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Isospin breaking in $K\pi$ vector form-factors for the weak and rare decays $K_{\ell 3}$, $K \to \pi\nu\overline{\nu}$ and $K \to \pi\ell^+\ell^-$

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Abstract

We calculate the two form-factors for the four Kaon to pion transitions via a vector current to order p^6 in Chiral Perturbation Theory to first order in isospin breaking via the quark masses. In addition we derive relations between these form-factors valid to first order in the up-down quark-mass difference but to all orders in Chiral Perturbation Theory.

We present numerical results for all eight form-factors at $t = 0$ and for varying t and for the scalar form-factors at the Callan-Treiman point.

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1 Introduction

The semileptonic decays of a Kaon to a pion and two leptons play a significant role in flavour physics. On the one hand, the weak decays $K_{\ell 3}$ are a main source of our knowledge of the CKM matrix element V_{us} and on the other hand, the rare decays to a lepton-anti-lepton or neutrino-antineutrino pair provide a good testing bed for loop effects in flavour physics. The form-factors themselves quantize the hadronic uncertainties as can be exemplified by the so-called master formulae. This is e.g. for $K_{\ell 3}$, see [1] and references therein,

$$\Gamma\left(K^i \to \pi^j \ell^+ \nu_l\right) = C_{ij}^2 \frac{G_F^2 S_{EW} m_K^5}{192\pi^3} \left|V_{us} f_+^{K^i\pi^j}(0)\right|^2 \mathcal{I}_\ell^{ij} \left(1 + 2\Delta_{EM}^{ij}\right) . \tag{1}$$

A similar formula exists for the rare decays, see [2] and references therein. We will refer to the $K_{\ell 3}$ decays as weak decays and the ones with a lepton-antilepton pair as rare decays.

Theoretical work on these form-factors goes back a long way. In Chiral Perturbation Theory (ChPT), the lowest-order (LO) result dates back to [3] while the next-to-leading-order (NLO) was evaluated by Gasser and Leutwyler [4]. They calculated the vector form-factor f_+ for the weak decays including the isospin breaking due to $m_u - m_d$ and the scalar form-factor f_0 in the isospin limit. The form-factors are known in the isospin limit to next-to-next-to-leading order (NNLO) in Chiral Perturbation Theory (ChPT) [5, 6]. In [5] a comparison with experimental results was done and a useful relation for the order p^6 constants needed for $f_+(0)$ obtained. The isospin breaking to NLO for the vector form-factors $f_+(t)$ for the rare decays was done in [2] to NLO. The electromagnetic corrections to NLO, i.e. order e^2p^2, are also known, for the weak decays [7] and for the rare decays [2]. In this paper we calculate the isospin breaking corrections due to the quark-mass difference $m_u - m_d$ to the vector and scalar form-factors to NNLO order in ChPT for all eight form-factors. The NNLO results are new for all form-factors while the NLO results are new for the scalar form-factors. Some preliminary results were reported in [8]. In addition we discuss the results on ratios of form-factors to NNLO. Some of these ratios were observed to have special features at NLO in [4] and [2]. We prove that the relations (11) and (12) are valid to all orders in ChPT to first order in $m_u - m_d$. The double ratio (12) was also discussed in [2] but not proven there. There exists also work using dispersion relation for the form-factors in the isospin limit, see [9, 10] and references therein.

This paper is organized as follows. We define the form-factors and derive the relations the form-factors should satisfy to all orders in ChPT and first order in $m_u - m_d$ in Sect. 2. Next we give a short discussion of ChPT in Sect. 3 and derive how π^0-η mixing can be taken into account to NNLO in ChPT in Sect. 4. Sect. 5 defines the various ratios of form-factors we use and discusses how they are obeyed at NLO and NNLO. We also discuss there some general aspects of our calculation and give the LO results. Explicit formulas are not provided at NNLO, they are simply too long but we present the NLO formulas in App. A and the dependence on the order p^6 low-energy constants (LECs) in App. B. The estimate of these order p^6 LECs we use is presented in Sect. 6. Our main results are presented numerically in Sect. 7. These include, numerical results on the values of

$f_+(0)$, its t-dependence, ratios as a function of t and the deviation from F_K/F_π at the Callan-Treiman point. A short summary is given in Sect. 8.

2 Form-factors and isospin relations

In this paper we deal with the four matrix-elements

$$\langle\pi^0(p')|\overline{s}\gamma_\mu u(0)|K^+(p)\rangle = \frac{1}{\sqrt{2}}\left[(p'+p)_\mu f_+^{K^+\pi^0}(t)+(p-p')_\mu f_-^{K^+\pi^0}(t)\right], \quad (2)$$
$$\langle\pi^-(p')|\overline{s}\gamma_\mu u(0)|K^0(p)\rangle = \left[(p'+p)_\mu f_+^{K^0\pi^-}(t)+(p-p')_\mu f_-^{K^0\pi^-}(t)\right], \quad (3)$$
$$\langle\pi^+(p')|\overline{s}\gamma_\mu d(0)|K^+(p)\rangle = \left[(p'+p)_\mu f_+^{K^+\pi^+}(t)+(p-p')_\mu f_-^{K^+\pi^+}(t)\right]. \quad (4)$$
$$\langle\pi^0(p')|\overline{s}\gamma_\mu d(0)|K^0(p)\rangle = \frac{-1}{\sqrt{2}}\left[(p'+p)_\mu f_+^{K^0\pi^0}(t)+(p-p')_\mu f_-^{K^0\pi^0}(t)\right]. \quad (5)$$

We have thus in total a set of 8 form-factors. They depend on

$$t=(p'-p)^2\,, \quad (6)$$

the square of the four momentum transfer to the leptons. The form-factors are normalized such that

$$f_+^{K^i\pi^j}(0)=1 \quad (7)$$

in the $SU(3)$ limit of $m_u=m_d=m_s$. In the isospin limit

$$f_\pm = f_\pm^{K\pi} = f_\pm^{K^+\pi^0} = f_\pm^{K^0\pi^-} = f_\pm^{K^+\pi^+} = f_\pm^{K^0\pi^0}\,. \quad (8)$$

$f_+^{K\pi}$ is referred to as the vector form-factor, because it specifies the P-wave projection of the crossed channel matrix-elements $\langle\overline{s}\gamma_\mu q(0)|\ |\ K^i,\pi^j \text{ in} >$. The S-wave projection is described by the scalar form-factor

$$f_0^{K^i\pi^j}(t) = f_+^{K^i\pi^j}(t) + \frac{t}{m_{K^i}^2-m_{\pi^j}^2} f_-^{K^i\pi^j}(t)\,. \quad (9)$$

We will refer to the decays as the charged weak for (2), neutral weak for (3), charged rare for (4) and neutral rare for (5).

In this paper we derive the isospin breaking due to the quark-mass difference m_u-m_d to NNLO for the eight form-factors defined above. We do this to first order in isospin breaking. Let us now derive first some general properties. The isospin-breaking operator $(1/2)\,(m_u-m_d)\left(\overline{u}u-\overline{d}d\right)$ has isospin one. The pions have isospin one and the Kaons as well as the vector operator are in an isospin $1/2$ multiplet. To first order in isospin breaking from $\delta=m_u-m_d$ the form-factors described above can be rewritten in the form

$$\begin{aligned}
f_\ell^{K^+\pi^0}(t) &= f_\ell^A(t)+\delta f_\ell^B(t)+\mathcal{O}(\delta^2)\,,\\
f_\ell^{K^0\pi^-}(t) &= f_\ell^A(t)-\delta f_\ell^D(t)+\mathcal{O}(\delta^2)\,,\\
f_\ell^{K^+\pi^+}(t) &= f_\ell^A(t)+\delta f_\ell^D(t)+\mathcal{O}(\delta^2)\,,\\
f_\ell^{K^0\pi^0}(t) &= f_\ell^A(t)-\delta f_\ell^B(t)+\mathcal{O}(\delta^2)\,,
\end{aligned} \quad (10)$$

for $\ell = +,-,0$. The form (10) is a direct consequence of the Wigner-Eckart theorem. This can be interpreted as that the size of isospin breaking depends on the final pion and the sign also depends on which kaon is in the initial state.

As a consequence of (10) we obtain the relations

$$f_\ell^{K^+\pi^0}(t) - f_\ell^{K^0\pi^-}(t) - f_\ell^{K^+\pi^+}(t) + f_\ell^{K^0\pi^0}(t) = 0 + \mathcal{O}(\delta^2)\,, \tag{11}$$

and

$$r(t) \equiv \frac{f_\ell^{K^+\pi^0}(t) f_\ell^{K^0\pi^0}(t)}{f_\ell^{K^0\pi^-}(t) f_\ell^{K^+\pi^+}(t)} = 1 + \mathcal{O}(\delta^2) \tag{12}$$

These relations do not have to be satisfied when electromagnetic corrections are included. Photon exchange contains isospin 0, 1 and 2 parts allowing different corrections to all four amplitudes. The isospin 0 and 1 parts do satisfy the same relations, but not the isospin 2 part.

The relations are valid for all three form-factors $f_\ell^{K^i\pi^j}$ with $\ell = +,-,0$. They are also valid if the currents in (2-5) are replaced by the scalar densities $\overline{s}u(0)$ and $\overline{s}d(0)$.

3 Chiral Perturbation Theory

Chiral Perturbation Theory (ChPT) is an effective field theory to describe the strong interactions at very low energy. The effective Lagrangian is constructed based on two important properties of the physical hadron spectrum. Pseudo-scalar mesons, the lowest-lying states in the spectrum are separated from the rest of the hadrons, i.e. there exists a mass gap. This allows the heavier particles to decouple from the dynamics of the pseudo-scalar mesons. Their influence can be described by point-like couplings. The other important fact is that the spectrum does not show the chiral symmetry of the underlying theory (QCD). The pseudo-scalars are assumed to be the the pseudo-Goldstone particles emerging from the spontaneous breaking of this chiral symmetry. The nonzero but small mass of the pseudo-scalar mesons are because quarks have a finite mass in, reality which breaks the chiral symmetry explicitly.

According to the Goldstone's theorem, the Goldstone particles do not interact at zero momentum. This immediately offers a weakly interacting theory as a basis for perturbation theory. The first systematic consideration on the applicability of the effective Lagrangians was made by Weinberg [11] and Gasser and Leutwyler [12]. The effective chiral Lagrangian is an expansion in momentum and quark masses. In the chiral power-counting, quark masses are of order p^2. Taking into account the Lorentz invariance and chiral symmetry, the lowest order chiral Lagrangian which also complies with the discrete symmetries can be written down as

$$\mathcal{L}_2 = \frac{F_0^2}{4}\left(\langle D_\mu U D^\mu U^\dagger\rangle + \langle \chi U^\dagger + U\chi^\dagger\rangle\right) \tag{13}$$

and the next-to-leading Lagrangian with the introduction of the external field technique was written down by Gasser and Leutwyler [13] and reads

$$\mathcal{L}_4 = L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle$$

$$
\begin{aligned}
&+L_3\langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U\rangle + L_4\langle D^\mu U^\dagger D_\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+L_5\langle D^\mu U^\dagger D_\mu U(\chi^\dagger U + U^\dagger\chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
&+L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U\chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
&-iL_9\langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle \\
&+L_{10}\langle U^\dagger F^R_{\mu\nu} U F^{L\mu\nu}\rangle + H_1\langle F^R_{\mu\nu}F^{R\mu\nu} + F^L_{\mu\nu}F^{L\mu\nu}\rangle + H_2\langle \chi^\dagger\chi\rangle\,. \qquad (14)
\end{aligned}
$$

The matrix $U \in SU(3)$ contains the pseudo-scalars and its exponential representation is

$$U(\phi) = \exp(i\sqrt{2}\phi/F_0)\,, \qquad (15)$$

where

$$\phi(x) = \begin{pmatrix} \frac{\pi_3}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi_3}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix}. \qquad (16)$$

The external fields are defined through the covariant derivatives and field strength tensor as

$$D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu\,, \quad F^L_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu - i\,[l_\mu, l_\nu]\,, \qquad (17)$$

The right-handed and left-handed external fields are denoted by r_μ and l_μ respectively. The Hermitian 3×3 matrix χ contains the scalar (s) and pseudo-scalar (p) external densities and is given as $\chi = 2B_0\,(s + ip)$. The constants F_0 and B_0 are related to the pion decay constant and quark condensate respectively. There are however, 10+2 unknown free parameters in the Lagrangian $\mathcal{L}_4$ where these effective constants contain the effects of heavy degrees of freedom and can be determined by invoking experimental data as well as by Lattice QCD technique. One of the theoretical approach, on the other hand, is the application of the resonance chiral perturbation which provides an approximate estimate of the low energy constants (LECs). The extention of the chiral Lagrangian to the next-to-next-to-leading order is also accomplished [14]. At this order there are a large number of LECs, 90+4.

The external scalar field s contains the quark masses and the mass terms in the lowest order Lagrangian $\mathcal{L}_2$ can be diagonalized exactly. In the presence of $m_u \neq m_d$ the physical π^0 and η differ from the triplet and octet states via a lowest-order mixing angle ϵ as

$$
\begin{aligned}
\pi_3 &= \pi^0\cos(\epsilon) - \eta\sin(\epsilon) \\
\eta_8 &= \pi^0\sin(\epsilon) + \eta\cos(\epsilon) \qquad (18)
\end{aligned}
$$

The lowest order mixing angle is

$$
\begin{aligned}
\tan(2\epsilon) &= \frac{\sqrt{3}}{2}\frac{m_d - m_u}{m_s - \hat{m}}\,, \\
\hat{m} &= (m_u + m_d)/2\,. \qquad (19)
\end{aligned}
$$

A review on ChPT to order p^6 is [15]. References to other recent reviews and lectures can be found there.

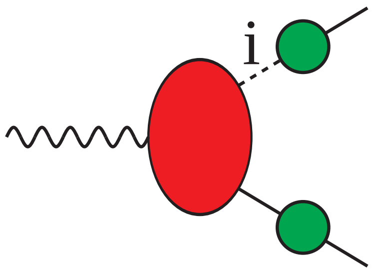

Figure 1: The full three-point Green function is represented. The oval stands for the amputated three-point Green function and circles indicate the full two-point functions. The solid lines are external mesons and the dashed line labeled by i , indicates the sum over states implied in the external leg where mixing occurs. The wiggly line indicates the external vector current.

4 Matrix-elements in the presence of mixing

For this work we need to work out the matrix elements defined earlier in the presence of mixing. These matrix elements can be determined from three-point Green functions. Two of the external legs are the meson propagators and the third one is the external field. The matrix element is obtained from the Green function using the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula. The matrix element is related to the residue of the Green function in momentum space where all the propagators are continued to the on-shell mass. The case of two point-function with one leg undergoing mixing is worked out in [16] and we generalized this to a four point-function with mixing on two external legs in [17]. In this article we study the form-factors in $K^i \to \pi^j$ transitions where, in case of the neutral pion in the decay product, mixing should also taken into account. Fig. 1 depicts the three-point Green function relevant for this work where we have only considered mixing in one external propagator. Amplitudes are obtained via

$$\mathcal{A}_{i_1\ldots i_n} = \left(\frac{(-i)^n}{\sqrt{Z_{i_1}\ldots Z_{i_n}}}\right) \prod_{i=1}^{n} \lim_{k_i^2\to m_i^2} (k_i^2 - m_i^2)\, G_{i_1\ldots i_n}(k_1,\ldots,k_n)\,. \tag{20}$$

This formula shows the general case with n-outgoing particles. The function $G_{i_1\ldots i_n}$ is the full n-point Green function where we now express it in terms of the amputated Green functions and only two meson propagators to suite for the current article as follows

$$G_{43,extv} = G_{44}(p^2 \approx m^2_{4\,phys}) G_{3i}(p^2 \approx m^2_{3\,phys}) \mathcal{G}_{4i,extv}\,. \tag{21}$$

Summation over index i runs over two possibilities of being a neutral pion or eta. 4 and 3 are the indices referring to the Kaon and the neutral pion respectively. $\mathcal{G}_{4i}$ is the amputated Green function that contains both on-shell and off-shell Feynman diagrams. The two-point functions are expanded near the physical poles as

$$G_{ii}(p^2 \approx m^2_{i\,phys}) = \frac{iZ_i}{p^2 - m^2_{i\,phys}}\,. \tag{22}$$

The function Z_i is called the wavefunction renormalization factor. The expansion of the off-diagonal two-point functions around the physical poles is somewhat more involved but can be done in terms of the one-particle irreducible two-point functions $\Pi_{ij}(m^2)$ and the mass differences needed in the propagator of the i-leg in Fig. 1 as explained in [16]. We now expand all quantities to the required chiral order and use the fact that we have exactly diagonalized the lowest Lagrangian to obtain for the full amplitudes to order p^6:

$$\begin{aligned}
\mathcal{A}_{43,extv} &= \mathcal{A}^{(2)}_{43,extv} + \mathcal{A}^{(4)}_{43,extv} + \mathcal{A}^{(6)}_{43,extv} + \cdots\,, && (23)\\
\mathcal{A}^{(2)}_{43,extv} &= \mathcal{G}^{(2)}_{43,extv}\,, && (24)\\
\mathcal{A}^{(4)}_{43,extv} &= \mathcal{G}^{(4)}_{43,extv} - \left(\frac{1}{2}Z^{(4)}_{44} + \frac{1}{2}Z^{(4)}_{33}\right)\mathcal{G}^{(2)}_{43,extv} - \frac{\Pi^{(4)}_{38}(3)}{\Delta m^2_2}\mathcal{G}^{(2)}_{48,extv}\,, && (25)\\
\mathcal{A}^{(6)}_{43,extv} &= \mathcal{G}^{(6)}_{43,extv} - \frac{1}{2}\left(Z^{(6)}_{33} + Z^{(6)}_{44}\right)\mathcal{G}^{(2)}_{43,ext} - \frac{1}{2}\left(Z^{(4)}_{33} + Z^{(4)}_{44}\right)\mathcal{G}^{(4)}_{43,extv}\\
&\quad + \frac{3}{8}\left(\left(Z^{(4)}_{33}\right)^2 + \left(Z^{(4)}_{44}\right)^2\right)\mathcal{G}^{(2)}_{43,ext} + \frac{1}{4}\left(Z^{(4)}_{33}\,Z^{(4)}_{44}\right)\mathcal{G}^{(2)}_{43,extv}\\
&\quad + \frac{\Pi_{38}(3)^{(4)}}{\Delta m^2_2}\mathcal{G}^{(4)}_{48,extv} + \frac{\Pi_{38}(3)^{(6)}}{\Delta m^2_2}\mathcal{G}^{(2)}_{48,extv} + \frac{\Pi_{38}(3)^{(4)}\,\Pi_{88}(3)^{(4)}}{\Delta m^2_2}\mathcal{G}^{(2)}_{48,extv}\\
&\quad - \frac{1}{2}\left(Z^{(4)}_{38}\frac{\Pi_{38}(3)^{(4)}}{\Delta m^2_2}\right)\mathcal{G}^{(2)}_{43,extv} - \frac{1}{2}\left(Z^{(4)}_{33}\frac{\Pi_{38}(3)^{(4)}}{\Delta m^2_2} + Z^{(4)}_{44}\frac{\Pi_{38}(3)^{(4)}}{\Delta m^2_2}\right)\mathcal{G}^{(2)}_{48,extv}\,. && (26)
\end{aligned}$$

The Z and Π factors have been valuated earlier [16, 17] and we thus need to evaluate the various $\mathcal{G}$ amputated amplitudes.

5 Analytical results and ratios of form-factors

To do the calculation, we need to calculate the tree level diagrams of Fig.2, the one- and two-loop diagrams of Fig. 3 and the two-loop diagrams with overlapping divergences of Fig. 4 with isospin breaking kept in the masses and vertices. These amplitudes should then be put together with the wave-function renormalization and mixing effects given in (23). The lowest order expressions are quite simple. The form-factors $f_-^{K^i\pi^j}$ all vanish and the others are

$$\begin{aligned}
f_+^{K^+\pi^0}(t) &= \cos\epsilon + \sqrt{3}\sin\epsilon\,,\\
f_+^{K^0\pi^-}(t) &= 1\,,\\
f_+^{K^+\pi^+}(t) &= 1\,,\\
f_+^{K^0\pi^0}(t) &= \cos\epsilon - \sqrt{3}\sin\epsilon\,. && (27)
\end{aligned}$$

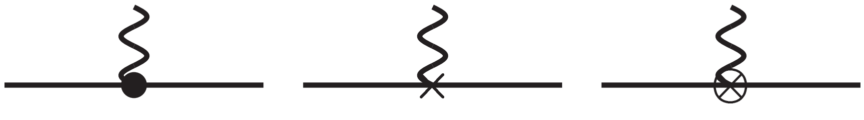

Figure 2: The tree level Feynman diagrams for the Kaon transition form-factors. The wiggly line indicates the insertion of the vector current, a dot an order p^2 vertex, a cross an order p^4 vertex and a crossed circle an order p^6 vertex.

The NLO expressions agree with the isospin breaking ones calculated in [4, 2] for the f_+ form-factors. The isospin breaking in the f_- and f_0 form-factors is new. The NLO results are given in App. A. The full NNLO results are very lengthy but we have performed two independent calculations that are in agreement. All eight form-factors are also finite using the general subtractions calculated in [18]. The nonlocal divergences and the other quantities that can be removed using $\overline{MS}$ subtraction also cancel as they should. These consistency checks are described in detail in [19]. The loop integrals are computed using the methods described in [20, 21].

The main existing previous work is for $K_{\ell 3}$ decays. Isospin breaking to order p^4 for f_+ was done in [4] and the electromagnetic parts worked out in [7] to order e^2p^2.

While this work was in progress, an analysis of the isospin breaking in the rare decay form-factors $f_+^{K^+\pi^+}$ and $f_+^{K^0\pi^0}$ to NLO and order e^2p^2 appeared [2]. They also noted that the relation (12) was satisfied but do not seem to have realized it is an immediate consequence of isospin.

Isospin breaking in $f_-^{K^i\pi^j}$ has not been discussed earlier within the context of ChPT.

In [4] another relation valid to NLO and first order in isospin breaking was found. The ratio of form-factors

$$\frac{f_+^{K^0\pi^-}(t)}{f_+^{K^+\pi^0}(t)} = \frac{f_+^{K^0\pi^0}(t)}{f_+^{K^+\pi^+}(t)} \tag{28}$$

is independent of momenta and can be cleanly predicted in terms of pseudo-scalar meson masses. The equality follows from the use of (12).

Our results satisfy the relations (11) and (12), we had to use a large number of relations between the various integrals to check this and obtained in this way another nontrivial check on our results. The NLO relation found by [4] is no longer true at NNLO. There are t-dependent corrections at order NNLO of all[3] types, pure two-loop, L_i^r-dependent and C_i^r-dependent ones. The relation (28) is also not true for the scalar form-factors $f_0^{K^i\pi^j}(t)$ nor for $f_-^{K^i\pi^j}(t)$ already at NLO.

We define here also two more ratios for later use, first the ratio of the two weak decay

[3]That it was not valid for the C_i^r contributions at order p^6 was also noticed in [2].

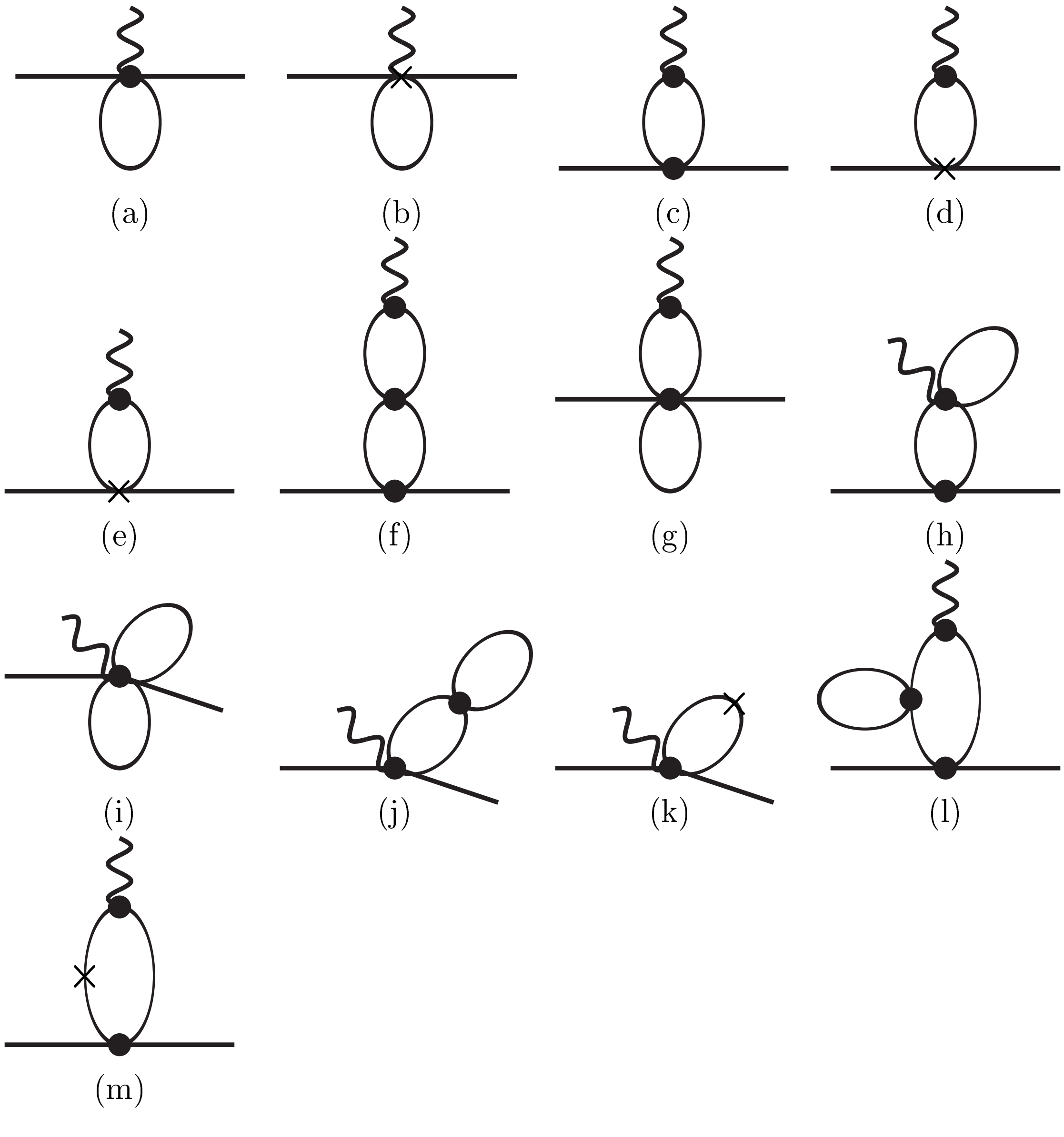

Figure 3: The one- and two-loop Feynman diagrams for the Kaon transition form-factors without overlapping divergences. Notation as in Fig. 2.

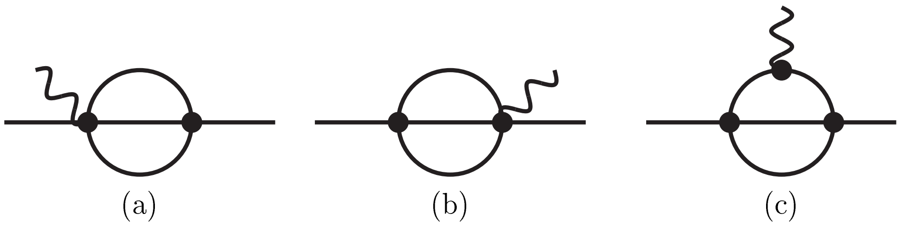

Figure 4: The two-loop Feynman diagrams for the Kaon transition form-factors with overlapping divergences. Notation as in Fig. 2.

form-factors

$$r_{0-}(t) = \frac{f_+^{K^+\pi^0}(t)}{f_+^{K^0\pi^-}(t)} \tag{29}$$

and second the ratio of the rare to the weak decay with charged pions in the final state

$$r_K(t) = \frac{f_+^{K^+\pi^+}(t)}{f_+^{K^0\pi^-}(t)}\,. \tag{30}$$

We define similarly definitions of $r^0(t)$, $r^0_{0-}(t)$ and $r^0_K(t)$ for ratios of the scalar form-factors $f_0^{K^i\pi^j}(t)$.

6 Resonance estimate of the contribution from the C_i^r

This contribution is the most difficult to estimate. In the isospin limit, $f_+^{K\pi}(0)$ only depends on the combination $(C_{12}^r + C_{34}^r)\left(m_K^2 - m_\pi^2\right)^2$ [5] and its estimate is the main uncertainty in the chiral prediction for $f_+^{K\pi}(0)$. A review can be found in [1]. The under lying reason for the factor $\left(m_K^2 - m_\pi^2\right)^2$ is the Ademollo-Gatto theorem[22]. The reasoning used there remains valid also in the case with isospin breaking for the form-factors that do not involve π^0-η mixing. The isospin conserving case is proportional to $(m_s - \hat{m})^2$, but including isospin breaking, the form-factor for the neutral weak decay is proportional to $(m_s - m_u)^2$ and for the charged rare decay it is proportional to $(m_s - m_d)^2$. The full order p^6 tree level contribution in these cases is again proportional to $C_{12}^r + C_{34}^r - (L_5^r)^2$ just as was found for the isospin conserved case in [5, 23].

The general method we use to estimate the C_i^r is of saturation by a finite number of resonances introduced by [24, 25]. We use the vector Lagrangian in the Proca formulation with parameters as determined in [19, 5]. The scalar effect was studied in detail in [23] and more generally in [26]. Some problems with this procedure are discussed in [27].

The vector exchange contribution does not contribute to the values at $t = 0$ for $f_+^{K^i\pi^j}(t)$. It does however contribute strongly away from zero. The estimate we use here for the C_i^r from vector exchange is described in [5]. In particular, the same estimate is in good agreement with the estimate of the curvature in the pion electromagnetic form-factor which leads to an experimental determination of[21]

$$-4\left(C^r_{88}-C^r_{90}\right)=(0.22\pm0.02)\;10^{-3} \tag{31}$$

compared with a prediction of $0.26\,10^{-3}$. This is the part that estimates the contribution from the C^r_i in Fig. 6. The way we have implemented it here is via the effect on the C^r_i directly as given in [28].

Second, we take into account the contribution from the singlet pseudo-scalar degree of freedom P_1. We use the simple Lagrangian

$$\mathcal{L}_{\eta'}=\frac{1}{2}\partial_\mu P_1\partial^\mu P_1-\frac{1}{2}M^2_{\eta'}P_1^2+i\tilde{d}_m P_1\langle\chi_-\rangle\,. \tag{32}$$

Integrating out P_1 leads to the order p^4 term with L_7 and the order p^6 Lagrangian

$$\mathcal{L}_{\eta'}=-\frac{\tilde{d}_m^2}{2M^4_{\eta'}}\partial_\mu\langle\chi_-\rangle\partial^\mu\langle\chi_-\rangle\text{ with }\tilde{d}_m=20\text{ MeV}. \tag{33}$$

The latter was rewritten in general in terms of the basis of operators of [14] in [17]. The result is[4]

$$\begin{aligned}\partial_\mu\langle\chi_-\rangle\partial_\mu\langle\chi_-\rangle &= O_{18}+\frac{2}{9}O_{19}-\frac{1}{3}O_{20}+\frac{1}{3}O_{21}+2O_{27}+\frac{2}{3}O_{31}-\frac{1}{3}O_{32}+\frac{1}{3}O_{33}\\ &\quad -2O_{35}+O_{37}-\frac{8}{3}O_{94}\,.\end{aligned} \tag{34}$$

The result is that the singlet P_1 contributes via the order p^6 constants C^r_i also to the isospin breaking in the values for $f_+^{K^i\pi^j}(0)$ but it does so only via π^0-η mixing. The numerical result is

$$\begin{aligned}f_+^{K^+\pi^0}(0)\Big|_{P_1} &= 0.00065\,,\\ f_+^{K^0\pi^0}(0)\Big|_{P_1} &= -0.00065\,.\end{aligned} \tag{35}$$

7 Numerical results

7.1 Input parameters

For the masses we use the particle data book masses except for the eta where we use for consistency the value 547.3 MeV. The input values for the order p^4 constants L^r_i we use are fit 10 of [16]. This fit used the K_{e4} data from E865, and input values $m_s/\hat{m}=24$ and $F_K/F_\pi=1.22$. For the masses it used the physical masses. Electromagnetic corrections

[4]This was derived by the authors of [29] but not included in the final manuscript. It also agrees with the expression shown by Kaiser[30].

to the Kaon mass were included with the estimate of the violation of Dashen's theorem of [31] included. An extensive discussion of this fit can be found in [29] using the older K_{e4} data and working fully in the isospin limit.

We will always quote results for the isospin conserving formulas of [21] where the kaon mass is taken to be the mass of the kaon involved and similarly, we use for the pion mass the mass of the particle involved in the matrix element. For the results with the formulas including isospin breaking, we have used for the Kaons their physical masses but for both charged and neutral pion the same mass, since to first order in $m_u - m_d$ these have the same mass. We have always taken the mass of the final state pion involved in the matrix element. The reason for this choice is to always have the kinematics right in the matrix elements. The effect of changing the pion mass can be judged by looking at the results for the isospin symmetric formulae which we quote for different input Kaon and pion masses in Tab. 1.. The order p^6 constants C_i^r have been put to zero at the scale $\mu = 770$ MeV unless otherwise noted in Sect. 7.2. In Sects. 7.3, 7.4 and 7.5 we have put the C_i^r at the value estimated by vector and singlet pseudo-scalar exchange at $\mu = 700$ MeV.

The main fit 10 with Dashen's violation gave $m_u/m_d = 0.45$ while removing the violation of Dashen's theorem gave $m_u/m_d = 0.52$. The standard values without order p^6 and without violation of Dashen's theorem gave $m_u/m_d = 0.585$ [16]. These values, together with the input value for $m_s/\hat{m}$ correspond to $\sin\epsilon = 0.0143$, 0.0119 and 0.00986 respectively. This can be compared with the value of 0.0106 ± 0.0008 used in [2] which used the input neglecting order p^6 effects. Note however that the recent evaluation from $\eta \to 3\pi$ [17] leads to somewhat different values.

7.2 $f_+^{K^i\pi^j}(0)$

Here we give the results for the form-factor values at zero. In Tab. 1 we first show the results for the isospin conserving formula of [5]. Here the only isospin breaking effect is the different kaon and pion mass used as described in Sect. 7.1. The results for the charged and neutral weak decay are in agreement with [5]. We have in fact checked that the formulas including isospin breaking numerically agree with the isospin conserving formula if the masses are set to the same isospin conserving masses and $\sin\epsilon = 0$. As is clear from the numbers in Tab. 1, the isospin breaking effects from varying the masses in the loops is quite small.

In contrast, we have shown the equivalent set of values for our amplitudes including isospin violation. It can be seen that effect is much larger for the amplitudes with a neutral pion in the final state. That is, as can already be seen at lowest order, pion-eta mixing is important for this decay. The values in Tab. 2 are with $m_u/m_d = 0.45$ or $\sin\epsilon = 0.01429$.

To show the variation with the input for m_u/m_d, we show in Tab. 3 using the same inputs as for Tab. 2 but with $m_u/m_d = 0.585$ or $\sin\epsilon = 0.009857$. This corresponds to the fit for m_u/m_d without violations of Dashen's theorem and using order p^4 expressions. Our results, except for the lowest order in (27), are explicitly linear in $\sin\epsilon$. The numbers are slightly different from the preliminary results quoted in [8]. This is due to a slightly different way of treating the pion masses.

	$f_+^{K^+\pi^0}$	$f_+^{K^0\pi^-}$	$f_+^{K^+\pi^+}$	$f_+^{K^0\pi^0}$
order p^2	1.00000	1.00000	1.00000	1.00000
order p^4	-0.02276	-0.02266	-0.02226	-0.02316
order p^6	0.01423	0.01462	0.01406	0.01480
p^6 2-loop	0.01104	0.01130	0.01090	0.01145
p^6 L_i^r-dependent	0.00320	0.00332	0.00316	0.00336
sum of p^2, p^4 and p^6	0.99156	0.99196	0.99180	0.99164

Table 1: The different contributions to $f_+^{K^i\pi^j}(0)$ using the isospin *conserving* amplitudes of [5]. We have also shown the break-up of the order p^6 expressions in the pure two-loop part and the L_i^r-dependent part. The part depending on the C_i^r is *not* included.

	$f_+^{K^+\pi^0}$	$f_+^{K^0\pi^-}$	$f_+^{K^+\pi^+}$	$f_+^{K^0\pi^0}$
order p^2	1.02465	1.00000	1.00000	0.97514
order p^4	-0.01775	-0.02292	-0.02197	-0.02838
order p^6	0.00809	0.01470	0.01391	0.02095
p^6 2-loop	0.00159	0.01145	0.01081	0.02092
p^6 L_i^r-dependent	0.00650	0.00325	0.00309	0.00004
sum of p^2, p^4 and p^6	1.01499	0.99177	0.99194	0.96772

Table 2: The different contributions to $f_+^{K^i\pi^j}(0)$ using the amplitudes *including isospin breaking*. We have also shown the break-up of the order p^6 expressions in the pure two-loop part and the L_i^r-dependent part. The part depending on the C_i^r is *not* included. We used here $m_u/m_d = 0.45$ corresponding to the two-loop fit of [16] including Dashen's theorem violations.

	$f_+^{K^+\pi^0}$	$f_+^{K^0\pi^-}$	$f_+^{K^+\pi^+}$	$f_+^{K^0\pi^0}$
order p^2	1.01702	1.00000	1.00000	0.98288
order p^4	-0.01931	-0.02282	-0.02202	-0.02675
order p^6	0.00986	0.01467	0.01395	0.01919
p^6 2-loop	0.00435	0.01142	0.01084	0.01815
p^6 L_i^r-dependent	0.00551	0.00325	0.00311	0.00104
sum of p^2, p^4 and p^6	1.00757	0.99186	0.99193	0.97532

Table 3: The different contributions to $f_+^{K^i\pi^j}(0)$ using the amplitudes *including isospin breaking*. We have also shown the break-up of the order p^6 expressions in the pure two-loop part and the L_i^r-dependent part. The part depending on the C_i^r is *not* included. We used here $m_u/m_d = 0.585$ corresponding to the one-loop fit of [16] without violations of Dashen's theorem.

Using the results of Tab. 2 we can also quote numerical results for the various ratios defined earlier at the point $t = 0$. First the ratio of charged to neutral weak decay. This is

$$r_{0-}(0) = 1.02465 + 0.00587 - 0.00711 = 1.02341\,, \tag{36}$$

where we see that the order p^6 contributions lower the result and essentially cancel the enhancement from the ratio at order p^4. If we add the contribution from singlet P_1 exchange we obtain $r_{0-} = 1.024068$. However, compared to the old order p^4 value, we get again an enhancement due to the larger value of $\sin\epsilon$ obtained from the order p^6 fit. We showed the contributions to the ratio at order p^2, p^4 and p^6. This should be compared to the experimental ratio as determined from the global FLAVIAnet fit [32]

$$r_{0-exp} = 1 + \Delta_{SU(2)} = 1.0284 \pm 0.0040\,. \tag{37}$$

As we see, we obtain a reasonable agreement.

We can also look at the double ratio r from (12). Our formulas satisfy it exactly. The main numerical source of the difference at higher orders results from the fact that we used a different pion mass in the denominator and the numerator. The result is

$$r = 0.99918 - 0.00161 + 0.00085 = 0.99842\,, \tag{38}$$

where a fairly sizable cancellation happens between the order p^4 and order p^6 contributions. We again showed the contributions to the ratio at order p^2, p^4 and p^6.

The final ratio, of weak to rare decays with a charged pion in the final state is

$$r_K = 1.00000 + 0.00097 - 0.00080 = 1.00017\,. \tag{39}$$

The three numbers in the middle part are once more the contributions to the ratio at order p^2, p^4 and p^6. Once more, we see a significant cancellation between the order p^4 and p^6 contributions.

7.3 $f_+^{K^i\pi^j}(t)$

In this subsection we show the results as a function of t for the the $f_+^{K^i\pi^j}$ form-factors. We first show the case for the neutral weak decay in Figs. 5 and 6. Fig. 5 shows the result to lowest order, NLO and NNLO. It can be seen that there is a nice convergence in the entire region shown.

We show the various subparts of the order p^6 contribution in Fig. 6. The contributions shown are the two-loop contribution, the part dependent on the order p^4 LECs L_i^r as well as the part that depends on the order p^6 LECs C_i^r.

The results shown so far for $f_+^{K^0\pi^-}$ are essentially the same as those in the isospin limit of [5]. We have included isospin breaking but it is a rather small effect for this form-factor. Rather than showing similar plots for the other three form-factors we show here the ratios as a function of t. First we show the variation of the full ratio r as a function of t. The ratio

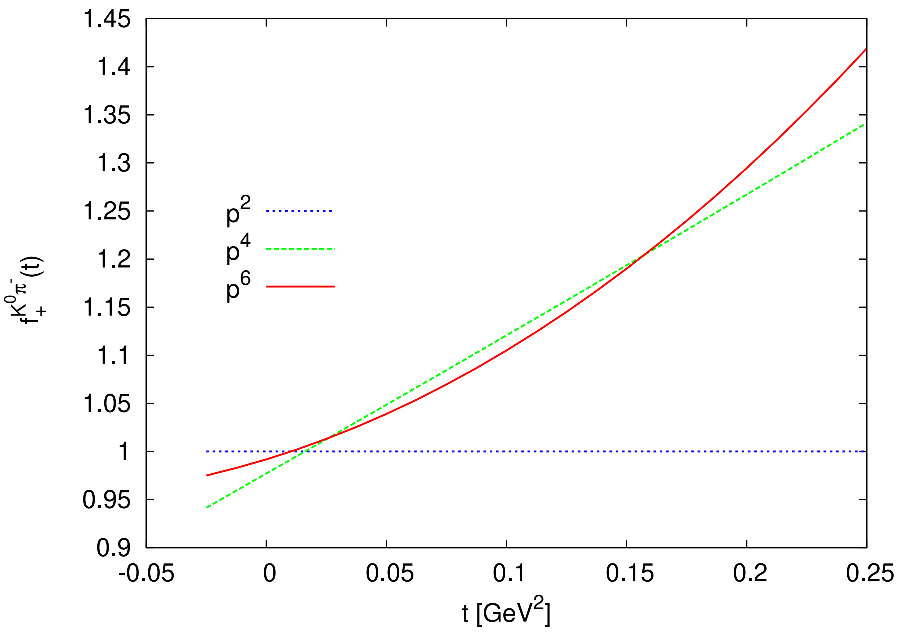

Figure 5: The form-factor $f_+^{K^0\pi^-}(t)$ as a function of t. Shown are the lowest order (p^2), NLO (p^4) and NNLO result (p^6). Isospin breaking is included.

r is somewhat more different from one than naively expected since we included different pion masses. The ratio r_{0-} defined in (29) is shown as a function of t in Fig. 8. It was shown in [4] that at NLO this ratio is independent of t. We have checked that this is no longer true at NNLO as is clearly visible in the figure. However, there is clearly no sign of an anomalously large isospin breaking effect in this ratio. We also show the similar ratio for the charged rare to neutral weak decay, r_K as defined in (30) in Fig. 9.

7.4 $f_0^{K^i\pi^j}(t)$

In this subsection we show the results as a function of t for the the $f_0^{K^i\pi^j}$ form-factors. We first show the case for the neutral weak decay in Figs. 10 and 11. Fig. 10 shows the result to lowest order, NLO and NNLO. It can be seen that there is a nice convergence in the entire region shown.

We show the various subparts of the order p^6 contribution in Fig. 11. The contributions shown are the two-loop contribution, the part dependent on the order p^4 LECs L_i^r as well as the part that depends on the order p^6 LECs C_i^r. The latter is essentially zero here since the vector exchange contribution to the scalar form-factor vanishes to the order considered here and the singlet pseudo-scalar doesn't contribute either. A scalar exchange would contribute but we have not included such an estimate here. The curvature visible is here mainly coming from the loops.

The results shown so far for $f_0^{K^0\pi^-}$ are essentially the same as those in the isospin limit

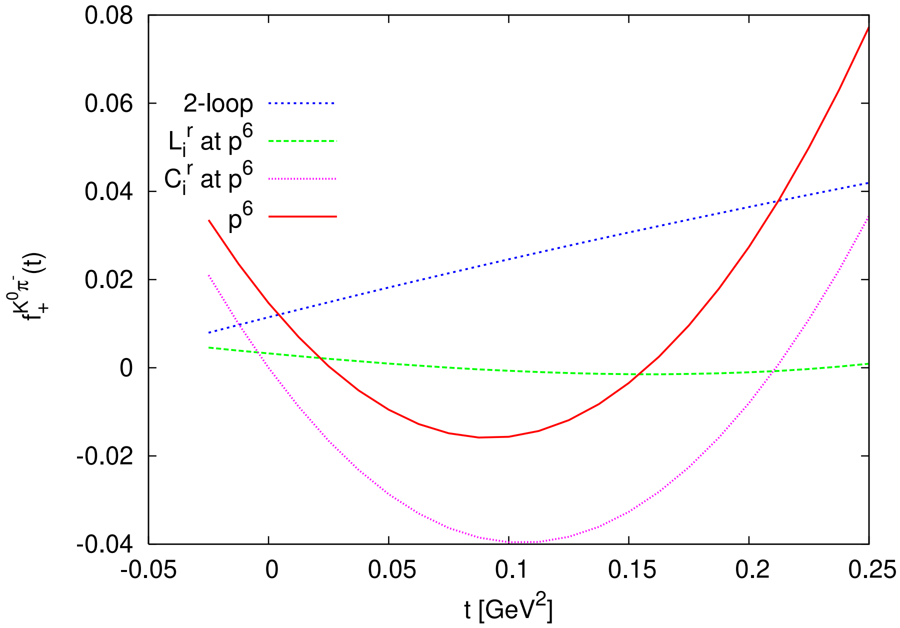

Figure 6: The form-factor $f_+^{K^0\pi^-}(t)$ as a function of t. Shown are the full order p^6 contribution and its three constituent parts, the pure two-loop contribution, the L_i^r-dependent part and the C_i^r-dependent part. The contribution to the quadratic slope comes mainly from the C_i^r dependent part but that is fixed from the pion electromagnetic form-factor [21]. Isospin breaking is included.

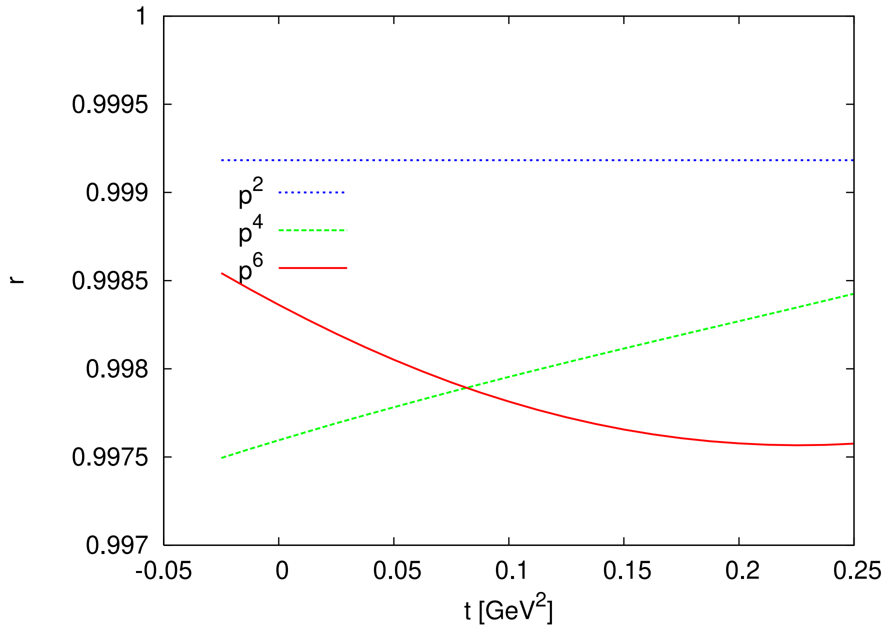

Figure 7: The ratio r as defined in (12) as a function of t. Both the deviation from 1 and the t dependence are effects of higher order in isospin breaking.

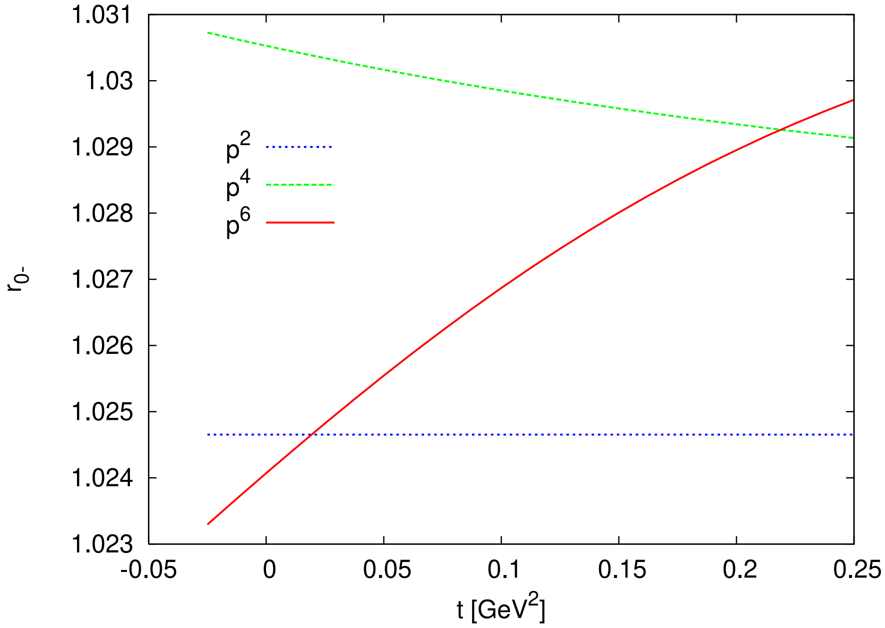

Figure 8: The ratio r_{0-} as defined in (29) as a function of t. This is the ratio of the charged to neutral weak decay. The t dependence for the NLO result is higher order in isospin breaking but is first order at NNLO.

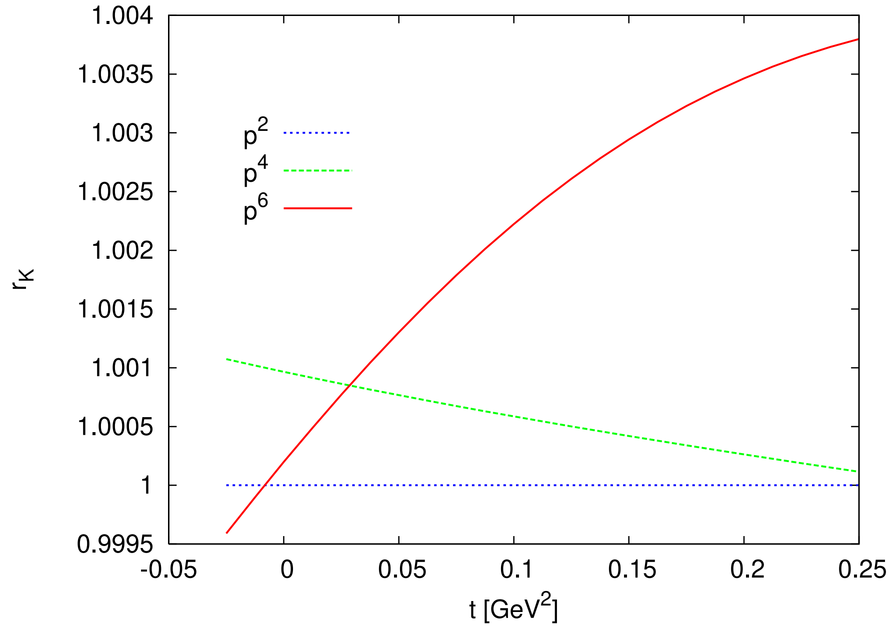

Figure 9: The ratio r_K as defined in (30) as a function of t. This is the ratio of the charged rare to neutral weak decay.

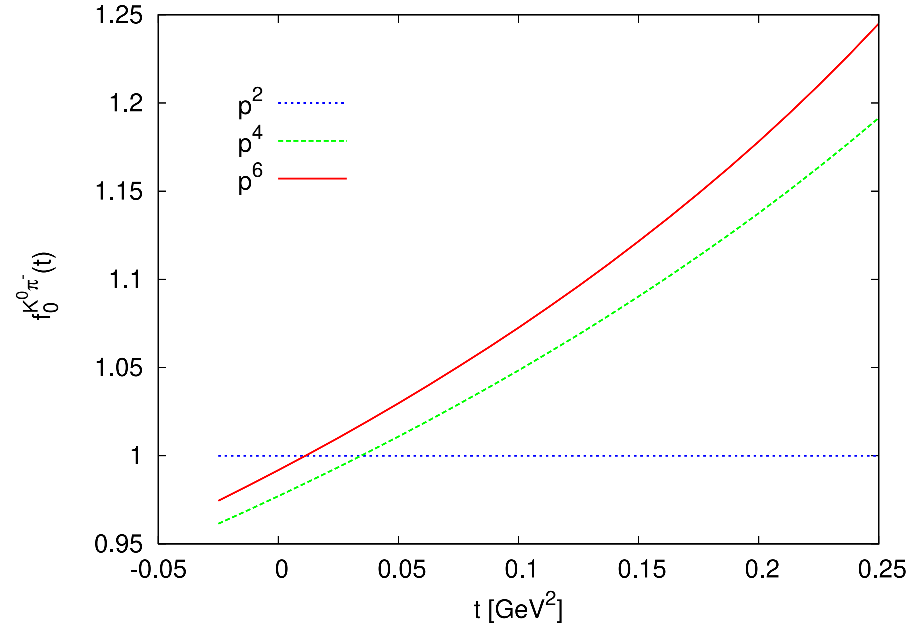

Figure 10: The form-factor $f_0^{K^0\pi^-}(t)$ as a function of t. Shown are the lowest order (p^2), NLO (p^4) and NNLO result (p^6). Isospin breaking is included.

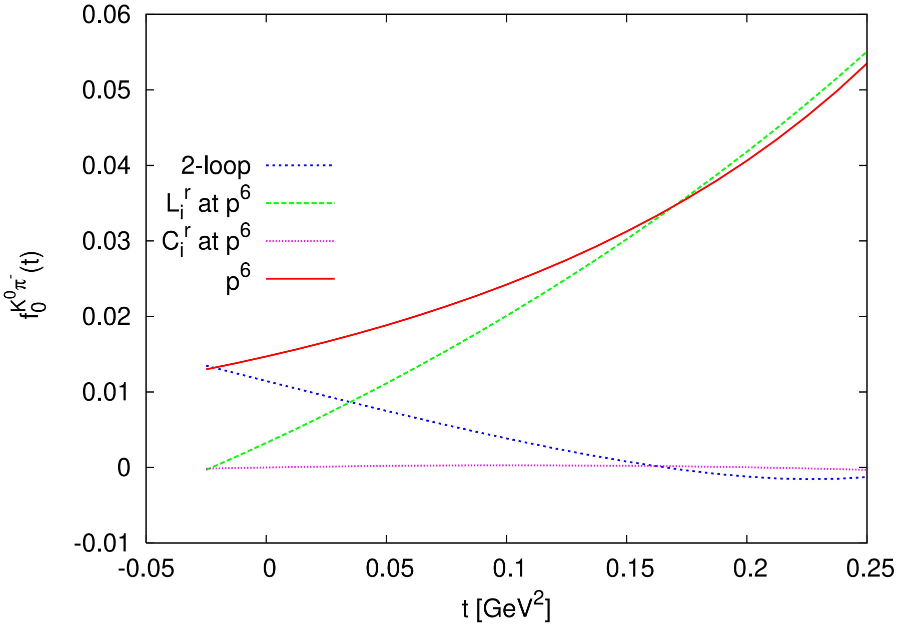

Figure 11: The form-factor $f_0^{K^0\pi^-}(t)$ as a function of t. Shown are the full order p^6 contribution and its three constituent parts, the pure two-loop contribution, the L_i^r-dependent part and the C_i^r-dependent part. Isospin breaking is included.

of [5]. We have included isospin breaking but it is a rather small effect for this form-factor. Rather than showing similar plots for the other three form-factors we show here the ratios as a function of t. First we show the variation of the full ratio r^0 as a function of t. The ratio r^0 is somewhat larger than naively expected since we included different pion masses The ratio r^0_{0-} defined in (29) but for the scalar form-factor is shown as a function of t in Fig. 13. This ratio can be t-dependent already at NLO which is clearly visible. However, there is no sign of an anomalously large isospin breaking effect in this ratio. We also show the similar ratio for the charged rare to neutral weak decay, r^0_K as defined in (30) in Fig. 14. The scalar form-factors are not needed for the weak decays to an electron or the rare decays to a neutrino-antineutrino pair. They do contribute to the weak decays to a muon and the rare decays with a muon-anti-muon pair via a the axial current couplings to the latter from the short-distance contributions.

7.5 Callan-Treiman point

The Callan-Treiman relation [33] states that the scalar form-factor at $t = m_K^2 - m_\pi^2$ satisfies

$$f_0\left(m_K^2 - m_\pi^2\right) = \frac{F_K}{F_\pi} + \mathcal{O}(m_u, m_d)\,. \tag{40}$$

This relation is derived using current algebra in the up-down–sector and should thus have rather small corrections of order m_π^2. The relation is exact when the up and down quark

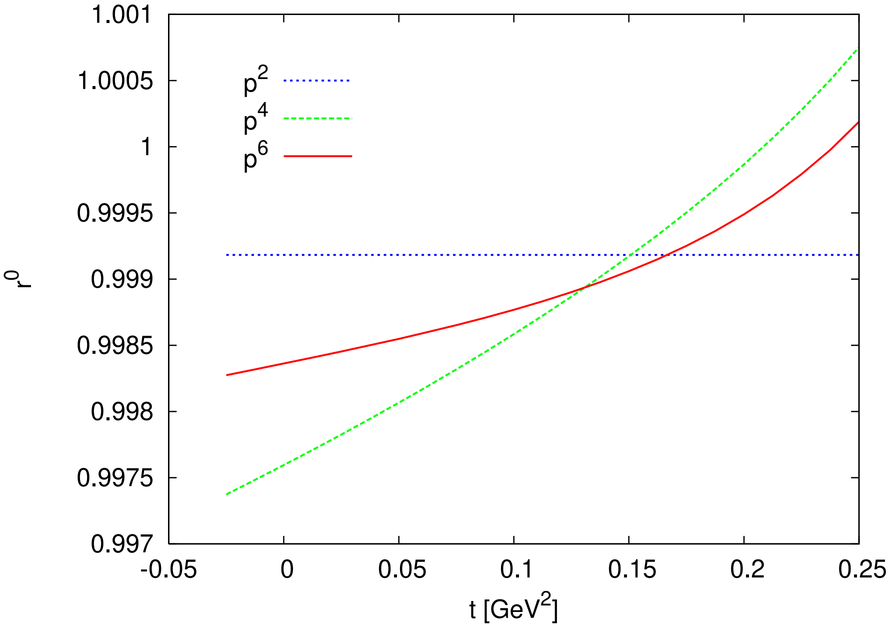

Figure 12: The ratio r^0 as defined in (12) bit for the scalar form-factor as a function of t. Both the deviation from 1 and the t dependence are effects of higher order in isospin breaking.

masses are zero. The correction at NLO was worked out in [4] and found to be

$$\Delta_{CT} \equiv f_0\left(m_K^2 - m_\pi^2\right) - \frac{F_K}{F_\pi} = -3.5\,10^{-3}[4]\,. \tag{41}$$

The correction in the isospin limit at NNLO was never presented in [5]. We have calculated this and also for the four different amplitudes at order p^4 but for the C_i^r estimates we have used the vector and pseudo-scalar singlet contributions as described in Sect. 6.

The inputs we used produce $F_K/F_\pi = 1.22$, which is what we have subtracted from the $f_0^{K^i\pi^j}(m_{K^i}^2 - m_{\pi^j}^2)$ in the numbers quoted below. The isospin symmetric expression with $m_K^2 = m_{K^0}^2$ and $m_\pi^2 = m_{\pi^+}^2$ gives

$$\Delta_{CT} = -6.2\,10^{-3}\,. \tag{42}$$

As we see there is a substantial NNLO correction. Note that we did not include the contributions from nonzero C_{12}^r, C_{34}^r in this expression. These read

$$\Delta_{CT}|_{C_i^r} = \frac{16}{F_\pi^4}\left(2C_{12}^r + C_{34}^r\right) m_\pi^2 \left(m_K^2 - m_\pi^2\right)\,. \tag{43}$$

It is clear that this satisfies the Callan-Treiman theorem. Notice that it is the same combination of order p^6 LECs that shows up in the scalar slope when the part via F_K/F_π is subtracted as in Eq. (5.2) of [5].

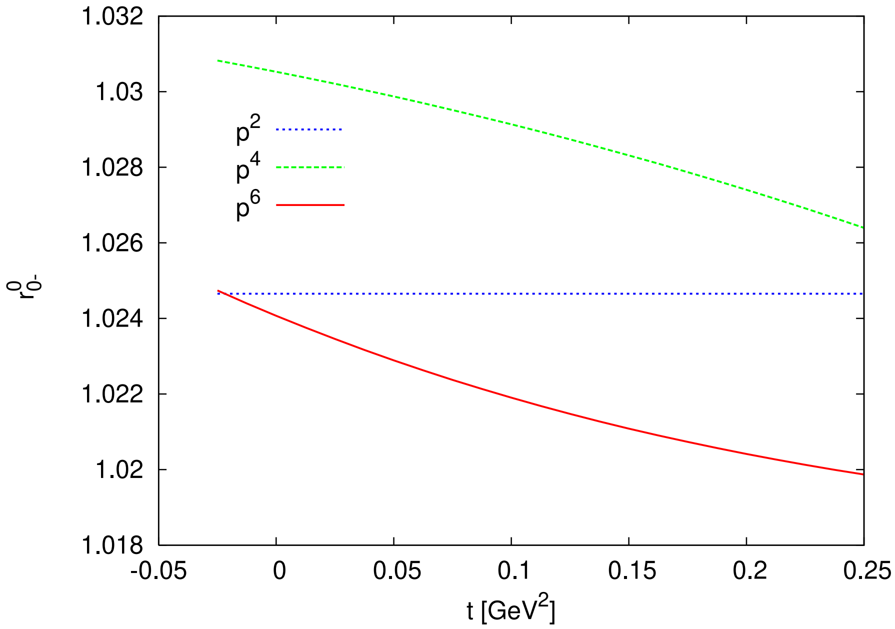

Figure 13: The ratio r^0_{0-} as defined in (29) but for the scalar form-factor as a function of t. This is the ratio of the charged to neutral weak decay. The t dependence is first order in isospin breaking both at NLO and NNLO.

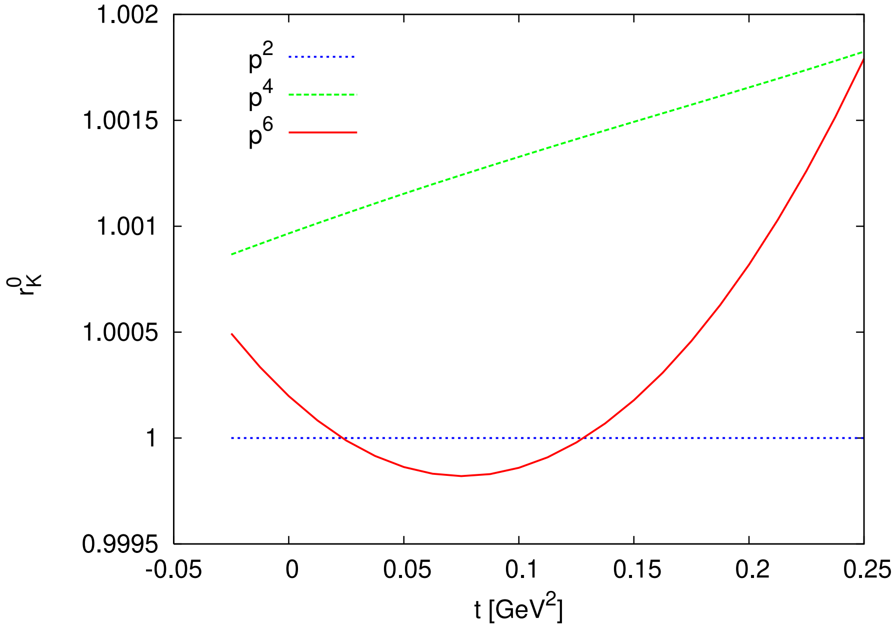

Figure 14: The ratio r^0_K as defined in (30) but for the scalar form-factor as a function of t. This is the ratio of the charged rare to neutral weak decay.

For the expression including isospin breaking we simply present the numerical results directly

$$
\begin{aligned}
\Delta_{CT}^{K^+\pi^0} &= 15.1\,10^{-3}\,,\\
\Delta_{CT}^{K^0\pi^-} &= -5.6\,10^{-3}\,,\\
\Delta_{CT}^{K^+\pi^+} &= -9.4\,10^{-3}\,,\\
\Delta_{CT}^{K^0\pi^0} &= -26.4\,10^{-3}\,.
\end{aligned}
\tag{44}
$$

Recently, Leutwyler discussed the experimental measurements and the extrapolation to the Callan-Treiman point [34]. The results we obtained are clearly not sufficient to explain the large value of $\Delta_{CT} = -0.071 \pm 0.014_{NA48} \pm 0.002_{theo} \pm 0.005_{ext}$ observed by NA48 in the charged weak decay[35] but are in reasonable agreement withe one observed by KLOE for the neutral weak decay [36].

8 Conclusions

In this paper we have calculated all the vector form-factors of Kaon to Pion transitions to first order in the quark mass difference $m_u - m_d$ and to NNLO in ChPT. We have thus calculated the eight different form-factors defined in Eqs. (2-5) to order $p^6(m_u - m_d)$. This complements the earlier calculations to order $p^4(m_u - m_d)$ done for the f_+ form-factors in [4] and [2] and to order p^6 in the isospin limit for the vector and scalar form-factor.

What we find in all cases is the the NNLO results diminish the effects of isospin breaking but due to the change in m_u/m_d from a NLO to a NNLO fit the total effect is to increase the isospin breaking in the form-factors. This goes some way towards reconciling the determinations of V_{us} from the charged and neutral weak $K_{\ell 3}$ decays but does not explain the full difference. We have also calculated isospin breaking in all the scalar form-factors. Here again, the effects are sizable but not unexpectedly large. In particular, they are not large enough to explain the discrepancy with the Callan-Treiman point observed by NA48[35].

We have presented numerical results for the values at $t = 0$ and for the t-dependence as well as for various ratios of the form-factors. In particular, we have shown that the relations (11) and (12) are valid to all orders in ChPT and to first order in $m_u - m_d$. We presented numerical results for the ratios r, r_{0-} and r_K as well as for their equivalents for the scalar form-factors.

Acknowledgments

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A The order p^4 expression

In this appendix we explicitly write out our order p^4 results. We have checked that they agree with the published results for the f_+ form-factors of [4] and [2]. They also satsify the relation (12) when the integrals are expanded to obtain a common Kaon mass, but we have quoted all eight formfactors here since by rewriting one can move things between the order p^4 and p^6. The expressions quoted here are the ones we used to define the order p^4 part. The integrals used below are the standard one-loop integrals defined in many places, see e.g. [20].

$$
\begin{aligned}
f_+^{K^+\pi^0(4)}(t) &= \frac{1}{F_\pi^2\left(m_{\pi^0}^2-m_\eta^2\right)}\Big(-2/3\,\overline{A}(m_{K^+}^2)\,m_{K^+}^2-1/3\,\overline{A}(m_{K^+}^2)\,m_{\pi^0}^2\\
&\quad +2/3\,\overline{A}(m_{K^0}^2)\,m_{K^+}^2+1/3\,\overline{A}(m_{K^0}^2)\,m_{\pi^0}^2\Big)\\
&\quad +\frac{\sin\epsilon}{\sqrt{3}\,F_\pi^2\left(m_{\pi^0}^2-m_\eta^2\right)}\Big(+128\,m_{K^+}^4\,L_8^r+384\,m_{K^+}^4\,L_7^r\\
&\quad -256\,m_{\pi^0}^2\,m_{K^+}^2\,L_8^r-768\,m_{\pi^0}^2\,m_{K^+}^2\,L_7^r+128\,m_{\pi^0}^4\,L_8^r+384\,m_{\pi^0}^4\,L_7^r\\
&\quad -4/3\,\overline{A}(m_{\pi^+}^2)\,m_{K^+}^2+16/3\,\overline{A}(m_{\pi^+}^2)\,m_{\pi^0}^2-2\,\overline{A}(m_{\pi^0}^2)\,m_{K^+}^2+2\,\overline{A}(m_{\pi^0}^2)\,m_{\pi^0}^2\\
&\quad -\overline{A}(m_{K^+}^2)\,m_{\pi^0}^2+4/3\,\overline{A}(m_{K^0}^2)\,m_{K^+}^2-13/3\,\overline{A}(m_{K^0}^2)\,m_{\pi^0}^2+2\,\overline{A}(m_\eta^2)\,m_{K^+}^2\\
&\quad -2\,\overline{A}(m_\eta^2)\,m_{\pi^0}^2\Big)\\
&\quad +\frac{\sin\epsilon}{\sqrt{3}\,F_\pi^2}\Big(+6\,L_9^r t-1/4\,\overline{A}(m_{\pi^+}^2)+9/8\,\overline{A}(m_{\pi^0}^2)+7/4\,\overline{A}(m_{K^+}^2)\\
&\quad +3/2\,\overline{A}(m_{K^0}^2)+3/8\,\overline{A}(m_\eta^2)\\
&\quad -3\,\overline{B}_{22}(m_{\pi^+}^2,m_{K^0}^2,t)-9/2\,\overline{B}_{22}(m_{\pi^0}^2,m_{K^+}^2,t)-3/2\,\overline{B}_{22}(m_{K^+}^2,m_\eta^2,t)\Big)\\
&\quad +\frac{1}{F_\pi^2}\Big(+2\,L_9^r t+1/4\,\overline{A}(m_{\pi^+}^2)+1/8\,\overline{A}(m_{\pi^0}^2)+3/4\,\overline{A}(m_{K^+}^2)+3/8\,\overline{A}(m_\eta^2)\\
&\quad -\overline{B}_{22}(m_{\pi^+}^2,m_{K^0}^2,t)-1/2\,\overline{B}_{22}(m_{\pi^0}^2,m_{K^+}^2,t)-3/2\,\overline{B}_{22}(m_{K^+}^2,m_\eta^2,t)\Big)\,,\\
f_+^{K^0\pi^-(4)}(t) &= \frac{\sin\epsilon}{\sqrt{3}\,F_\pi^2}\Big(+3/4\,\overline{A}(m_{\pi^0}^2)-3/4\,\overline{A}(m_\eta^2)-3\,\overline{B}_{22}(m_{\pi^0}^2,m_{K^+}^2,t)\\
&\quad +3\,\overline{B}_{22}(m_{K^+}^2,m_\eta^2,t)\Big)\\
&\quad +\frac{1}{F_\pi^2}\Big(2\,L_9^r t+1/4\,\overline{A}(m_{\pi^+}^2)+1/8\,\overline{A}(m_{\pi^0}^2)+1/2\,\overline{A}(m_{K^+}^2)+1/4\,\overline{A}(m_{K^0}^2)\\
&\quad +3/8\,\overline{A}(m_\eta^2)-\overline{B}_{22}(m_{\pi^+}^2,m_{K^0}^2,t)-1/2\,\overline{B}_{22}(m_{\pi^0}^2,m_{K^+}^2,t)\\
&\quad -3/2\,\overline{B}_{22}(m_{K^+}^2,m_\eta^2,t)\Big)\,,\\
f_+^{K^+\pi^+(4)}(t) &= +\frac{\sin\epsilon}{\sqrt{3}F_\pi^2}\Big(-3/4\,\overline{A}(m_{\pi^0}^2)+3/4\,\overline{A}(m_\eta^2)+3\,\overline{B}_{22}(m_{\pi^0}^2,m_{K^0}^2,t)\\
&\quad -3\,\overline{B}_{22}(m_{K^0}^2,m_\eta^2,t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{F_\pi^2}\Big(2\,L_9^r t + 1/4\,\overline{A}(m_{\pi^+}^2) + 1/8\,\overline{A}(m_{\pi^0}^2) + 1/4\,\overline{A}(m_{K^+}^2) + 1/2\,\overline{A}(m_{K^0}^2) \\
& +3/8\,\overline{A}(m_\eta^2) - \overline{B}_{22}(m_{\pi^+}^2, m_{K^+}^2, t) - 1/2\,\overline{B}_{22}(m_{\pi^0}^2, m_{K^0}^2, t) \\
& -3/2\,\overline{B}_{22}(m_{K^0}^2, m_\eta^2, t)\Big)\,, \\
f_+^{K^0\pi^0(4)}(t) \;=\;& \frac{1}{F_\pi^2\,\left(m_{\pi^0}^2 - m_\eta^2\right)}\Big(+ 2/3\,\overline{A}(m_{K^+}^2)\, m_{K^0}^2 + 1/3\,\overline{A}(m_{K^+}^2)\, m_{\pi^0}^2 \\
& -2/3\,\overline{A}(m_{K^0}^2)\, m_{K^0}^2 - 1/3\,\overline{A}(m_{K^0}^2)\, m_{\pi^0}^2\Big) \\
& +\frac{\sin\epsilon}{F_\pi^2\sqrt{3}\,\left(m_{\pi^0}^2 - m_\eta^2\right)}\Big(- 128\, m_{K^0}^4\, L_8^r - 384\, m_{K^0}^4\, L_7^r + 256\, m_{\pi^0}^2\, m_{K^0}^2\, L_8^r \\
& +768\, m_{\pi^0}^2\, m_{K^0}^2\, L_7^r - 128\, m_{\pi^0}^4\, L_8^r - 384\, m_{\pi^0}^4\, L_7^r + 4/3\,\overline{A}(m_{\pi^+}^2)\, m_{K^0}^2 \\
& -16/3\,\overline{A}(m_{\pi^+}^2)\, m_{\pi^0}^2 + 2\,\overline{A}(m_{\pi^0}^2)\, m_{K^0}^2 - 2\,\overline{A}(m_{\pi^0}^2)\, m_{\pi^0}^2 - 4/3\,\overline{A}(m_{K^+}^2)\, m_{K^0}^2 \\
& +13/3\,\overline{A}(m_{K^+}^2)\, m_{\pi^0}^2 + \overline{A}(m_{K^0}^2)\, m_{\pi^0}^2 - 2\,\overline{A}(m_\eta^2)\, m_{K^0}^2 + 2\,\overline{A}(m_\eta^2)\, m_{\pi^0}^2\Big) \\
& +\frac{\sin\epsilon}{F_\pi^2\sqrt{3}}\Big(- 6\,L_9^r t + 1/4\,\overline{A}(m_{\pi^+}^2) - 9/8\,\overline{A}(m_{\pi^0}^2) - 3/2\,\overline{A}(m_{K^+}^2) \\
& -7/4\,\overline{A}(m_{K^0}^2) - 3/8\,\overline{A}(m_\eta^2) + 3\,\overline{B}_{22}(m_{\pi^+}^2, m_{K^+}^2, t) + 9/2\,\overline{B}_{22}(m_{\pi^0}^2, m_{K^0}^2, t) \\
& +3/2\,\overline{B}_{22}(m_{K^0}^2, m_\eta^2, t)\Big) \\
& +\frac{1}{F_\pi^2}\Big(2\,L_9^r t + 1/4\,\overline{A}(m_{\pi^+}^2) + 1/8\,\overline{A}(m_{\pi^0}^2) + 3/4\,\overline{A}(m_{K^0}^2) + 3/8\,\overline{A}(m_\eta^2) \\
& -\overline{B}_{22}(m_{\pi^+}^2, m_{K^+}^2, t) - 1/2\,\overline{B}_{22}(m_{\pi^0}^2, m_{K^0}^2, t) - 3/2\,\overline{B}_{22}(m_{K^0}^2, m_\eta^2, t)\Big)\,,; \\
f_-^{K^+\pi^0(4)}(t) \;=\;& \frac{\sin\epsilon}{F_\pi^2\sqrt{3}}\Big(- 6\, m_{K^+}^2\, L_9^r + 12\, m_{K^+}^2\, L_5^r + 6\, m_{\pi^0}^2\, L_9^r - 12\, m_{\pi^0}^2\, L_5^r + 1/2\,\overline{A}(m_{\pi^+}^2) \\
& +3/4\,\overline{A}(m_{\pi^0}^2) - 7/4\,\overline{A}(m_{K^+}^2) - \overline{A}(m_{K^0}^2) + 1/2\,\overline{A}(m_\eta^2) \\
& +\overline{B}(m_{\pi^+}^2, m_{K^0}^2, t)\,(-1/2\;t - 3/2\, m_{K^+}^2 + 5/2\, m_{\pi^0}^2) \\
& +\overline{B}(m_{\pi^0}^2, m_{K^+}^2, t)\,(-3/4\, t - 5/4\, m_{K^+}^2 + 15/4\, m_{\pi^0}^2 \\
& +\overline{B}(m_{K^+}^2, m_\eta^2, t)\,(+1/4\, t - 5/4\; m_{K^+}^2 + 3/4\; m_{\pi^0}^2) \\
& +\overline{B}_1(m_{\pi^+}^2, m_{K^0}^2, t)\,(+1/2\, t + 9/2\, m_{K^+}^2 - 13/2\, m_{\pi^0}^2) \\
& +\overline{B}_1(m_{\pi^0}^2, m_{K^+}^2, t)\,(+3/4\, t + 19/4\, m_{K^+}^2 - 39/4\, m_{\pi^0}^2) \\
& +\overline{B}_1(m_{K^+}^2, m_\eta^2, t)(-1/4\, t + 13/4\, m_{K^+}^2 - 9/4\, m_{\pi^0}^2) \\
& +\overline{B}_{21}(m_{\pi^+}^2, m_{K^0}^2, t)\,(+ t - 3\, m_{K^+}^2 + 3\, m_{\pi^0}^2) \\
& +\overline{B}_{21}(m_{\pi^0}^2, m_{K^+}^2, t)\,(+3/2\, t - 9/2\, m_{K^+}^2 + 9/2\, m_{\pi^0}^2) \\
& +\overline{B}_{21}(m_{K^+}^2, m_\eta^2, t)\,(-1/2\, t - 3/2\, m_{K^+}^2 + 3/2\, m_{\pi^0}^2) \\
& +\overline{B}_{22}(m_{\pi^+}^2, m_{K^0}^2, t) + 3/2\,\overline{B}_{22}(m_{\pi^0}^2, m_{K^+}^2, t) - 1/2\,\overline{B}_{22}(m_{K^+}^2, m_\eta^2, t)\Big) \\
& +\frac{1}{F_\pi^2}\Big(- 2\, m_{K^+}^2\, L_9^r + 4\, m_{K^+}^2\, L_5^r + 2\, m_{\pi^0}^2\, L_9^r - 4\, m_{\pi^0}^2\, L_5^r - 1/2\,\overline{A}(m_{\pi^+}^2)
\end{aligned}
$$

$$
\begin{aligned}
& +1/12\,\overline{A}(m_{\pi^0}^2) - 5/12\,\overline{A}(m_{K^+}^2) + \overline{A}(m_{K^0}^2) + 1/2\,\overline{A}(m_\eta^2) \\
& + \overline{B}(m_{\pi^+}^2, m_{K^0}^2, t)\,(+1/2\,t - 1/2\,m_{K^+}^2 - 1/2\,m_{\pi^0}^2) \\
& + \overline{B}(m_{\pi^0}^2, m_{K^+}^2, t)\,(-1/12\,t + 1/12\,m_{K^+}^2 + 5/12\,m_{\pi^0}^2) \\
& + \overline{B}(m_{K^+}^2, m_\eta^2, t)\,(+1/4\,t - 7/12\,m_{K^+}^2 + 1/12\,m_{\pi^0}^2) \\
& + \overline{B}_1(m_{\pi^+}^2, m_{K^0}^2, t)\,(-1/2\,t + 3/2\,m_{K^+}^2 + 1/2\,m_{\pi^0}^2) \\
& + \overline{B}_1(m_{\pi^0}^2, m_{K^+}^2, t)\,(+1/12\,t + 1/12\,m_{K^+}^2 - 13/12\,m_{\pi^0}^2) \\
& + \overline{B}_1(m_{K^+}^2, m_\eta^2, t)\,(-1/4\,t + 23/12\,m_{K^+}^2 - 11/12\,m_{\pi^0}^2) \\
& +\overline{B}_{21}(m_{\pi^+}^2, m_{K^0}^2, t)\,(-\,t -\ m_{K^+}^2 +\ m_{\pi^0}^2) \\
& + \overline{B}_{21}(m_{\pi^0}^2, m_{K^+}^2, t)\,(+1/6\,t - 1/2\,m_{K^+}^2 + 1/2\,m_{\pi^0}^2) \\
& \overline{B}_{21}(m_{K^+}^2, m_\eta^2, t)\,(-1/2\,t - 3/2\,m_{K^+}^2 + 3/2\,m_{\pi^0}^2) \\
& -\overline{B}_{22}(m_{\pi^+}^2, m_{K^0}^2, t) + 1/6\,\overline{B}_{22}(m_{\pi^0}^2, m_{K^+}^2, t) - 1/2\,\overline{B}_{22}(m_{K^+}^2, m_\eta^2, t)\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
f_-^{K^0\pi^-(4)}(t) \;=\; & \frac{\sin\epsilon}{F_\pi^2\sqrt{3}}\Big(- 1/2\,\overline{A}(m_{\pi^0}^2) + 1/2\,\overline{A}(m_{K^+}^2) + \overline{A}(m_\eta^2) \\
& +\overline{B}(m_{\pi^0}^2, m_{K^+}^2, t)\,(+1/2\,t - 3/2\,m_{K^0}^2 + 1/2\,m_{\pi^+}^2) \\
& +\overline{B}(m_{K^+}^2, m_\eta^2, t)\,(+1/2\,t + 1/2\,m_{K^0}^2 - 1/2\,m_{\pi^0}^2 -\ m_{\pi^+}^2) \\
& +\overline{B}_1(m_{\pi^0}^2, m_{K^+}^2, t)\,(-1/2\,t + 9/2\,m_{K^0}^2 - 5/2\,m_{\pi^+}^2) \\
& +\overline{B}_1(m_{K^+}^2, m_\eta^2, t)\,(-1/2\,t - 5/2\,m_{K^0}^2 +\ m_{\pi^0}^2 + 7/2\,m_{\pi^+}^2) \\
& +\overline{B}_{21}(m_{\pi^0}^2, m_{K^+}^2, t)\,(-\,t - 3\,m_{K^0}^2 + 3\,m_{\pi^+}^2) \\
& +\overline{B}_{21}(m_{K^+}^2, m_\eta^2, t)\,(-\,t + 3\,m_{K^0}^2 - 3\,m_{\pi^+}^2) \\
& -\overline{B}_{22}(m_{\pi^0}^2, m_{K^+}^2, t) - \overline{B}_{22}(m_{K^+}^2, m_\eta^2, t)\Big) \\
& +\frac{1}{F_\pi^2}\Big(- 2\,m_{K^0}^2\,L_9^r + 4\,m_{K^0}^2\,L_5^r - 4\,m_{\pi^0}^2\,L_5^r + 2\,m_{\pi^+}^2\,L_9^r - 1/6\,\overline{A}(m_{\pi^+}^2) \\
& -1/4\,\overline{A}(m_{\pi^0}^2) + 1/4\,\overline{A}(m_{K^+}^2) + 1/3\,\overline{A}(m_{K^0}^2) + 1/2\,\overline{A}(m_\eta^2) \\
& +\overline{B}(m_{\pi^+}^2, m_{K^0}^2, t)\,(+1/6\,t - 1/6\,m_{K^0}^2 + 1/6\,m_{\pi^+}^2) \\
& +\overline{B}(m_{\pi^0}^2, m_{K^+}^2, t)\,(+1/4\,t - 1/4\,m_{K^0}^2 - 1/4\,m_{\pi^0}^2) \\
& +\overline{B}(m_{K^+}^2, m_\eta^2, t)\,(+1/4\,t - 7/12\,m_{K^0}^2 - 1/6\,m_{\pi^0}^2 + 1/4\,m_{\pi^+}^2) \\
& + \overline{B}_1(m_{\pi^+}^2, m_{K^0}^2, t)\,(-1/6\,t + 5/6\,m_{K^0}^2 - 5/6\,m_{\pi^+}^2) \\
& + \overline{B}_1(m_{\pi^0}^2, m_{K^+}^2, t)\,(-1/4\,t + 3/4\,m_{K^0}^2 + 1/2\,m_{\pi^0}^2 - 1/4\,m_{\pi^+}^2) \\
& + \overline{B}_1(m_{K^+}^2, m_\eta^2, t)\,(-1/4\,t + 23/12\,m_{K^0}^2 + 1/3\,m_{\pi^0}^2 - 5/4\,m_{\pi^+}^2) \\
& + \overline{B}_{21}(m_{\pi^+}^2, m_{K^0}^2, t)\,(-1/3\,t -\ m_{K^0}^2 +\ m_{\pi^+}^2) \\
& + \overline{B}_{21}(m_{\pi^0}^2, m_{K^+}^2, t)\,(-1/2\,t - 1/2\,m_{K^0}^2 + 1/2\,m_{\pi^+}^2) \\
& +\overline{B}_{21}(m_{K^+}^2, m_\eta^2, t)\,(-1/2\,t - 3/2\,m_{K^0}^2 + 3/2\,m_{\pi^+}^2) \\
& -1/3\,\overline{B}_{22}(m_{\pi^+}^2, m_{K^0}^2, t) - 1/2\,\overline{B}_{22}(m_{\pi^0}^2, m_{K^+}^2, t) - 1/2\,\overline{B}_{22}(m_{K^+}^2, m_\eta^2, t)\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
f_-^{K^+\pi^+(4)}(t) \;=\;& \frac{\sin\epsilon}{F_\pi^2\sqrt{3}}\Big(+ 1/2\,\overline{A}(m_{\pi^0}^2) - 1/2\,\overline{A}(m_{K^0}^2) - \overline{A}(m_\eta^2) \\
&+ \overline{B}(m_{\pi^0}^2, m_{K^0}^2, t)\,(-1/2\,t + 3/2\,m_{K^+}^2 - 1/2\,m_{\pi^+}^2) \\
&+ \overline{B}(m_{K^0}^2, m_\eta^2, t)\,(-1/2\,t - 1/2\,m_{K^+}^2 + 1/2\,m_{\pi^0}^2 + \; m_{\pi^+}^2) \\
&+ \overline{B}_1(m_{\pi^0}^2, m_{K^0}^2, t)\,(+1/2\,t - 9/2\,m_{K^+}^2 + 5/2\,m_{\pi^+}^2) \\
&+, \overline{B}_1(m_{K^0}^2, m_\eta^2, t)\,(+1/2\,t + 5/2\,m_{K^+}^2 - \; m_{\pi^0}^2 - \; m_{\pi^+}^2) \\
&+\overline{B}_{21}(m_{\pi^0}^2, m_{K^0}^2, t)\,(+\,t + 3\,m_{K^+}^2 - 3\,m_{\pi^+}^2) \\
&+\overline{B}_{21}(m_{K^0}^2, m_\eta^2, t)\,(+\,t - 3\,m_{K^+}^2 + 3\,m_{\pi^+}^2) \\
&+\overline{B}_{22}(m_{\pi^0}^2, m_{K^0}^2, t) + \overline{B}_{22}(m_{K^0}^2, m_\eta^2, t)\Big) \\
&+\frac{1}{F_\pi^2}\Big(- 2\,m_{K^+}^2\,L_9^r + 4\,m_{K^+}^2\,L_5^r - 4\,m_{\pi^0}^2\,L_5^r + 2\,m_{\pi^+}^2\,L_9^r - 1/6\,\overline{A}(m_{\pi^+}^2) \\
&-1/4\,\overline{A}(m_{\pi^0}^2) + 1/3\,\overline{A}(m_{K^+}^2) + 1/4\,\overline{A}(m_{K^0}^2) + 1/2\,\overline{A}(m_\eta^2) \\
&+1/6\,\overline{B}(m_{\pi^+}^2, m_{K^+}^2, t)\,(+1/6\,t - 1/6\,m_{K^+}^2 + 1/6\,m_{\pi^+}^2) \\
&+1/4\,\overline{B}(m_{\pi^0}^2, m_{K^0}^2, t)\,(+1/4\,t - 1/4\,m_{K^+}^2 - 1/4\,m_{\pi^0}^2) \\
&+ \overline{B}(m_{K^0}^2, m_\eta^2, t)\,(+1/4\,t - 7/12\,m_{K^+}^2 - 1/6\,m_{\pi^0}^2 + 1/4\,m_{\pi^+}^2) \\
&+ \overline{B}_1(m_{\pi^+}^2, m_{K^+}^2, t)\,(-1/6\,t + 5/6\,m_{K^+}^2 - 5/6\,m_{\pi^+}^2) \\
&+ \overline{B}_1(m_{\pi^0}^2, m_{K^0}^2, t)\,(-1/4\,t + 3/4\,m_{K^+}^2 + 1/2\,m_{\pi^0}^2 - 1/4\,m_{\pi^+}^2) \\
&+ \overline{B}_1(m_{K^0}^2, m_\eta^2, t)\,(-1/4\,t + 23/12\,m_{K^+}^2 + 1/3\,m_{\pi^0}^2 - 5/4\,m_{\pi^+}^2) \\
&+ \overline{B}_{21}(m_{\pi^+}^2, m_{K^+}^2, t)\,(-1/3\,t - \; m_{K^+}^2 + \; m_{\pi^+}^2) \\
&+ \overline{B}_{21}(m_{\pi^0}^2, m_{K^0}^2, t)\,(-1/2\,t - 1/2\,m_{K^+}^2 + 1/2\,m_{\pi^+}^2) \\
&+ \overline{B}_{21}(m_{K^0}^2, m_\eta^2, t)\,(-1/2\,t - 3/2\,m_{K^+}^2 + 3/2\,m_{\pi^+}^2) \\
&-1/3\,\overline{B}_{22}(m_{\pi^+}^2, m_{K^+}^2, t) - 1/2\,\overline{B}_{22}(m_{\pi^0}^2, m_{K^0}^2, t) - 1/2\,\overline{B}_{22}(m_{K^0}^2, m_\eta^2, t)\Big) \\
f_-^{K^0\pi^0(4)}(t) \;=\;& \frac{\sin\epsilon}{F_\pi^2\sqrt{3}}\Big(+ 6\,m_{K^0}^2\,L_9^r - 12\,m_{K^0}^2\,L_5^r - 6\,m_{\pi^0}^2\,L_9^r + 12\,m_{\pi^0}^2\,L_5^r - 1/2\,\overline{A}(m_{\pi^+}^2) \\
&-3/4\,\overline{A}(m_{\pi^0}^2) + \overline{A}(m_{K^+}^2) + 7/4\,\overline{A}(m_{K^0}^2) - 1/2\,\overline{A}(m_\eta^2) \\
&+ \overline{B}(m_{\pi^+}^2, m_{K^+}^2, t)\,(+1/2\,t + 3/2\,m_{K^0}^2 - 5/2\,m_{\pi^0}^2) \\
&+ \overline{B}(m_{\pi^0}^2, m_{K^0}^2, t)\,(+3/4\,t + 5/4\,m_{K^0}^2 - 15/4\,m_{\pi^0}^2) \\
&+ \overline{B}(m_{K^0}^2, m_\eta^2, t)\,(-1/4\,t + 5/4\,m_{K^0}^2 - 3/4\,m_{\pi^0}^2) \\
&+ \overline{B}_1(m_{\pi^+}^2, m_{K^+}^2, t)\,(-1/2\,t - 9/2\,m_{K^0}^2 + 13/2\,m_{\pi^0}^2) \\
&+ \overline{B}_1(m_{\pi^0}^2, m_{K^0}^2, t)\,(-3/4\,t - 19/4\,m_{K^0}^2 + 39/4\,m_{\pi^0}^2) \\
&+ \overline{B}_1(m_{K^0}^2, m_\eta^2, t)\,(+1/4\,t - 13/4\,m_{K^0}^2 + 9/4\,m_{\pi^0}^2) \\
&+\overline{B}_{21}(m_{\pi^+}^2, m_{K^+}^2, t)\,(-\,t + 3\,m_{K^0}^2 - 3\,m_{\pi^0}^2) \\
&+ \overline{B}_{21}(m_{\pi^0}^2, m_{K^0}^2, t)\,(-3/2\,t + 9/2\,m_{K^0}^2 - 9/2\,m_{\pi^0}^2) \\
&+ \overline{B}_{21}(m_{K^0}^2, m_\eta^2, t)\,(+1/2\,t + 3/2\,m_{K^0}^2 - 3/2\,m_{\pi^0}^2) \\
&-\overline{B}_{22}(m_{\pi^+}^2, m_{K^+}^2, t) - 3/2\,\overline{B}_{22}(m_{\pi^0}^2, m_{K^0}^2, t) + 1/2\,\overline{B}_{22}(m_{K^0}^2, m_\eta^2, t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{F_\pi^2}\Big(-2\,m_{K^0}^2\,L_9^r+4\,m_{K^0}^2\,L_5^r+2\,m_{\pi^0}^2\,L_9^r-4\,m_{\pi^0}^2\,L_5^r-1/2\,\overline{A}(m_{\pi^+}^2)\\
&+1/12\,\overline{A}(m_{\pi^0}^2)+\overline{A}(m_{K^+}^2)-5/12\,\overline{A}(m_{K^0}^2)+1/2\,\overline{A}(m_\eta^2)\\
&+,\overline{B}(m_{\pi^+}^2,m_{K^+}^2,t)\,(+1/2\,t-1/2\,m_{K^0}^2-1/2\,m_{\pi^0}^2)\\
&+\overline{B}(m_{\pi^0}^2,m_{K^0}^2,t)\,(-1/12\,t+1/12\,m_{K^0}^2+5/12\,m_{\pi^0}^2)\\
&+,\overline{B}(m_{K^0}^2,m_\eta^2,t)\,(+1/4\,t-7/12\,m_{K^0}^2+1/12\,m_{\pi^0}^2)\\
&+\overline{B}_1(m_{\pi^+}^2,m_{K^+}^2,t)\,(-1/2\,t+3/2\,m_{K^0}^2+1/2\,m_{\pi^0}^2)\\
&+\overline{B}_1(m_{\pi^0}^2,m_{K^0}^2,t)\,(+1/12\,t+1/12\,m_{K^0}^2-13/12\,m_{\pi^0}^2)\\
&+\overline{B}_1(m_{K^0}^2,m_\eta^2,t)\,(-1/4\,t+23/12\,m_{K^0}^2-11/12\,m_{\pi^0}^2)\\
&+\overline{B}_{21}(m_{\pi^+}^2,m_{K^+}^2,t)\,(-\,t-\,m_{K^0}^2+\,m_{\pi^0}^2)\\
&+\overline{B}_{21}(m_{\pi^0}^2,m_{K^0}^2,t)\,(+1/6\,t-1/2\,m_{K^0}^2+1/2\,m_{\pi^0}^2)\\
&+\overline{B}_{21}(m_{K^0}^2,m_\eta^2,t)\,(-1/2\,t-3/2\,m_{K^0}^2+3/2\,m_{\pi^0}^2)\\
&-\overline{B}_{22}(m_{\pi^+}^2,m_{K^+}^2,t)+1/6\,\overline{B}_{22}(m_{\pi^0}^2,m_{K^0}^2,t)-1/2\,\overline{B}_{22}(m_{K^0}^2,m_\eta^2,t)\Big)\,. \qquad (45)
\end{aligned}
$$

B The order p^6 LECs dependent part

In this appendix we write out explicitly the part dependent on the order p^6 LECs C_i^r. We use here a notation which uses the property (10).

$$
\begin{aligned}
f_\ell^{K^+\pi^0}(t)\Big|_{C_i^r} &= \frac{1}{F_\pi^4}\left(f_\ell^A(t)+\frac{\sin\epsilon}{\sqrt{3}}f_\ell^B(t)+\frac{\sin\epsilon}{\sqrt{3}\left(m_{\pi^0}^2-m_\eta^2\right)}f_\ell^E(t)\right)\,,\\
f_\ell^{K^0\pi^-}(t)\Big|_{C_i^r} &= \frac{1}{F_\pi^4}\left(f_\ell^A(t)-\frac{\sin\epsilon}{\sqrt{3}}f_\ell^D(t)\right)\,,\\
f_\ell^{K^+\pi^+}(t)\Big|_{C_i^r} &= \frac{1}{F_\pi^4}\left(f_\ell^A(t)+\frac{\sin\epsilon}{\sqrt{3}}f_\ell^D(t)\right)\,,\\
f_\ell^{K^0\pi^0}(t)\Big|_{C_i^r} &= \frac{1}{F_\pi^4}\left(f_\ell^A(t)-\frac{\sin\epsilon}{\sqrt{3}}f_\ell^B(t)-\frac{\sin\epsilon}{\sqrt{3}\left(m_{\pi^0}^2-m_\eta^2\right)}f_\ell^E(t)\right)\,. \qquad (46)
\end{aligned}
$$

We also use the notation

$$
m_\sigma^2=m_{K^+}^2+m_{K^0}^2-m_\pi^2\,. \qquad (47)
$$

The pion mass we have used generically since they are the same to the order of our calculation.

The C_i^r dependence is now given by

$$
\begin{aligned}
f_+^A(t) \;=\; &+t^2\,(-4\,C_{88}^r+4\,C_{90}^r)+m_\sigma^2\,t\,(-4\,C_{12}^r-16\,C_{13}^r-4\,C_{63}^r-4\,C_{64}^r-2\,C_{90}^r)\\
&+m_\pi^2\,t\,(-12\,C_{12}^r-32\,C_{13}^r-4\,C_{63}^r-8\,C_{64}^r-4\,C_{65}^r-6\,C_{90}^r)+m_\sigma^4\,(-2\,C_{12}^r-2\,C_{34}^r)\\
&+m_\pi^2\,m_\sigma^2\,(4\,C_{12}^r+4\,C_{34}^r)+m_\pi^4\,(-2\,C_{12}^r-2\,C_{34}^r)\,,
\end{aligned}
$$

$$
\begin{aligned}
f_+^B(t) &= +t^2\,(-12\,C_{88}^r + 12\,C_{90}^r) + m_\sigma^2\,t\,(-4\,C_{12}^r - 48\,C_{13}^r - 4\,C_{63}^r - 12\,C_{64}^r - 2\,C_{90}^r)\\
&\quad + m_\pi^2\,t\,(-44\,C_{12}^r - 96\,C_{13}^r - 20\,C_{63}^r - 24\,C_{64}^r - 12\,C_{65}^r - 22\,C_{90}^r)\\
&\quad + m_\sigma^4\,(2\,C_{12}^r + 16\,C_{14}^r + 16\,C_{17}^r + 48\,C_{18}^r - 14\,C_{34}^r - 24\,C_{35}^r)\\
&\quad + m_\pi^2\,m_\sigma^2\,(-4\,C_{12}^r - 32\,C_{14}^r - 32\,C_{17}^r - 96\,C_{18}^r + 28\,C_{34}^r + 48\,C_{35}^r)\\
&\quad + m_\pi^4\,(2\,C_{12}^r + 16\,C_{14}^r + 16\,C_{17}^r + 48\,C_{18}^r - 14\,C_{34}^r - 24\,C_{35}^r)\,,\\
f_+^E(t) &= +m_\sigma^6\,(96\,C_{19}^r + 64\,C_{20}^r + 64\,C_{31}^r + 64\,C_{32}^r + 128\,C_{33}^r)\\
&\quad + m_\pi^2\,m_\sigma^4\,(-32\,C_{14}^r - 32\,C_{17}^r - 96\,C_{18}^r) + m_\pi^4\,m_\sigma^2\,(64\,C_{14}^r + 64\,C_{17}^r + 192\,C_{18}^r\\
&\quad - 288\,C_{19}^r - 192\,C_{20}^r - 192\,C_{31}^r - 192\,C_{32}^r - 384\,C_{33}^r)\\
&\quad + m_\pi^6\,(-32\,C_{14}^r - 32\,C_{17}^r - 96\,C_{18}^r + 192\,C_{19}^r + 128\,C_{20}^r + 128\,C_{31}^r + 128\,C_{32}^r\\
&\quad + 256\,C_{33}^r)\,,\\
f_+^D(t) &= +m_\sigma^2\,t\,(8\,C_{12}^r - 8\,C_{63}^r + 8\,C_{65}^r + 4\,C_{90}^r) + m_\pi^2\,t\,(-8\,C_{12}^r + 8\,C_{63}^r - 8\,C_{65}^r - 4\,C_{90}^r)\\
&\quad + m_\sigma^4\,(8\,C_{12}^r + 8\,C_{34}^r) + m_\pi^2\,m_\sigma^2\,(-16\,C_{12}^r - 16\,C_{34}^r) + m_\pi^4\,(8\,C_{12}^r + 8\,C_{34}^r)\,,\\
f_-^A(t) &= +m_\sigma^2\,t\,(-4\,C_{12}^r + 2\,C_{88}^r - 2\,C_{90}^r) + m_\pi^2\,t\,(4\,C_{12}^r - 2\,C_{88}^r + 2\,C_{90}^r)\\
&\quad + m_\sigma^4\,(6\,C_{12}^r + 8\,C_{13}^r + 4\,C_{14}^r + 4\,C_{15}^r + 2\,C_{34}^r + 2\,C_{63}^r + 2\,C_{64}^r + C_{90}^r)\\
&\quad + m_\pi^2\,m_\sigma^2\,(12\,C_{12}^r + 8\,C_{13}^r + 4\,C_{15}^r + 8\,C_{17}^r + 4\,C_{34}^r + 2\,C_{64}^r + 2\,C_{65}^r + 2\,C_{90}^r)\\
&\quad + m_\pi^4\,(-18\,C_{12}^r - 16\,C_{13}^r - 4\,C_{14}^r - 8\,C_{15}^r - 8\,C_{17}^r - 6\,C_{34}^r - 2\,C_{63}^r - 4\,C_{64}^r\\
&\quad - 2\,C_{65}^r - 3\,C_{90}^r)\,,\\
f_-^B(t) &= +m_\sigma^2\,t\,(-4\,C_{12}^r + 2\,C_{88}^r - 2\,C_{90}^r) + m_\pi^2\,t\,(4\,C_{12}^r - 2\,C_{88}^r + 2\,C_{90}^r)\\
&\quad + m_\sigma^4\,(-6\,C_{12}^r + 8\,C_{13}^r - 4\,C_{14}^r + 4\,C_{15}^r - 32\,C_{17}^r - 48\,C_{18}^r - 18\,C_{34}^r - 24\,C_{35}^r - 2\,C_{63}^r\\
&\quad + 2\,C_{64}^r - C_{90}^r)\\
&\quad + m_\pi^2\,m_\sigma^2\,(36\,C_{12}^r + 8\,C_{13}^r + 16\,C_{14}^r + 4\,C_{15}^r + 72\,C_{17}^r + 96\,C_{18}^r + 44\,C_{34}^r + 48\,C_{35}^r\\
&\quad + 8\,C_{63}^r + 2\,C_{64}^r + 2\,C_{65}^r + 6\,C_{90}^r)\\
&\quad + m_\pi^4\,(-30\,C_{12}^r - 16\,C_{13}^r - 12\,C_{14}^r - 8\,C_{15}^r - 40\,C_{17}^r - 48\,C_{18}^r - 26\,C_{34}^r - 24\,C_{35}^r\\
&\quad - 6\,C_{63}^r - 4\,C_{64}^r - 2\,C_{65}^r - 5\,C_{90}^r)\,,\\
f_-^E(t) &= 0\,,\\
f_-^D(t) &= +m_\sigma^2\,t\,(8\,C_{12}^r - 4\,C_{88}^r + 4\,C_{90}^r) + m_\pi^2\,t\,(-8\,C_{12}^r + 4\,C_{88}^r - 4\,C_{90}^r)\\
&\quad + m_\sigma^4\,(-24\,C_{12}^r - 16\,C_{13}^r - 8\,C_{15}^r - 16\,C_{17}^r - 8\,C_{34}^r - 4\,C_{64}^r - 4\,C_{65}^r - 4\,C_{90}^r)\\
&\quad + m_\pi^2\,m_\sigma^2\,(-16\,C_{13}^r - 16\,C_{14}^r - 8\,C_{15}^r + 16\,C_{17}^r - 8\,C_{63}^r - 4\,C_{64}^r + 4\,C_{65}^r)\\
&\quad + m_\pi^4\,(24\,C_{12}^r + 32\,C_{13}^r + 16\,C_{14}^r + 16\,C_{15}^r + 8\,C_{34}^r + 8\,C_{63}^r + 8\,C_{64}^r + 4\,C_{90}^r)\,.
\end{aligned}
\tag{48}
$$

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