y_N)(x_{p(N)}, y_{p(N)}) \to (x_N, y_{p(N)})(x_{p(N)}, y_N)$
2 $(x_{N-1}, y_{N-1})(x_{p(N-1)}, y_{p(N-1)}) \to (x_{N-1}, y_{p(N-1)})(x_{p(N-1)}, y_{N-1})$

etc. Then, after at most $N-2$ swings we are either finished, or we have

$$(x_1, y_{p(2)})(x_2, y_{p(1)}) \prod_{i=3}^{N} (x_i, y_{p(i)}). \tag{5.5}$$

We then need only one more swing $(x_1, y_{p(2)})(x_2, y_{p(1)}) \to (x_1, y_{p(1)})(x_2, y_{p(2)})$, and so after at most $N-1$ swings we are finished.

The problem is that generally the dipoles are not spatially independent, and one then has to be careful in performing swings, since they might generate zero size dipoles, *i.e.* colour singlet gluons which cannot be allowed.

5.2.2 Dipole states initiated by a $q\bar{q}$ dipole

Representation of the dipole states and the swing in terms of permutations

Consider the evolution initiated by a $q\bar{q}$ colour dipole. In the original formulation of the dipole model, the dipole state at each rapidity Y consists of an open chain, $\mathcal{C}$, of colour dipoles which are linked together via the gluons. Note the dual role played by the gluons

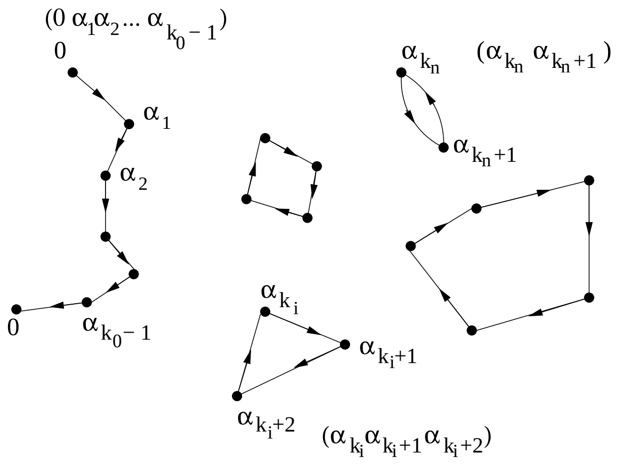

Figure 6: A generic dipole state formed after a rapidity evolution of Y, starting from a $q\bar{q}$ pair. The initial quark and the antiquark are both denoted by 0 while the gluons are denoted by α_i as explained in the text and in eq (5.8). The arrows on the dipoles indicate the colour flow, which goes from colour to anti-colour as before.

and the dipoles, each gluon links together two dipoles, and each dipole links together two gluons.

The inclusion of the dipole swing generates closed dipole loops, $\mathcal{L}$, in addition to the open chain, $\mathcal{C}$. The swing induces the transformations

$$\mathcal{C} \leftrightarrow \mathcal{C}' + \mathcal{L}, \tag{5.6}$$

$$\mathcal{L} \leftrightarrow \mathcal{L}_1 + \mathcal{L}_2. \tag{5.7}$$

In what follows, we will denote each N-dipole state as an element of the permutation group $\mathcal{P}_N$. For simplicity we suppress the transverse coordinates in the notation, and each gluon is denoted by a number indicating the order in which it was emitted, the first emitted gluon is denoted 1, the second 2 and so on. The initial $q\bar{q}$ pair is simply denoted by 0. A generic N-dipole state containing $N-1$ gluons, with $k_0 - 1$ gluons in the open chain, and the rest in m closed loops each containing k_i gluons, is denoted

$$\mathscr{A}_N = (0\,\alpha_1 \ldots \alpha_{k_0-1})(\alpha_{k_0} \ldots \alpha_{k_0+k_1-1}) \ldots (\alpha_{\sum_{i=0}^{m-1} k_i} \ldots \alpha_{N-1}) \in \mathcal{P}_N. \tag{5.8}$$

Here $\{\alpha_i\}_{i=1}^{N-1}$ is a permutation of $i = 1, \ldots, N-1$. The generic dipole state in (5.8) is illustrated in 6. Each arrow indicates the colour flow, and in the group theoretical notation in (5.8), each gluon α_i points to the gluon to the right of it. The open chain is always represented by the cycle containing the element 0 (the $q\bar{q}$ pair), and each cycle in (5.8) corresponds to a colour singlet. Since we cannot have colour singlet gluons, the numbers α_i cannot appear as 1-cycles. The only 1-cycle allowed is (0), which corresponds to a dipole formed by the initial $q\bar{q}$ pair.

Every element in the group $\mathcal{P}_N$ belongs to a certain class, which is determined by the cyclic structure of the element. The group $\mathcal{P}_4$ has 5 classes: 1111, 211, 31, 22 and 4. Here each n-cycle is represented by the number n. The state $\mathscr{A}_N$ in (5.8) belongs to the class

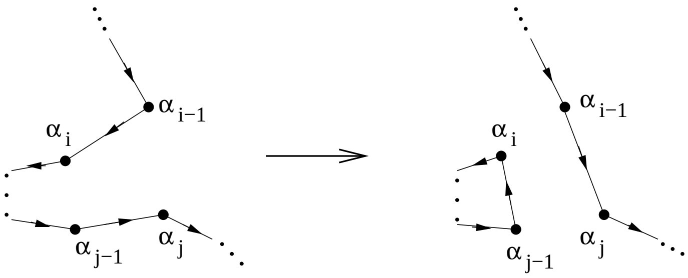

Figure 7: Illustration of eq (5.9).

$k_0\, k_1 \dots k_m$. The identity element is the permutation which takes every number onto itself, and has the cyclic structure $11\dots1$.

A swing operation can be represented by an element of $\mathcal{P}_N$ which consist of one 2-cycle and $(N-2)$ 1-cycles, *i.e.* by an element belonging to the class $211\dots1$. Thus for example, the swing illustrated in fig 7 is represented by $S(\alpha_i,\alpha_j) = (\alpha_i\alpha_j)\prod_{k\neq i,j}(\alpha_k)$, and we have

$$\begin{aligned} S(\alpha_i,\alpha_j)\otimes(\dots\alpha_{i-1}\alpha_i\dots\alpha_{j-1}\alpha_j\dots) &= (\alpha_i\alpha_j)\prod_{k\neq i,j}(\alpha_k)\otimes(\dots\alpha_{i-1}\alpha_i\dots\alpha_{j-1}\alpha_j\dots) \\ &= (\dots\alpha_{i-1}\alpha_j\dots)(\alpha_i\dots\alpha_{j-1}). \end{aligned} \tag{5.9}$$

Here $\otimes$ denotes the group multiplication. The action of $S(\alpha_i,\alpha_j)$ makes α_{i-1} point at α_j, and α_{j-1} point at α_i, leaving all other α_k unchanged as shown in the figure.

Due to the fact that not every swing leads to a physically acceptable state, the number of allowed swings for a state containing N dipoles is not simply $\frac{1}{2}N(N-1)$. This would e.g. be the the number of pairs in a reaction-diffusion type of formalism. In most formulations this is not taken into account, but we here wish to emphasize the importance of keeping track of the correct topology of the evolution. While this is not important in the original formulation of the dipole model where the dipole state evolves through the $1\to2$ splitting only, it is very necessary for transitions involving more than one initial dipole. In the appendix we show that the the number of physically possible states $\mathcal{N}_D$, and the number of possible swings $\mathcal{N}_S$ are for N dipoles given by

$$\mathcal{N}_D(N) = (N-1)!\sum_{l=0}^{N-1}\frac{(-1)^l}{l!}(N-l), \tag{5.10}$$

$$\mathcal{N}_S(N) = \frac{1}{2}(N-1)(N-2)+n_2, \tag{5.11}$$

where n_2 is the number of closed loops containing 2 dipoles.

Multiple swings in the $N\to N+1$ evolution

The classes of the group $\mathcal{P}_N$ are connected to each other via the swing as illustrated in fig 8, where each line means that two elements from the respective classes can be transformed into one another using one swing. Note that the longest distance is that between

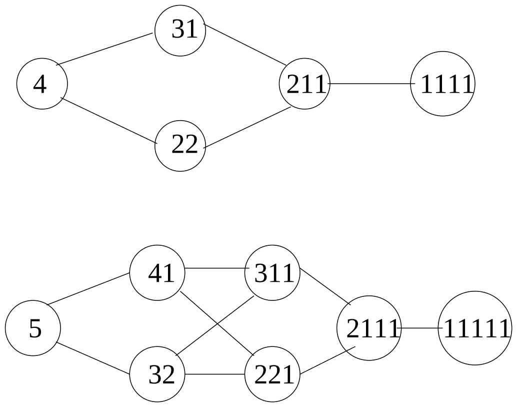

Figure 8: Class diagrams for $\mathcal{P}_4$ and $\mathcal{P}_5$.

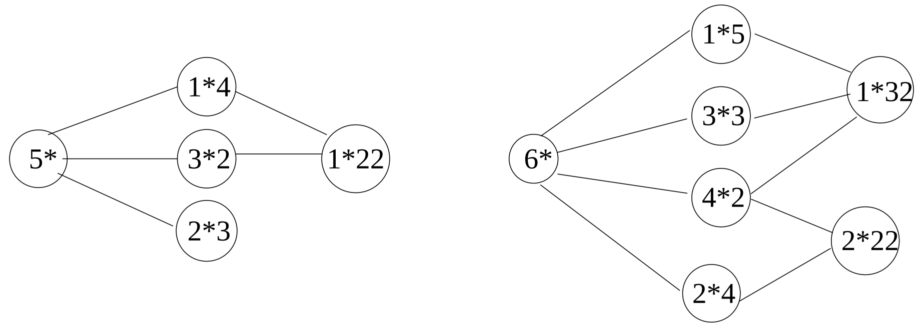

Figure 9: Class diagrams for the subset of physical states of $\mathcal{P}_5$ and $\mathcal{P}_6$. The open chain is marked by $*$.

4 and 1111, which requires 3 swings. In $\mathcal{P}_5$, we need 4 swings to go from 11111 to 5, as is also shown in fig 8.

Generally, for $\mathcal{P}_N$, any element in the class N can be reached from the identity element using $N-1$ swings. Explicitely, we can write the N-cycle $(j_1 \dots j_N)$ as $\prod_{i=1}^{N-1}(j_1 j_{i+1})$. This also implies that, given any arbitrary element $a \in \mathcal{P}_N$, we can reach any other element $b \in \mathcal{P}_N$ using at most $N-1$ swings. This is so since we can always find $N-1$ swings such that their product equals ba^{-1}.

However, not all classes fall into the subset of physically acceptable states, which in particular does not contain the identity element. Therefore we cannot a priori say whether or not the result above also holds for this subset. In fig 9, we show the class diagrams of physically acceptable states for $\mathcal{P}_5$ and $\mathcal{P}_6$. Here n^* denotes the open chain containing $n-1$ gluons. Thus using this notation we would say that $\mathscr{A}_N$ in (5.8) belongs to the "class" $k_0^* k_1 \dots k_m$. With a slight abuse of nomenclature, we will for simplicity continue to refer to these quantities as "classes", even though they do not constitute classes in the group theoretical sense.

Actually, the dipole splitting can be represented by the same class of elements as the dipole swing. Assume we are in the state $\mathscr{A}_N$. We then regard the splitting as a two-step process; first, we add the Nth gluon as a 1-cycle into the state $\mathscr{A}_N$, formally writing $\mathscr{A}_N$

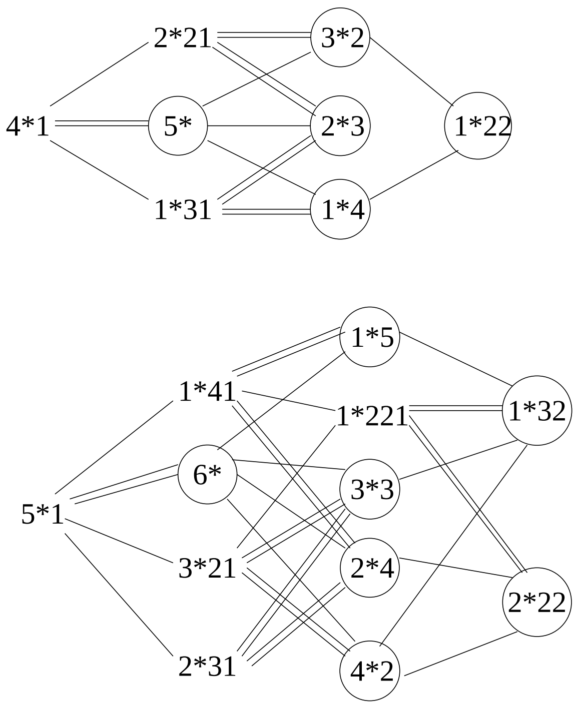

Figure 10: Class diagrams representing the evolutions $N = 4 \to N = 5$, and $N = 5 \to N = 6$. Here the double lines represent the dipole splitting. Note that for these we can only go in one direction, from an un-circled class to a circled one.

as an element of $\mathcal{P}_{N+1}$,

$$\bar{\mathscr{A}_N} = (0\,\alpha_1 \ldots \alpha_{k_0-1})(\alpha_{k_0} \ldots \alpha_{k_0+k_1-1}) \ldots (\alpha_{\sum_{i=0}^{l-1} k_i} \ldots \alpha_{N-1})(N) \in \mathcal{P}_{N+1}. \quad (5.12)$$

We put a bar on $\bar{\mathscr{A}_N}$ since, written in this way, it is not a physically acceptable state. Then, in the second step, we represent the emission of N from the dipole spanned between α_i and α_{i+1} by operating on $\bar{\mathscr{A}_N}$ with $S(N, \alpha_{i+1})$ (see eq (5.9)), since in that case $(\ldots \alpha_i\,\alpha_{i+1} \ldots)$ is replaced by $(\ldots \alpha_i\,N\,\alpha_{i+1} \ldots)$.

The class diagrams for the generic $N \to N + 1$ evolution can be drawn in a similar fashion as before. In fig 10, we show examples for $N = 4$ and $N = 5$. Here only the circled classes are physically acceptable, and it is one of these that we must end up in, starting from one of the un-circled ones. The maximal distance between any two circled classes in an $N \to N + 1$ evolution is 2 for $N = 3$, 3 for $N = 4$, and also 3 for $N = 5$. Thus in this case this distance is not equal to $N - 1$ for a N-dipole state. This does, however, not automatically imply that we can reach any given state in less than $N - 1$ swings.

We here conjecture that one can also for the subset of physical states go from $\mathscr{A}_N$ to any $\mathscr{A}_{N+1}$ by combining a dipole splitting with at most $N - 1$ swings. In the appendix we show explicitely that this statement is true for $N = 4$ (the cases $N = 1, 2$ are trivial, and $N = 3$ can be checked very easily). We have also checked this result for $N = 5$, $N = 6$ and $N = 7$, but we will for simplicity not present the calculations for these cases. In a process where $k - 1$ swings take place, k dipoles in the cascade get replaced by $k + 1$ dipoles, thus giving a $k \to k+1$ transition.

The only exception to the statement above is when there are states containing an isolated triangle. For example, if we wish to go from the state $(0)(1\,2\,3)(4)$ in 1^*31, to the state $(0\,4)(1\,3\,2)$ in 2^*3, we need 5 steps totally, 1 splitting and 4 swings. Thus for these states, $N-1$ swings are not enough (one needs $N+1$ swings). This is directly related to the fact that the physical set of states does not allow the steps: $(1\,2\,3) \to (1)(2\,3) \to (1\,3\,2)$. Therefore we need to use 4 swings rather than only 2 swings, which implies that we generally need $N+1$ swings for these states. Note, however, that this problem does not appear for higher order cycles. The 4-cycle $(1\,2\,3\,4)$ can for example be transformed into $(1\,4\,3\,2)$ easily: $(1\,2\,3\,4) \to (1\,4)(2\,3) \to (1\,4\,3\,2)$.

However, we also note that the only difference between the configurations $(1\,2\,3)$ and $(1\,3\,2)$ is in the orientation of the dipoles. Moreover, the states $\mathscr{A}_N = (\alpha_1\,\alpha_2\,\alpha_3)\mathscr{B}_{N-3}$ and $\mathscr{A}'_N = (\alpha_1\,\alpha_3\,\alpha_2)\mathscr{B}_{N-3}$ have exactly the same weights in the cascade evolution. They are therefore always produced equally, and it is therefore not a problem if we cannot go between them using $N-1$ swings. Finally, we note that the semi-classical approximation represented by the cascade evolution cannot take into account all quantum-mechanical interference effects. The quantum-mechanical states corresponding to the configurations $(\alpha_1\,\alpha_2\,\alpha_3)$ and $(\alpha_1\,\alpha_3\,\alpha_2)$ have colour factors $\mathrm{Tr}(T^aT^bT^c)$ and $\mathrm{Tr}(T^aT^cT^b)$ respectively, and for finite N_c these states are not orthogonal. Although the interference is suppressed by $1/N_c^2$, it is enhanced in for example the decay process $\Upsilon \to 3g$, and is in this case quite large [22], which is also confirmed experimentally.

6. Conclusions

Mueller's dipole model gives a simple picture of the small-x evolution which is also very suitable to use in a MC simulation. While it is known that it gives the correct evolution for dilute systems, a fully consistent version for dense systems, where saturation effects during the evolution cannot be neglected, is not known. There have been some attempts to interpret these saturation effects in terms of dipole mergings but it has not been possible to present a consistent probabilistic formulation.

A consequence of neglecting the saturation effects during the dipole evolution is that the model is not frame independent. In a previous paper [15] (see also [17] for a more detailed account) we demonstrated that approximate frame independence can be achieved by including a so called dipole swing in the evolution (the swing was also suggested in [16] as a mechanism to generate pomeron loops). Based on this, we constructed a phenomenological model which, implemented in a MC simulation, gives an almost frame-independent formalism.

It has been quite difficult to analytically derive the relevant dipole interactions which would give rise to saturation effects in the dipole model in a way consistent with boost invariance. A very simplified treatment of the dipole evolution is offered by the toy model introduced in [3, 11], and later also studied in [12, 13]. In this model it is possible to modify the evolution so that the formalism is explicitely frame independent. The evolution proceeds here by the addition of a new dipole at each step, in such a way that the total splitting rate saturates as the dipole occupation number gets large. As discussed in [12], this is actually quite similar to the way saturation occurs in the CGC formalism.

In this paper we have first shown that it is possible to give a probabilistic interpretation to the toy model evolutions in terms of positive definite $k \to k+1$ transitions. These transitions describe the coherent evolution of the dipoles, which is an unavoidable consequence of the requirement that the transition rates be positive definite. The evolution can also be formulated in a more close analogy with a standard reaction-diffusion picture, where the $k \to k+1$ transition rates only depend on the k dipoles involved in the transition. In this case, however, these rates appear with alternating signs which implies that a probabilistic treatment is not possible.

In the real dipole model such positive definite vertices can be generated by combining the dipole splitting with the dipole swing. In a $k \to k+1$ transition, a splitting is combined with $k-1$ simultaneous swings. In the approximation where each single dipole only scatters once, we have seen that it is enough to combine each splitting with a single swing in order to generate the necessary the colour correlations.

When each single dipole is allowed to scatter multiply, one needs to include more than one simultaneous swing. In this case the evolution proceeds by the $k \to k+1$ transitions as in the toy models mentioned above, and we have further shown that for a system of N dipoles, one needs at most $N-1$ simultaneous swings in order to generate all colour correlations induced by the multiple dipole interactions. We therefore obtain a close analogy with the toy model evolutions, and the dipole swing furthermore gives a probabilistic interpretation of the evolution. This is easy to show for spatially disconnected dipoles, but it is also the case in the more relevant situation when the dipoles are connected in chains.

This statement is strictly speaking not true for states containing a triangular loop, where only one orientation of this loop can be reached using at most $N-1$ swings. This is, however, no problem because the two possible orientations always appear with the same weight.

Acknowledgments

I would like to thank Gösta Gustafson for valuable discussions and critical reading of the manuscript. I am also thankful to Leif Lönnblad for useful comments, and to Bo Söderberg for useful discussions on mathematical issues.

A. The Number of Dipole States

In this section we will demonstrate that the number of possible states for a system containing $N-1$ gluons, together with the initial $q\bar{q}$ pair, is given by formula (5.10).

We start by considering n gluons in a closed topology, *i.e.* a state containing one or more closed dipole loops. If all n gluons are in the same loop we obviously have $(n-1)!$ possible states. Next we might have n gluons in two loops. The number of such states is given by the number of elements in $\mathcal{P}_n$ which consists of one k- and one $(n-k)$-cycle.

There are $\frac{1}{2}\binom{n}{k}(k-1)!(n-k-1)!$ such elements. The symmetry factor $1/2$ comes from the fact that we can write the k-cycle either to the left or the right of the $(n-k)$-cycle. For a closed topology consisting of m loops, each containing k_i dipoles, the number of possible states is given by

$$\frac{1}{m!}\prod_{j=1}^{m-1}\binom{\sum_{i=j}^{m} k_i}{k_j}\prod_{i=1}^{m}(k_i-1)! = \frac{1}{m!}\frac{n!}{\prod_{i=1}^{m} k_i}, \tag{A.1}$$

where

$$\sum_{i=1}^{m} k_i = n. \tag{A.2}$$

For each fixed closed topology with n gluons we also have $(N-1-n)!$ possible states in the open chain. To write down the total number of states it is convenient to introduce a generating function $G(z)$ whose series expansion give the dipole state multiplicity. We have

$$\begin{aligned} G(z) &= \binom{N-1}{n}(N-1-n)!n!\sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{k=2}^{\infty}\frac{z^k}{k}\right)^m\sum_{k_0=0}^{\infty} z^{k_0}, \\ &= (N-1)!\frac{e^{-z}}{(1-z)^2} \end{aligned} \tag{A.3}$$

where k_0 is the number of gluons in the open chain. We also demand that each closed loop contain at least 2 dipoles as we do not allow colour singlet gluons. The constraint $\sum_{i=0}^{m} k_i = N-1$ is automatically ensured since we are looking for the $(N-1)$th coefficient in the expansion of G. The expansion of G gives

$$G(z) = (N-1)!\sum_{M=0}^{\infty}\sum_{l=0}^{M}\frac{(-1)^l}{l!}(M-l+1)z^M. \tag{A.4}$$

We then immediately see that the $M = N-1$ coefficient is equal to eq (5.10). Notice also that for large N the number of states approaches $\frac{N!}{e}$. This can be compared to the number of possible states for N dipoles formed by N spatially independent charge–anti-charge pairs, which is $N!$, and to the number of states in a system consisting of a single open dipole chain, as in the original formulation of the dipole model, which is $(N-1)!$.

B. The Number of Possible Swings

In this section we demonstrate that the number of possible swings for a system containing $N-1$ gluons, together with the initial $q\bar{q}$ pair, is given by eq (5.11).

Assume again that we have m closed loops each containing k_i dipoles ($i = 1, \ldots, m$) with $k_i \geqslant 2$. The open chain contains k_0 gluons, and thus $\sum_{i=0}^{m} k_i = N-1$. Within each closed loop we then have

$$\frac{1}{2}\sum_{i=1}^{m} k_i(k_i-3)\theta(k_i \geqslant 3) \tag{B.1}$$

swings. The theta function takes into account the fact that we cannot have any swings in a loop containing only two or three dipoles. The number of swings between the closed loops is given by

$$\frac{1}{2}\sum_{i\neq j}^{m} k_i k_j \tag{B.2}$$

since there are no restrictions in this case. The number of swings between the open chain and the closed loops is given by

$$\sum_{i=1}^{m}(k_0+1)k_i, \tag{B.3}$$

and finally, the number of swings within the open chain is given by

$$\frac{k_0(k_0-1)}{2}. \tag{B.4}$$

The total number of swings is then given by

$$\begin{aligned}
&\frac{1}{2}\left\{\sum_{i=1}^{m}\{k_i(k_i-3)-k_i(k_i-3)\delta_{k_i\,2}\}+\sum_{i\neq j}^{m}k_ik_j+2(k_0+1)(N-1-k_0)+k_0(k_0-1)\right\}\\
&=\frac{1}{2}\left\{(N-1-k_0)^2-3(N-1-k_0)+2n_2+2(k_0+1)(N-1-k_0)+k_0(k_0-1)\right\}\\
&=\frac{1}{2}(N-1)(N-2)+n_2
\end{aligned} \tag{B.5}$$

where n_2 is the number of closed loops containing 2 dipoles.

C. More Details on the $N \to N+1$ Evolution

In this last appendix, we will explicitely prove that $N-1$ swings are enough to reach any arbitrary state for $N=4$. We have also explicitely checked the cases $N=5$, but we will for simplicity not present these calculations. We will very briefly try to sketch the case when $N=7$. The cases $N=1,2$ are trivial, and we also omit the case $N=3$ which is very easy to work out.

C.1 $N=4$

For this case, the class diagram is shown in fig 10. Assume first that we are in the class 4^*1. An arbitrary element in this class is given by $(0p_1\,p_2\,p_3)(4)$, for some permutation $\{p(i)\}$. We must reach any arbitrary element using at most 3 swings, and thus using at most 4 steps, counting the splitting as one step.

4*1 → 1*22: We see from fig 10 that we have to reach any element in 1^*22 using at most 3 steps (or else we would need at least 5 steps). An arbitrary element in 1^*22 can be written $(0)(p'_1\,p'_2)(p'_3\,p'_4)$ where $\{p'(i)\}$ is some other permutation. Without any loss of generality we might as well assume $p'_4=4$. Then we can always start by putting 4 next to p'_3 in the step $4^*1 \to 5^*$. In the next step we can then always isolate $(p'_3 4) = (p'_3 p'_4)$ in a 2-cycle. We then have an element $(0\pi(1)\pi(2))(p'_3\, p'_4)$ where $(\pi(1), \pi(2)) = (p'_1, p'_2)$ or (p'_2, p'_1). Finally we can separate$(0\pi(1)\pi(2)) \to (0)(\pi(1)\pi(2))$ to reach $(0)(p'_1\, p'_2)(p'_3\, p'_4)$. To summarize, we can go through the following steps

$$4^*1 \to 5^* \to 3^*2 \to 1^*22, \tag{C.1}$$

and reach any element in 1^*22 using at most 3 steps.

4*1 → 3*2: Here we can go in either 2 or 4 steps. We then want to go to an element $(0p'_1\, p'_2)(p'_3\, p'_4)$. Again we start by putting 4 to the right of the number which appears to the left of it in $(0p'_1\, p'_2)(p'_3\, p'_4)$. If this number is 0 (*i.e.* if $p'_1 = 4$), we can go from 5^* to 2^*3, putting 0 and 4 in 2^*. Then we just go back to 5^* to obtain $(0p'_1\, p'_2\, p'_3\, p'_4)$ or $(0p'_1\, p'_2\, p'_4\, p'_3)$. In either we case we can split this chain into $(0p'_1\, p'_2)(p'_3\, p'_4)$. Thus we go through

$$4^*1 \to 5^* \to 2^*3 \to 5^* \to 3^*2. \tag{C.2}$$

If $p'_1 \neq 4$, we go from 5^* to 3^*2, isolating 4 and its partner in 2. If the three elements in 3^* are not in the right order, we can split 3^* into 1^*2 and then go back to 3^*. Thus we can go through the steps

$$4^*1 \to 5^* \to 3^*2 \to 1^*23 \to 3^*2. \tag{C.3}$$

4*1 → 2*3: We here want to go to the element $(0p'_1)(p'_2\, p'_3\, p'_4)$. Again we start by putting 4 together with its final partner (4 is always put to the right of its partner). If its partner is 0, we can first go from 5^* to 3^*2 and then go back to 5^* so that the elements $(p'_2\, p'_3\, p'_4)$ have the correct permutation. Then we can in one step go the final configuration, and thus complete the process

$$4^*1 \to 5^* \to 3^*2 \to 5^* \to 2^*3. \tag{C.4}$$

If 4 is in the 3-cycle in the final element, we can go from 5^* to 3^*2 and then to 5^* again to put all the elements together in the correct positions. This is possible since all three elements in the final 3-cycle must have the correct permutation. Then we can finish by going from 5^* to 2^*3. Thus we can choose the path

$$4^*1 \to 5^* \to 3^*2 \to 5^* \to 2^*3. \tag{C.5}$$

Here the 2-cycle in 3^*2 contains 4 and its partner.

4*1 → 1*4: Here we want to reach an element $(0)(p'_1\, p'_2\, p'_3\, p'_4)$. Since we always put 4 (without loss of generality we can assume $p'_4 = 4$) next to its partner in the first step, all we need to do is to isolate them (p'_3 and p'_4) in a 2-cycle by going from 5^* to 3^*2. Then we go to 1^*22, after which we can simply join 22 to 4, to obtain any desired state. We thus have the path

$$4^*1 \to 5^* \to 3^*2 \to 1^*22 \to 1^*4. \tag{C.6}$$

4*1 → 5*: We can here use at most 3 steps. The final element we want to reach has the form $(0p'_1\, p'_2\, p'_3\, p'_4)$. There are two cases, either 4 and its partner are linked to 0, or they are not. If they are, we can split 5^* into 3^*2 where 3^* contains 0, 4 and its partner. Then the other two elements can always be put back in 5^* in the right position, so that we reach any 5^* element by

$$4^*1 \to 5^* \to 3^*2 \to 5^*. \tag{C.7}$$

In the second case, 4 and its partner are not linked to 0 (they are p_2' and p_3'). Then we can split 5^* into 2^*3 where 3 contains 4, its partner and one of the other two elements. They will automatically have the correct permutation, and we can then get the desired state by joining 3 and 2^* into 5^*. Then we have used the path

$$4^*1 \to 5^* \to 2^*3 \to 5^*. \tag{C.8}$$

We have thus seen that we can reach any arbitrary state in $N = 5$ from 4^*1 by combining at most 3 swings with a splitting.

Below we list the cases where we start from 2^*21. In this case we have an initial element $(0p_1)(p_2\, p_3)(4)$. By using a splitting first, we can either go to 3^*2, or to 2^*3.

$2^*21 \to 5^*$: First we fix 4 and its partner as usual. If the partner is p_1, we can separate $(0p_1\, 4)$ into $(0)(p_1\, 4)$, and then we can join $(p_1\, 4)$ with $(p_2\, p_3)$ to obtain an element in 1^*4. Then in one step we can go to the desired 5^* state. If its partner is 0, and the other three elements do not have the correct permutation, we can isolate two of them in a 2-cycle (after putting 3^* and 2 into 5^*), and then put them back into the 5^* state in the correct position. Thus we can through the two paths

$$2^*21 \to 3^*2 \to 1^*22 \to 1^*4 \to 5^*, \tag{C.9}$$

$$2^*21 \to 3^*2 \to 5^* \to 3^*2 \to 5^*. \tag{C.10}$$

If the final partner of 4 is either p_2 or p_3, we first go to 2^*3. Then it is easily seen that the two paths,

$$2^*21 \to 2^*3 \to 5^* \to 3^*2 \to 5^* \tag{C.11}$$

$$2^*21 \to 2^*3 \to 5^* \to 2^*3 \to 5^*, \tag{C.12}$$

can take us to any arbitrary element in 5^*.

$2^*21 \to 1^*4$: If 4 is next to either p_2 or p_3, we can directly from 2^*3 go to 5^*, and then to 1^*4. If 4 is next to p_1, we can from 3^*2 go to 1^*22 and then in one more step we can reach any 1^*4 state. Thus we can follow the paths

$$2^*21 \to 2^*3 \to 5^* \to 1^*4 \tag{C.13}$$

$$2^*21 \to 3^*2 \to 1^*22 \to 1^*4, \tag{C.14}$$

to reach any state in 1^*4 in maximum 3 steps.

$2^*21 \to 1^*22$: If 4 is next to p_1 we can finish in 2 steps, $2^*21 \to 3^*2 \to 1^*22$. If 4 is next to either p_2 or p_3, we can first join 2^*3 into 5^*, and then isolate 4 and its partner in the 2-cycle in 3^*2. Then we need only one more step. We thus have the steps

$$2^*21 \to 2^*3 \to 5^* \to 3^*2 \to 1^*22. \tag{C.15}$$

2*21 → 3*2: If p_1 is partner to 4, we need at most go to 1^*22 from 3^*2, and then back to 3^*2 to finish. If 4 is next to 0, we can first go to 5^* from 3^*2, and then split 5^* into the desired 3^*2 state. If 4 is next to p_2 or p_3, we can again go to 5^* and then directly to 3^*2. Thus we have the steps

$$2^*21 \to 3^*2 \to 1^*22 \to 3^*2 \tag{C.16}$$

$$2^*21 \to 3^*2 \to 5^* \to 3^*2 \tag{C.17}$$

$$2^*21 \to 2^*3 \to 5^* \to 3^*2. \tag{C.18}$$

2*21 → 2*3: If 4 is in the final 2-cycle (*i.e.* next to 0), all we need to do is to join 3^*2 into 5^*, after which we can extract the final 3-cycle in one step. If one the other hand 4 is in the final 3-cycle, we are after one step either finished, or we can from 2^*3 go to 5^*, putting 0 and its final partner together, after which we can split 5^* split into 2^*3, obtaining the desired state. Thus we can go choose one of the paths,

$$2^*21 \to 3^*2 \to 5^* \to 2^*3 \tag{C.19}$$

$$2^*21 \to 2^*3 \to 5^* \to 2^*3. \tag{C.20}$$

Finally, we check the case when we start from 1^*31.

1*31 → 1*22: Here we only need two steps:

$$1^*31 \to 1^*4 \to 1^*22 \tag{C.21}$$

which can be easily seen.

1*31 → 1*4: This case is almost trivial, and we can see that we need at most three steps:

$$1^*31 \to 1^*4 \to 1^*22 \to 1^*4. \tag{C.22}$$

1*31 → 3*2: If 4 appears in the 2-cycle in 3^*2, all we need is to take the steps

$$1^*31 \to 1^*4 \to 1^*22 \to 3^*2. \tag{C.23}$$

In the second step we here isolate 4 and its partner in one of the 2-cycles. If 4 instead appears in 3^*, we can go through either $1^*31 \to 1^*4 \to 5^* \to 3^*2$, or $1^*31 \to 2^*3 \to 5^* \to 3^*2$, depending whether or not 4 appears next to 0 in the final configuration.

1*31 → 5*: Here we can again have 4 either to the right of 0 or not. If not, we just take the steps $1^*31 \to 1^*4 \to 1^*22 \to 3^*2 \to 5^*$. If it is next to 0, we instead take the steps $1^*31 \to 2^*3 \to 5^* \to 3^*2 \to 5^*$.

1*31 → 2*3: If 4 appears in 3, we can just go through $1^*31 \to 1^*4 \to 5^* \to 2^*3$ and finish. However, if 4 is next to 0, $N-1$ swings are not enough to go to $(04)(p_1\, p_3\, p_2)$ as we have already discussed in the main text. We have also already noted that this is not a problem for the frame independence. With this remark we finish the case $N = 4$.

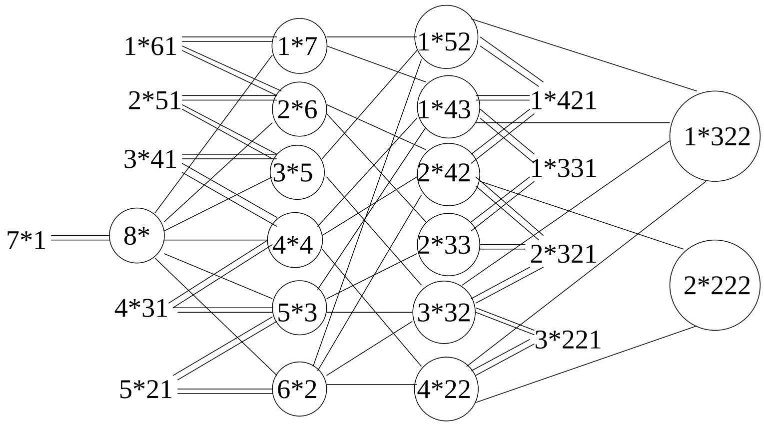

Figure 11: Class diagrams for the evolution $N = 7 \rightarrow N = 8$. Here for simplicity we do not draw lines between the un-circled classes.

C.2 $N = 7$

The class diagram for the evolution $N : 7 \rightarrow 8$ is shown in fig 11. Here it is obviously too tedious to explicitely check all possible connections. However, one can now use the results from the previous cases to simplify the analysis. For example, let us consider the case where we want to go from 7*1 to 1*322. If 7 appears in the final 2-cycle, we can pull it out together with its partner after 2 steps, and the question is then whether we can go from 6^* to 1*32 in 5 steps, and we know from above that this is indeed possible. If 7 appears in the final 3-cycle, we can after 4 steps isolate it with its partners, and then we need to go from 5^* to 1*22 in 3 steps, which we also know is possible. Another example is if we want to go to a final configuration in 2*42, where 7 appears in 2. After 2 steps, 7 and its partner can again be isolated, and we then need to go from 6^* to 2*4 in 5 steps. Again we know that this is indeed possible. We can similarly work out the rest of the cases. It is also interesting to note that we can here go from 1*331 to 2*33 in 7 steps, even if the initial and final states differ by the orientations of the two triangular loops.

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