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Status of Strong ChPT

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The present status of Chiral Perturbation Theory (ChPT) in the strong mesonic sector is discussed. A very short introduction to ChPT is followed by an overview of existing two-loop calculations and the determination of the Low-Energy-Constants (LECs). I discuss the case of $\eta \to \pi\pi\pi$ decays in somewhat more detail and finish by mentioning some recent work on partially quenched ChPT and on the renormalization group in ChPT.

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In this talk I will restrict myself to the light pseudo-scalar mesons (two, three or more flavours) and strong interaction only, including interaction with external currents and densities. This excludes the work including baryons, heavy quarks, vector mesons, structure functions and related quantities and non-leptonic weak interactions as well as effects of photon loops. I start by mentioning a few important historical papers which have roughly a jubilee this year, Section 1 followed by a short introduction to Chiral Perturbation Theory (ChPT) including discussions on chiral logarithms and what exactly to expand in. The next two sections give an overview of two and three flavour ChPT respectively, where in the latter case I go into some detail into how the LECs are determined experimentally and the assumptions on the order p^6 LECs. Section 5 discusses the decay $\eta \to 3\pi$ while at the end I make some comments about recent work in partially quenched ChPT and applications of the renormalization group to ChPT. Many part are also in my earlier talk [1].

1. Some History: 50, 40, 35, 30, 25, 20 and 15 years ago

Since in this conference we celebrate the scientific achievements of Gerhard Ecker and Jürg Gasser, it is appropriate to look back at the history of Chiral Perturbation Theory. I have picked out a few papers which fell at or close to jubileum years. About 50 years ago the subject was started with the Goldberger-Treiman relation [2] and the advent of PCAC, the partially conserved axial-current [3], and how this reproduced the Goldberger-Treiman relation. About 40 years ago a lot of work had been done within the framework of PCAC but 1968 and 1969 saw some very important papers: the Gell-Mann–Oakes–Renner relation [4] and the proper way how to implement chiral symmetry in all generality in phenomenological Lagrangians [5]. Shortly afterwards loop calculations started with e.g. loop results for $\pi\pi$ scattering [6] and $\eta \to 3\pi$ [7]. 30 years ago the start with the modern way of including higher order Lagrangians and performing a consistent renormalization came with [8]. At the same time there was also the beautiful paper by Gasser and Zepeda about the types of non-analytical corrections that can appear [9]. The seminal papers by Gasser and Leutwyler of 25 years ago then put the entire subject on a modern firm footing [10, 11]. The same period also had my own entry into the subject [12]. Lots of one-loop calculations were done and the understanding that the coefficients in the higher-order Lagrangians could be understood from the contributions of resonances was put on a firm footing [10, 13]. Let me close this historical part with two 15 year old papers, a very clear discussion of the basics of ChPT [14] and the first full two-loop calculation [15].

2. Chiral Perturbation Theory: ChPT, CHPT or χPT

ChPT is best described as " *Exploring the consequences of the chiral symmetry of QCD and its spontaneous breaking using effective field theory techniques*" and a particularly clear discussion about its derivation and underlying assumptions can be found in [14]. Some reviews are [16, 17]. More reviews and references to introductory lectures can be found on the webpage [18].

For effective field theories, there are three principles that are needed and for ChPT they are

- **Degrees of freedom:** Goldstone Bosons from the spontaneous chiral symmetry breakdown.
- **Power counting:** This is what allows a systematic ordering of terms and is here essentially dimensional counting in momenta and masses.

- **Expected breakdown scale:** The scale of the not explicitly included physics, here resonances, so the scale is of order M_ρ, but this is channel dependent.

Chiral symmetry is the (continuous) interchange of quarks. If we look at the QCD Lagrangian

$$\mathscr{L}_{QCD} = \sum_{q=u,d,s} \left[i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L + \bar{q}_L q_R \right) \right] - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \tag{2.1}$$

we see that we have an $SU(3)_V$ symmetry for equal quark masses but for $m_q = 0$ we can change left- and right-handed separately giving a $SU(3)_L \times SU(3)_R$ symmetry with $q^T = (\,u\;\; d\;\; s\,)$ transforming as $q_L \to g_L q_L$ and $q_R \to g_R q_R$.

The chiral symmetry is broken spontaneously by vacuum condensates $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$. This breaks the eight axial generators of the symmetry group but leaves the vector part unbroken: $SU(3)_L \times SU(3)_R \to SU(3)_V$. This produces eight massless Goldstone Bosons *and* their interaction vanishes at zero momentum. The latter is very important, it is the reason why there exists a proper power-counting in ChPT. This is illustrated in Fig. 1.

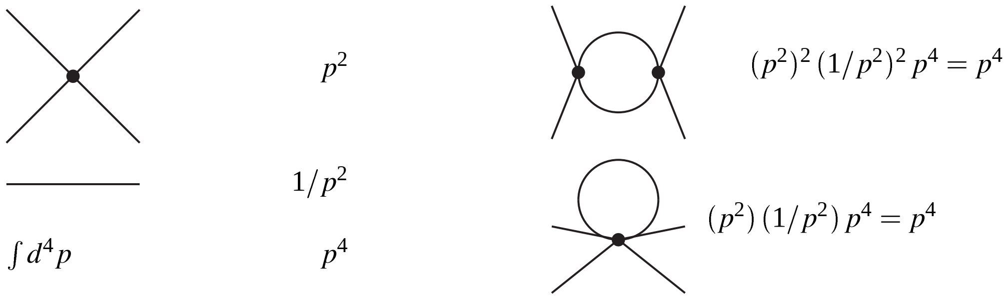

Figure 1: An illustration of the power-counting in ChPT. On the left we have the lowest order vertex with two powers of momenta or masses, the meson propagator with two inverse powers and the loop integration leading to four powers. On the right hand-side we see two one-loop contributions and how the counting on the left leads to the same power p^4 for both diagrams. This counting can be generalized to all orders [8].

We now start to look at the needed Lagrangians. The $SU(3)_L \times SU(3)_R / SU(3)_V$ manifold is parameterized by a matrix

$$U = \exp\left(i\sqrt{(}2)\Phi/F_0 \right) \quad \text{with} \quad \Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix}. \tag{2.2}$$

The traceless Hermitian matrix Φ is written in the usual pseudo-scalar fields. With the covariant derivative $D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu$, which includes the left and right external currents: $r(l)_\mu = v_\mu + (-) a_\mu$ and the matrix $\chi = 2B_0(s + ip)$ that contains the external scalar and pseudo-scalar fields s and p, the lowest order Lagrangian is

$$\mathscr{L}_2 = \left(F_0^2/4 \right) \left\{ \langle D_\mu U^\dagger D^\mu U \rangle + \langle \chi^\dagger U + \chi U^\dagger \rangle \right\} . \tag{2.3}$$

$\langle A \rangle$ is the trace over flavours $Tr_F\,(A)$. Quark masses are included via $s = \mathrm{diag}(m_u, m_d, m_s) + \cdots$.

At higher orders the number of terms in the Lagrangian increases rapidly. There exist two types, those representing contact terms, i.e. without pseudo-scalar bosons, and those with. The former can never be measured but are the reflection in ChPT of the definition of currents and densities

	2 flavour		3 flavour		3+3 PQChPT	
p^2	F,B	2	F_0,B_0	2	F_0,B_0	2
p^4	l_i^r,h_i^r	7+3	L_i^r,H_i^r	10+2	$\hat{L}_i^r,\hat{H}_i^r$	11+2
p^6	c_i^r	52+4	C_i^r	90+4	K_i^r	112+3

Table 1: The number of parameters+contact terms for the various types of ChPT.

in QCD. The latter are usually called low-energy-constants (LECs). The number of parameters at the various orders is shown in Tab. 1. The order p^2 is from [19], order p^4 from [10, 11], order p^6 from [20] after an earlier partial result [21]. The partially quenched results are derived from the n_F flavour case [22]. The difficulty in obtaining a minimal set can be seen from the recent discovery of a new relation for two flavours [23]. Since the normal case is a continuous limit of the partially quenched case, the resulting LECs are linear combinations of partially quenched LECs using the Cayley-Hamilton relations given in [20]. The general divergence structure at this order is known [24]. The parameters $B \neq B_0$ and $F \neq F_0$ are the two versus three-flavour lowest order constants, these are different quantities.

The main predictions of ChPT are twofold. 1) It relates processes with different numbers of pseudo-scalars. 2) It predicts nonanalytic dependences at higher orders, often referred to generically as *Chiral Log(arithm)s*. As an example, the pion mass for $n_F = 2$ is given at NLO by [10]

$$m_\pi^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2}\log\frac{(2B\hat{m})}{\mu^2} + 2l_3^r(\mu)\right] + \cdots \tag{2.4}$$

The implicit μ dependence in l_3^r and the explicit dependence in the logarithm cancel.

The LECs, like l_3^r in (2.4), have to be determined experimentally or from lattice calculations. For $n_F = 2$ Ref. [10] introduced the μ independent $\overline{l_i} = \left(32\pi^2/\gamma_i\right)\, l_i^r(\mu) - \log\left(M_\pi^2/\mu^2\right)$, which are proportional to the LECs $l_i^r(\mu = m_\pi)$. For $n_F = 3$ some of the corresponding γ_i are zero and no good equivalent definition of $\overline{L_i}$ exists. Here we always quote the $L_i^r(\mu)$. The scale μ is arbitrary but becomes relevant when using estimates for higher order constants.

A question which is often misunderstood is what quantities to expand in. The ChPT expansion is in momenta and masses. However, one first has to decide whether to expand in lowest order quantities, like $F, 2B\hat{m}$, or physical masses and decay constants, like $m_\pi, m_K, m_\eta, F_\pi, F_K$. The latter is not unique either since the Gell-Mann–Okubo relation and kinematical relations like $s+t+u = 2m_\pi^2 + 2m_K^2$ for πK-scattering can be (and are heavily) used to rewrite expressions. This sounds trivial but can change much how a series convergence looks as shown below for a simple example. I prefer to use physical masses and decay constants rather than the lowest order quantities. The physical quantities are typically better known and the chiral logs are created by particles propagating with their physical momentum. Also, thresholds appear in the right places at each order in perturbation theory. The differences are higher order, but can be numerically important.

Take as a simple example the relations $m_\pi = m_0/\left(1 + am_0/f_0\right)$, $\quad m_\pi = f_0/\left(1 + bm_0/f_0\right)$, as exact. We can expand to NNLO in several ways, some examples are

$$m_\pi = m_0 - a\frac{m_0^2}{f_0} + a^2\frac{m_0^3}{f_0^2} + \cdots \qquad f_\pi = f_0\left(1 - b\frac{m_0}{f_0} + b^2\frac{m_0^2}{f_0^2} + \cdots\right) \tag{2.5}$$

$$m_\pi = m_0 - a\frac{m_\pi^2}{f_\pi} + a(b-a)\frac{m_\pi^3}{f_\pi^2} + \cdots \qquad f_\pi = f_0\left(1 - b\frac{m_\pi}{f_\pi} + b(2b-a)\frac{m_\pi^2}{f_\pi^2} + \cdots\right) \tag{2.6}$$

The coefficients in the expansion and the actual numerical values clearly depend on the way we write the results. The plots in Fig. 2 show the convergence for $a = 1$, $b = 0.5$ and $f_0 = 1$. Only

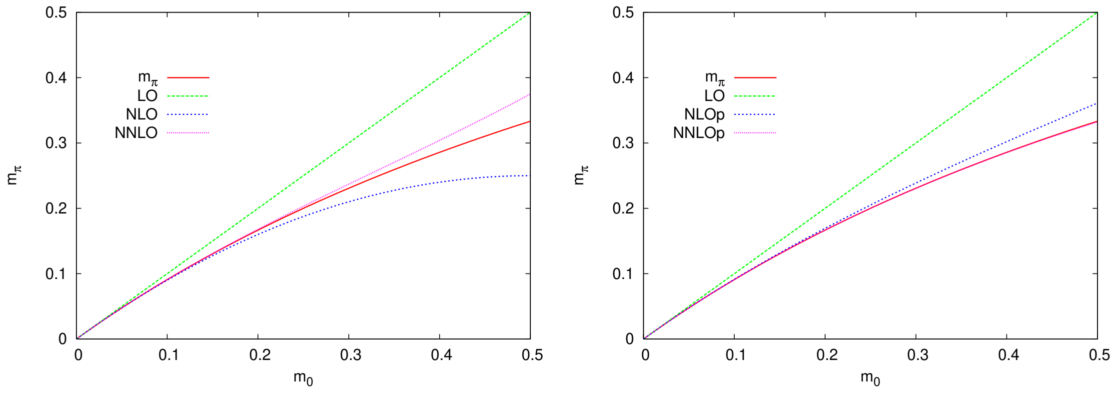

Figure 2: On the left the expansion of m_π in terms of m_o/f_0 (2.5) and on the right in terms of m_π/f_π (2.6). Shown are the full results (m_π) and the first three approximations.

knowing the first three terms one would draw very different conclusions on the quality of the convergence from Fig. 2 for the different ways of expanding.

3. Two-flavour ChPT at NNLO

References to order p^2 and p^4 work can be found in [17]. The first work at NNLO used dispersive methods to obtain the nonanalytic dependence on kinematical quantities, q^2, s, t, u at NNLO. This was done for the vector (electromagnetic) and scalar form-factor of the pion in [25] (numerically) and [26] (analytically) and for $\pi\pi$-scattering analytically in [27]. The work of [27] allowed to put many full NNLO calculations in two-flavour ChPT in a simple analytical form.

Essentially all processes of interest are calculated to NNLO fully in ChPT starting with $\gamma\gamma \to \pi^0\pi^0$ [15, 28], $\gamma\gamma \to \pi^+\pi^-$ [29, 30], F_π and m_π [29, 31, 32], $\pi\pi$-scattering [31], the pion scalar and vector form-factors [32] and pion radiative decay $\pi \to \ell\nu\gamma$ [33]. The pion mass is known at order p^6 in finite volume [34]. Recently $\pi^0 \to \gamma\gamma$ has been done to this order as discussed in the talk by Kampf [35].

The LECs have been fitted in several processes. $\bar{l}_4$ from fitting to the pion scalar radius [33, 36], $\bar{l}_3$ from an estimate of the pion mass dependence on the quark masses [10, 36] and $\bar{l}_1$, $\bar{l}_2$ from the agreement with $\pi\pi$-scattering [36], $\bar{l}_6$ from the pion charge radius [32] and $\bar{l}_6 - \bar{l}_5$ from the axial form-factor in $\pi \to \ell\nu\gamma$. There is also a recent determination of $\bar{l}_5$ from hadronic tau decays [37]. The final best values are [32, 33, 36, 37]

$$\begin{array}{llll} \bar{l}_1 = -0.4 \pm 0.6\,, & \bar{l}_2 = 4.3 \pm 0.1\,, & \bar{l}_3 = 2.9 \pm 2.4\,, & \bar{l}_4 = 4.4 \pm 0.2\,, \\ \bar{l}_6 - \bar{l}_5 = 3.0 \pm 0.3\,, & \bar{l}_6 = 16.0 \pm 0.5 \pm 0.7\,, & \bar{l}_5 = 12.24 \pm 0.21\,. & \end{array} \tag{3.1}$$

There is information on some combinations of p^6 LECs. These are basically via the curvature in the vector and scalar form-factor of the pion [32] and two combinations from $\pi\pi$-scattering [36]

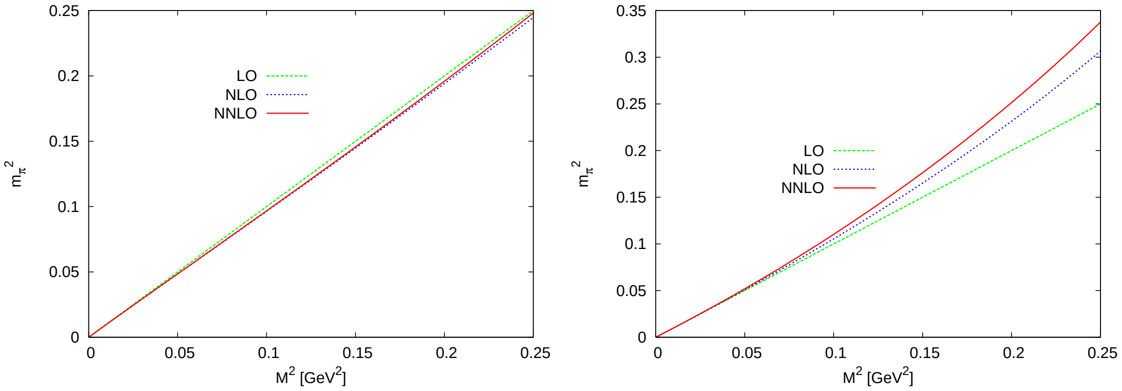

Figure 3: The pion mass squared as a function of the quark mass via $M^2 = 2B\hat{m}$, left with inputs as in (3.1) and right with $\bar{l}_3 = 0$, both are for $n_F = 2$ ChPT.

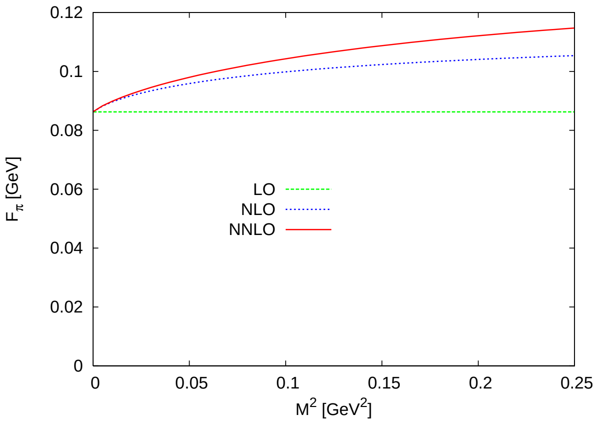

Figure 4: The pion decay constant as a function of the quark mass via $M^2 = 2B\hat{m}$, for $n_F = 2$ ChPT.

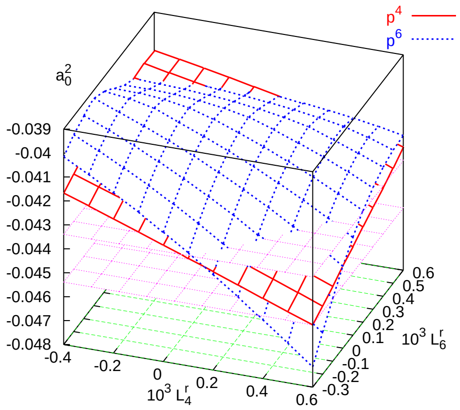

Figure 5: The $\pi\pi$ scattering length a_0^2 in three-flavour ChPT as a function of the input values of L_4^r, L_6^r used in the fits, from [38].

from the knowledge of b_5 and b_6 in that reference. The order p^6 LECs c_i^r are estimated to have a small effect for m_π, f_π and $\pi\pi$-scattering.

Let me now show the dependence on the quark mass via $M^2 = 2B\hat{m}$ for m_π^2 with surprisingly small NLO and NNLO corrections for the values of the input parameters in (3.1) and $c_i^r(\mu = 0.77\text{ GeV}) = 0$. The full result is extremely linear as can be seen in the left plot in Fig. 3. The linearity is a consequence of the fitting parameters as can be seen in the right figure in Fig. 3. Similarly, F_π as a function of M^2 expanded as in (2.6) is shown in Fig. 4. The values of m_π^2, F_π and M^2 are determined selfconsistently via an iterative method from the ChPT formulas quoted in [32].

4. Three-flavour ChPT

4.1 Calculations

In this section I discuss several results at NNLO in mesonic three-flavour ChPT. The formulas here are much more involved than in two-flavour ChPT and while the expressions have been reduced to a series of well-defined two-loop integrals, the latter are evaluated numerically. Both are the consequence of the different masses present. The vector two-point functions [39, 40] and the isospin breaking in the $\rho\omega$ channel [41] were among the first calculated. The disconnected scalar two-point function relevant for bounds on L_4^r and L_6^r was worked out in [42] The remaining scalar two-point functions are known but unpublished [43]. Masses and decay constants as well as axial-vector two-point functions were the first calculations which required full two-loop integrals, done in the π and η [39, 44] and the K channel [39]. Including isospin breaking contributions to masses and decay constants was done in [45]. After $K_{\ell 4}$ had also been evaluated to NNLO [46] a fit to the LECs was done as described below. The vacuum expectation values in the isospin limit were done in [46], with isospin breaking in [45] and at finite volume in [47].

Vector (electromagnetic) form-factors for pions and kaons were calculated in [48, 49] and in [49] a NNLO fit for L_9^r was performed. L_{10}^r can be had from hadronic tau decays [37] or the axial form-factor in $\pi, K \to \ell\nu\gamma$. The NNLO calculation is done, but no data fitting was performed[50]. A rather important calculation is the $K_{\ell 3}$ form-factor. This calculation was done by [51, 52] and a rather interesting relation between the value at zero, the slope and the curvature for the scalar form-factor obtained [51]. Isospin-breaking has been included as well [53].

Scalar form-factors including sigma terms and scalar radii [54] and $\pi\pi$ [38] and πK-scattering [55] have been performed as well and used to place limits on L_4^r and L_6^r. Finally, the relations between the l_i^r, c_i^r and L_i^r, C_i^r have been extended to the accuracy needed to compare order p^6 results in two and three-flavour calculations [56] and there has been some progress towards fully analytical results for m_π^2 [57] and πK-scattering lengths [58]. The most recent results are $\eta \to 3\pi$ [59], isospin breaking in $K_{\ell 3}$ [53].

4.2 The fitting and results

The inputs used for the fitting, as discussed more extensively in [45, 46], are

- $K_{\ell 4}$: $F(0)$, $G(0)$, λ from E865 at BNL[60].
- $m_{\pi^0}^2$, m_η^2, $m_{K^+}^2$, $m_{K^0}^2$, electromagnetic corrections include the violation of Dashen's theorem.
- F_{π^+} and F_{K^+}/F_{π^+}.
- $m_s/\hat{m} = 24$. Variations with $m_s/\hat{m}$ were studied in [45, 46].
- L_4^r, L_6^r the main fit, 10, has them equal to zero, but see below and the arguments in [42].

Some results of this fit are given in Tab. 2. The errors are very correlated, see Fig. 6 in [46] for an example. Varying the values of L_4^r, L_6^r as input can be done with a reasonable fitting chi-squared when varying $10^3 L_4^r$ from -0.4 to 0.6 and L_6^r from -0.3 to 0.6 [54]. The variation of many quantities with L_4^r, L_6^r (including the changes via the changed values of the other L_i^r) are shown in [54, 38, 55]. Fit B was one of the fits with a good fit to the pion scalar radius and fairly small corrections to the sigma terms [54] while fit D [61] is the one that gave agreement with $\pi\pi$ and πK-scattering threshold quantities.

Note that $m_u/m_d = 0$ is never even close to the best fit and this remains true for the entire variation with L_4^r, L_6^r. The value of F_0, the pion decay constant in the three-flavour chiral limit, can vary significantly, even though I believe that fit B is an extreme case.

	fit 10	same p^4	fit B	fit D
$10^3 L_1^r$	0.43 ± 0.12	0.38	0.44	0.44
$10^3 L_2^r$	0.73 ± 0.12	1.59	0.60	0.69
$10^3 L_3^r$	-2.53 ± 0.37	-2.91	-2.31	-2.33
$10^3 L_4^r$	$\equiv 0$	$\equiv 0$	$\equiv 0.5$	$\equiv 0.2$
$10^3 L_5^r$	0.97 ± 0.11	1.46	0.82	0.88
$10^3 L_6^r$	$\equiv 0$	$\equiv 0$	$\equiv 0.1$	$\equiv 0$
$10^3 L_7^r$	-0.31 ± 0.14	-0.49	-0.26	-0.28
$10^3 L_8^r$	0.60 ± 0.18	1.00	0.50	0.54
$10^3 L_9^r$	5.93 ± 0.43	7.0	–	–
$2B_0\hat{m}/m_\pi^2$	0.736	0.991	1.129	0.958
m_π^2: p^4, p^6	0.006,0.258	0.009,$\equiv 0$	-0.138,0.009	-0.091,0.133
m_K^2: p^4, p^6	0.007,0.306	0.075,$\equiv 0$	-0.149,0.094	-0.096,0.201
m_η^2: p^4, p^6	-0.052,0.318	0.013,$\equiv 0$	-0.197,0.073	-0.151,0.197
m_u/m_d	0.45±0.05	0.52	0.52	0.50
F_0 [MeV]	87.7	81.1	70.4	80.4
$\frac{F_K}{F_\pi}$: p^4, p^6	0.169,0.051	0.22,$\equiv 0$	0.153,0.067	0.159,0.061

Table 2: The fits of the L_i^r and some results, see text for a detailed description. They are all quoted at $\mu = 0.77$ GeV. Table with values from [45, 49, 54, 55, 61].

In Fig. 5 I show how the threshold parameter a_0^2 depend on the variation with L_4^r, L_6^r. a_0^0 always agrees well with the result of [36] while a_0^2 only agrees well within a limited region [38]. For comparison, the order p^2 values are $a_0^0 = 0.159$ and $a_0^2 = -0.0454$. The planes in Fig. 5 indicate the results $a_0^0 = 0.220 \pm 0.005$, $a_0^2 = -0.0444 \pm 0.0010$ [36]. The same study was performed for πK scattering lengths in [55] with the results of the Roy-Steiner analysis [62]. The resulting limits on the input values of L_4^r, L_6^r are shown in Fig. 6. The resulting region called fit D in Tab. 2 is

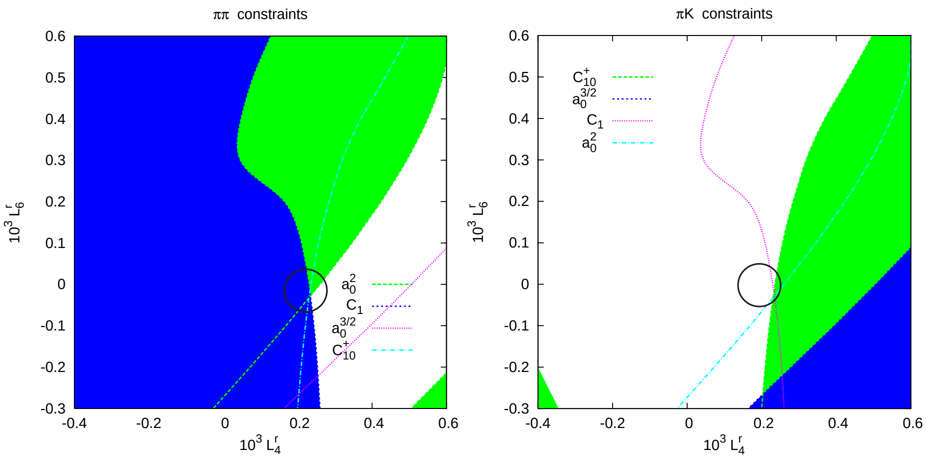

Figure 6: The bounds on L_4^r, L_6^r from $\pi\pi$ and πK-scattering threshold parameters. Left $\pi\pi$ where the bound from a_0^2 shown in Fig. 5 is the most stringent. Right πK. White regions are allowed. The region of fit D, compatible with both, is indicated by the circle. From [55].

$10^3 L_4^r \approx 0.2$, $10^3 L_6^r \approx 0.0$. This general fitting obviously needs more work and systematic studies

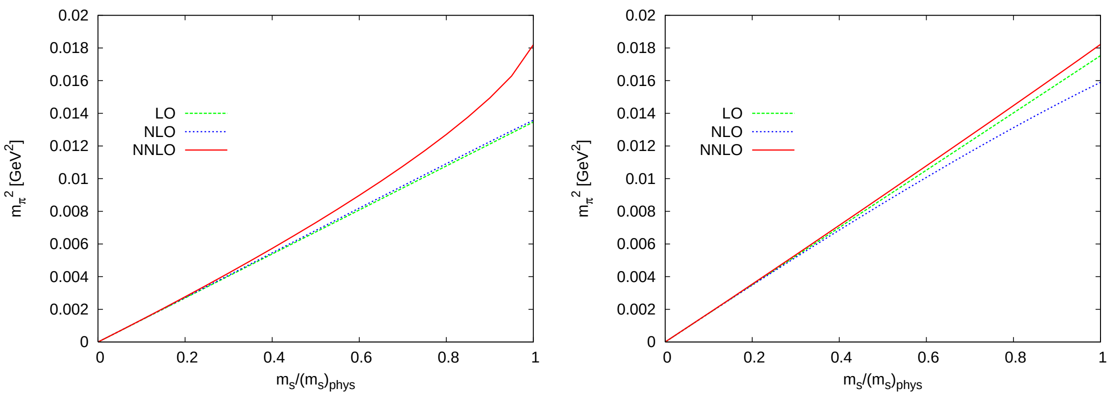

Figure 7: m_π^2 as a function of m_s for fit 10 (left) and fit D (right) of Tab. 2 with $m_s/\hat{m}$ fixed. Note the difference in convergence properties between the two fits.

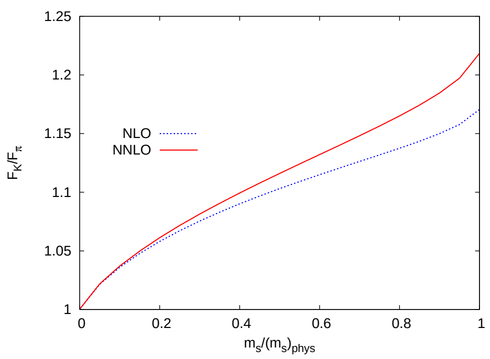

Figure 8: For fit 10: F_K/F_π as a function of m_s with $m_s/\hat{m}$ fixed.

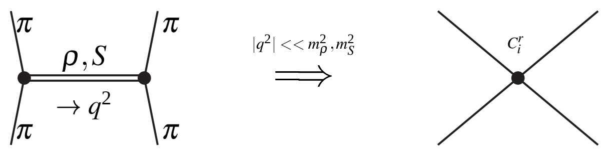

Figure 9: A schematic indication of the estimate of the order p^6 LECs by resonance exchange.

and constraints from lattice QCD on L_4^r, L_6^r will be very useful.

I now show the dependence of a few quantities on the input masses. These are updates of the plots shown in [46], more can be found in [1]. A selfconsistent set of m_π^2, m_K^2, m_η^2, F_π, $B_0 m_s$ and $B_0\hat{m}$ with the fitted values of L_i^r and F_0 is determined for each input value of two masses. This is done by iterating the formulas till convergence is reached. I show m_π^2 for fit 10 and fit D keeping $m_s/\hat{m} = 24$ and varying m_s in Fig. 7. The large corrections for fit 10 come from the kaon mass. The decay constants ratio F_K/F_π is shown as a function of m_s with $m_s/\hat{m} = 24$ as well.

4.3 C_i^r: estimates of order p^6 LECs

Most numerical analysis at order p^6 use a (single) resonance approximation to the order p^6 LECs. This is schematically shown in Fig. 9. The main underlying motivation is the large N_c limit and phenomenological success at order p^4 [13]. There is a large volume of work on this, some references are [63]. The numerical work I will report has used a rather simple resonance Lagrangian [13, 31, 45, 46, 13] only. The estimates of the C_i^r is the weakest point in the numerical fitting at present, however, many results are not very sensitive to this. The main problem is that the C_i^r which contribute to the masses, are estimated to be zero except for η' effects and how these might affect the determination of the others. The estimate is μ-independent while the C_i^r are not.

input	$C_1^r + 4C_3^r$	C_2^r	$C_4^r + 3C_3^r$	$C_1^r + 4C_3^r + 2C_2^r$
$\pi K : C_{30}^+, C_{11}^+, C_{20}^-$	20.7 ± 4.9	-9.2 ± 4.9	9.9 ± 2.5	2.3 ± 10.8
$\pi K : C_{30}^+, C_{11}^+, C_{01}^-$	28.1 ± 4.9	-7.4 ± 4.9	21.0 ± 2.5	13.4 ± 10.8
$\pi\pi$			23.5 ± 2.3	18.8 ± 7.2
Resonance model	7.2	-0.5	10.0	6.2

Table 3: Different determinations of the same combinations of the C_i^r from $\pi\pi$ and πK scattering [65].

The fits done here in [45, 46, 54] try to check this by varying the total resonance contribution by a factor of two, varying the scale μ from 550 to 1000 MeV and compare estimated C_i^r to experimentally determined ones. The latter works well, but the experimentally well determined ones are those with dependence on kinematic variables only, not ones relevant for quark-mass dependence.

We are at present [64] working on a new fit and trying to find how can be done without these estimates. That there might be some strain can be seen from the different C_i estimates from [65] shown in Table 3 using the results of $\pi\pi$ and πK scattering of [38, 55].

5. $\eta \to \pi\pi\pi$

In the limit of conserved isospin, no electromagnetism and $m_u = m_d$, the η is stable. Direct electromagnetic effects are small [66]. The decay thus proceeds mainly through the quark-mass difference $m_u - m_d$. The lowest order was done in [67], order p^4 in [68] and recently the full order p^6 has been evaluated [59]. The momenta for the decay $\eta \to \pi^+\pi^-\pi^0$ we label as p_η, p_+, p_- and p_0 respectively and we introduce the kinematical Mandelstam variables $s = (p_+ + p_-)^2$, $t = (p_+ + p_0)^2$, $u = (p_- + p_0)^2$. These are linearly dependent, $s + t + u = m_{\pi^o}^2 + m_{\pi^-}^2 + m_{\pi^+}^2 + m_\eta^2 \equiv 3s_0$. The amplitudes for the charged, $A(s,t,u)$, and neutral, $\overline{A}(s,t,u)$ are related

$$\overline{A}(s_1,s_2,s_3) = A(s_1,s_2,s_3) + A(s_2,s_3,s_1) + A(s_3,s_1,s_2)\,. \tag{5.1}$$

The relation in (5.1) is only valid to first order in $m_u - m_d$. The overall factor of $m_u - m_d$ can be put in different quantities, two common choices are

$$A(s,t,u) = \frac{\sqrt{3}}{4R} M(s,t,u) \quad \text{or} \quad A(s,t,u) = \frac{1}{Q^2}\frac{m_K^2}{m_\pi^2}(m_\pi^2 - m_K^2)\frac{\mathcal{M}(s,t,u)}{3\sqrt{3}F_\pi^2}\,, \tag{5.2}$$

with $R = (m_s - \hat{m})/(m_d - m_u)$ or $Q^2 = R(m_s + m_d)/(2\hat{m})$ pulled out. The lowest order result is

$$M(s,t,u)_{LO} = \left((4/3)\,m_\pi^2 - s\right)/F_\pi^2\,. \tag{5.3}$$

The tree level determination of R in terms of meson masses gives with (5.3) a decay rate of 66 eV which should be compared with the experimental results of 295±17 eV[69]. In principle, since the decay rate is proportional to $1/R^2$ or $1/Q^4$, this should allow for a precise determination of R and Q. However, the change required seems large. The order p^4 calculation [68] increased the predicted decay rate to 150 eV albeit with a large error. About half of the enhancement in the amplitude came from $\pi\pi$ rescattering and the other half from other effects like the chiral logarithms[68]. The rescattering effects have been studied at higher orders using dispersive methods in [70] and [71].

Both calculations found an enhancement in the decay rate to about 220 eV but differ in the way the Dalitz plot distributions look. This can be seen in Fig. 10 where I show the real part of the amplitude as a function of s along the line $s = u$. The calculations use different formalisms but make similar approximations, they mainly differ in the determination of the subtraction constants. That discrepancy and the facts that in $K_{\ell 4}$ the dispersive estimate [73] was about half the full ChPT

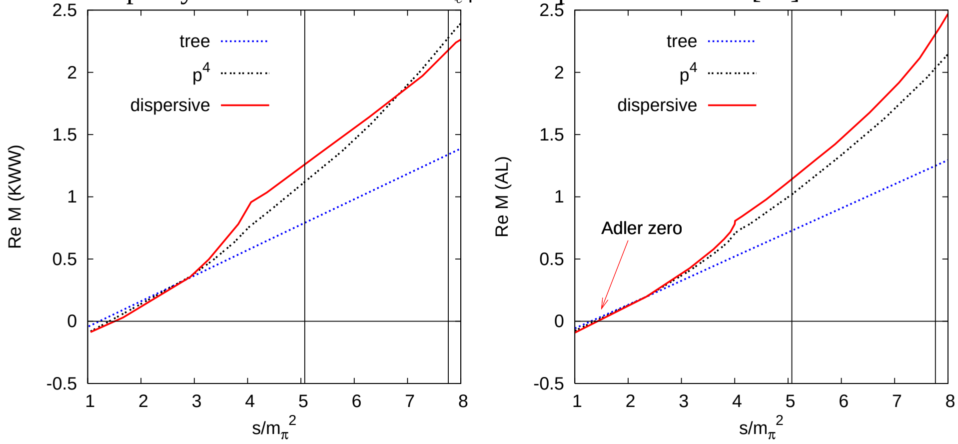

Figure 10: Left: Decay amplitude obtained by use of Khuri-Treiman equations[70] along the line $s = u$. Right: Alternative dispersive analysis for the decay amplitude[71]. Figs. from [72], adapted from [70, 71].

calculation [46] and at order p^4 the dispersive effect was about half of the correction for $\eta \to 3\pi$ makes it clear that also for this process a full order p^6 calculation was desirable.

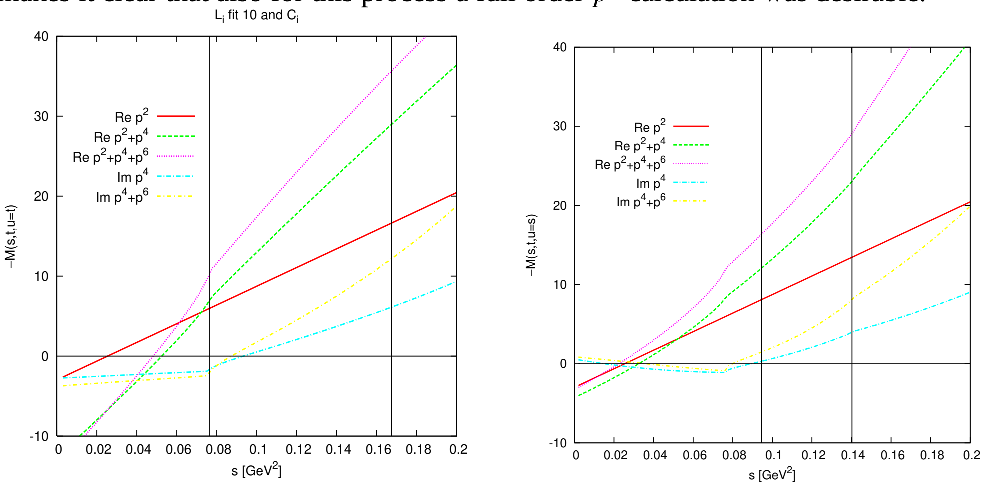

Figure 11: Left: The amplitude $M(s,t,u)$ along the line $t = u$. The vertical lines indicate the physical region. Shown are the real and imaginary parts with all parts summed up to the given order. Right: Similar plot but along the line $s = u$. Figs. from [59].

The calculation [59] generalized the methods of [45] to deal with π^0-η mixing. Here I show only results. In Fig. 11 I show the numerical result for the amplitude along two lines in the Dalitz plot, $t = u$ and $s = u$. The latter can be compared directly with the dispersive result of Fig. 10. The correction found in [59] at order p^6 is 20-30% in amplitude, larger in magnitude than the dispersive estimates [70, 71] but with a shape similar to [71].

The Dalitz plot in $\eta \to 3\pi$ is parameterized in terms of x and y defined in terms of the kinetic energies of the pions T_i and $Q_\eta = m_\eta - 2m_{\pi^+} - m_{\pi^0}$ for the charged decay and z defined in terms

Exp.	a	b	d
KLOE	$-1.090\pm0.005^{+0.008}_{-0.019}$	$0.124\pm0.006\pm0.010$	$0.057\pm0.006^{+0.007}_{-0.016}$
Crystal Barrel	-1.22 ± 0.07	0.22 ± 0.11	0.06 ± 0.04 (input)
Layter *et al.*	-1.08 ± 0.014	0.034 ± 0.027	0.046 ± 0.031
Gormley *et al*	-1.17 ± 0.02	0.21 ± 0.03	0.06 ± 0.04

Table 4: Measurements of the Dalitz plot distributions in $\eta\to\pi^+\pi^-\pi^0$. Quoted in the order cited in [74]. The KLOE result f is $f=0.14\pm0.01\pm0.02$.

	A_0^2	a	b	d	f
LO	120	-1.039	0.270	0.000	0.000
NLO	314	-1.371	0.452	0.053	0.027
NLO ($L_i^r=0$)	235	-1.263	0.407	0.050	0.015
NNLO	538	-1.271	0.394	0.055	0.025
NNLO ($\mu=0.6$ GeV)	543	-1.300	0.415	0.055	0.024
NNLO ($\mu=0.9$ GeV)	548	-1.241	0.374	0.054	0.025
NNLO ($C_i^r=0$)	465	-1.297	0.404	0.058	0.032
NNLO ($L_i^r=C_i^r=0$)	251	-1.241	0.424	0.050	0.007

Table 5: Theoretical estimate of the Dalitz plot distributions in $\eta\to\pi^+\pi^-\pi^0$.

of the pion energies E_i. The amplitudes are expanded in $x=\sqrt{3}\left(T_+-T_-\right)/Q_\eta$, $y=3T_0/Q_\eta-1$, $z=(2/3)\sum_{i=1,3}\left(3E_i-m_\eta\right)^2/\left(m_\eta-3m_{\pi^0}\right)^2$, via

$$|M(s,t,u)|^2 \;=\; A_0^2\left(1+ay+by^2+dx^2+fy^3+\cdots\right)\,,\quad |\overline{M}(s,t,u)|^2=\overline{A}_0^2\left(1+2\alpha_2+\cdots\right)\,. \tag{5.4}$$

Recent experimental results for these parameters are shown in Tabs. 4 and 6. There are discrepancies among the experiments but the latest precision measurements of α agree. The predictions from ChPT to order p^6 with the input parameters as described earlier are given in Tabs. 5 and 7. The different lines corresponds to variations on the input and the order of ChPT. The lines labeled NNLO are the central results. The agreement with experiment is not too good and clearly needs further study. Especially puzzling is that α is consistently positive while the dispersive calculations as well as [75] give a negative value. The inequality $\alpha\le\left(d+b-a^2/4\right)/4$ derived in [59] shows that α has rather large cancellations inherent in its prediction and that the overestimate of b is a likely cause of the wrong sign for α. The fairly large correction gives in the end larger values of Q compared to those derived from the masses [59].

6. Even more flavours at NNLO (or PQChPT)

NLO Partially Quenched ChPT has been studied by many people and found to be very useful, see [78] and references therein. The masses and decay constants are known to NNLO for almost all possible mass combinations. Formulas were kept in terms of the quark-mass expansion to avoid the proliferation in physical masses appearing in this case. The three sea flavour masses and decay constants are in [22, 79] and the two sea flavour results are in [80]. Numerical programs are available from the authors. The formulas are in the papers but can be downloaded from [18]. PQChPT NNLO results for neutral masses are in [81].

Exp.	α
Crystal Ball (MAMI C)	-0.032 ± 0.003
Crystal Ball (MAMI B)	$-0.032 \pm 0.002 \pm 0.002$
WASA/COSY	$-0.027 \pm 0.008 \pm 0.005$
KLOE	$-0.027 \pm 0.004^{+0.004}_{-0.006}$
Crystal Ball (BNL)	-0.031 ± 0.004
WASA/CELSIUS	$-0.026 \pm 0.010 \pm 0.010$
Crystal Barrel	$-0.052 \pm 0.017 \pm 0.010$
GAMS2000	-0.022 ± 0.023
SND	$-0.010 \pm 0.021 \pm 0.010$

Table 6: Measurements of the Dalitz plot distribution in $\eta \to \pi^0\pi^0\pi^0$. Quoted in the order cited in [76].

	$\overline{A}_0^2$	α
LO	1090	0.000
NLO	2810	0.013
NLO ($L_i^r = 0$)	2100	0.016
NNLO	4790	0.013
NNLO ($C_i^r = 0$)	4140	0.011
NNLO ($L_i^r, C_i^r = 0$)	2220	0.016
dispersive [70]	—	$-(0.007$—$0.014)$
tree dispersive	—	-0.0065
absolute dispersive	—	-0.007
Borasoy [75]	—	-0.031
error	160	0.032

Table 7: Theoretical estimates of the Dalitz plot distribution in $\eta \to \pi^0\pi^0\pi^0$. [77, 59]

7. Renormalization Group

In ChPT the renormalization group is not quite as useful as in renormalizable theories but Weinberg [8] already showed that some predictions can indeed be made. In particular leading logarithms at two loops can be had from only one-loop calculations. This was used for obtaining the leading double logarithms in $\pi\pi$ scattering [82] and in general [83]. The extension to all orders was proven in [84]. A leading log to five loops was obtained in [85] and recently recursion relations in the massless case were used to get results to very high orders [86].

The latter papers solved the practical problem of keeping track of all-order Lagrangians using two observations. First, in the massless limit, tadpoles vanish, and thus the number of external legs needed at any order does not increase. Second, the main vertex needed is then the four meson vertex. Here they found a useful general expressions using Legendre polynomials allowing them to do all needed loop integrals and obtain a fairly simple iterative algebraic recursion relation.

8. Conclusions

Modern ChPT is doing fine. Two flavour ChPT is in good shape: it is now precision science in many ways. For three flavour ChPT the corrections are larger and there seem to be some problems, but many parameters, especially in the scalar sector are rather uncertain and errors are very quantity dependent. Many partially quenched NNLO calculations have been done with an eye on lattice calculations and their extrapolations. A final comment is that new application areas continue to be found for ChPT and EFT in general.

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References

[1] J. Bijnens, PoS **LAT2007** (2007) 004 [arXiv:0708.1377 [hep-lat]].

[2] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110** (1958) 1178.

[3] Y. Nambu, Phys. Rev. Lett. **4** (1960) 380; M. Gell-Mann and M. Levy, Nuovo Cim. **16** (1960) 705.

[4] M. Gell-Mann, R. J. Oakes and B. Renner, Phys. Rev. **175** (1968) 2195.

[5] S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2247 (1969).

[6] G. Ecker and J. Honerkamp, Nucl. Phys. B **52**, 211 (1973).

[7] P. Langacker and H. Pagels, Phys. Rev. D **10**, 2904 (1974).

[8] S. Weinberg, Physica A **96**, 327 (1979).

[9] J. Gasser and A. Zepeda, Nucl. Phys. B **174**, 445 (1980).

[10] J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142.

[11] J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.

[12] J. Bijnens, H. Sonoda and M. B. Wise, Phys. Rev. Lett. **53**, 2367 (1984).

[13] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B **321** (1989) 311; G. Ecker *et al.*, Phys. Lett. B **223** (1989) 425; J. F. Donoghue, C. Ramirez and G. Valencia, Phys. Rev. D **39** (1989) 1947.

[14] H. Leutwyler, Annals Phys. **235** (1994) 165. [hep-ph/9311274].

[15] S. Bellucci, J. Gasser and M. E. Sainio, Nucl. Phys. B **423** (1994) 80 [Erratum-ibid. B **431** (1994) 413] [hep-ph/9401206].

[16] G. Ecker, Prog. Part. Nucl. Phys. **35** (1995) 1 [hep-ph/9501357]; A. Pich, Rept. Prog. Phys. **58** (1995) 563 [hep-ph/9502366]; V. Bernard *et al.*, Int. J. Mod. Phys. E **4** (1995) 193 [hep-ph/9501384]; V. Bernard and U. G. Meissner, Ann. Rev. Nucl. Part. Sci. **57** (2007) 33 [hep-ph/0611231].

[17] J. Bijnens, Prog. Part. Nucl. Phys. **58** (2007) 521 [hep-ph/0604043].

[18] `http://www.thep.lu.se/~bijnens/chpt.html`

[19] S. Weinberg, *Phys. Rev.* **166** (1968) 1568.

[20] J. Bijnens, G. Colangelo and G. Ecker, *J. High Energy Phys.* **9902** (1999) 020 [hep-ph/9902437].

[21] H. W. Fearing and S. Scherer, *Phys. Rev.* D **53** (1996) 315 [hep-ph/9408346].

[22] J. Bijnens *et al.*, *Phys. Rev.* D **70** (2004) 111503 [hep-lat/0406017], D **73** (2006) 074509 [hep-lat/0602003]

[23] C. Haefeli, M. A. Ivanov, M. Schmid and G. Ecker, arXiv:0705.0576 [hep-ph].

[24] J. Bijnens, G. Colangelo and G. Ecker, *Annals Phys.* **280** (2000) 100 [hep-ph/9907333].

[25] J. Gasser and U. G. Meissner, *Nucl. Phys.* B **357** (1991) 90.

[26] G. Colangelo, M. Finkemeier and R. Urech, *Phys. Rev.* D **54** (1996) 4403 [hep-ph/9604279].

[27] M. Knecht *et al.*, *Nucl. Phys.* B **457** (1995) 513 [hep-ph/9507319].

[28] J. Gasser, M. A. Ivanov and M. E. Sainio, *Nucl. Phys.* B **728** (2005) 31 [hep-ph/0506265].

[29] U. Burgi, *Phys. Lett.* B **377** (1996) 147 [hep-ph/9602421]; *Nucl. Phys.* B **479** (1996) 392 [hep-ph/9602429].

[30] J. Gasser, M. A. Ivanov and M. E. Sainio, Nucl. Phys. B **745** (2006) 84 [hep-ph/0602234].

[31] J. Bijnens *et al.*, *Phys. Lett.* B **374** (1996) 210 [hep-ph/9511397]; *Nucl. Phys.* B **508** (1997) 263 [Erratum-ibid. B **517** (1998) 639] [hep-ph/9707291].

[32] J. Bijnens, G. Colangelo and P. Talavera, *J. High Energy Phys.* **9805** (1998) 014 [hep-ph/9805389].

[33] J. Bijnens and P. Talavera, *Nucl. Phys.* B **489** (1997) 387 [hep-ph/9610269].

[34] G. Colangelo and C. Haefeli, *Nucl. Phys.* B **744** (2006) 14 [hep-lat/0602017].

[35] K. Kampf and B. Moussallam, 0901.4688 [hep-ph], K. Kamp, talk at this conference.

[36] G. Colangelo, J. Gasser and H. Leutwyler, *Nucl. Phys.* B **603** (2001) 125 [hep-ph/0103088].

[37] M. Gonzalez-Alonso *et al.*, Phys. Rev. D **78** (2008) 116012 [arXiv:0810.0760 [hep-ph]].

[38] J. Bijnens, P. Dhonte and P. Talavera, *J. High Energy Phys.* **0401** (2004) 050 [hep-ph/0401039].

[39] G. Amorós *et al.*, *Nucl. Phys.* B **568** (2000) 319 [hep-ph/9907264].

[40] E. Golowich and J. Kambor, *Nucl. Phys.* B **447** (1995) 373 [hep-ph/9501318]; S. Dürr and J. Kambor, *Phys. Rev.* D **61** (2000) 114025 [hep-ph/9907539].

[41] K. Maltman, *Phys. Rev.* D **53** (1996) 2573 [hep-ph/9504404].

[42] B. Moussallam, *J. High Energy Phys.* **0008** (2000) 005 [hep-ph/0005245].

[43] J. Bijnens, unpublished.

[44] E. Golowich and J. Kambor, *Phys. Rev.* D **58** (1998) 036004 [hep-ph/9710214].

[45] G. Amorós *et al.*, *Nucl. Phys.* B **602** (2001) 87 [hep-ph/0101127].

[46] G. Amorós *et al.*, *Phys. Lett.* B **480** (2000) 71 [hep-ph/9912398]; *Nucl. Phys.* B **585** (2000) 293 [Erratum-ibid. B **598** (2001) 665] [hep-ph/0003258].

[47] J. Bijnens and K. Ghorbani, *Phys. Lett.* B **636** (2006) 51 [hep-lat/0602019].

[48] P. Post and K. Schilcher, *Phys. Rev. Lett.* **79** (1997) 4088 [hep-ph/9701422]; *Nucl. Phys.* B **599** (2001) 30 [hep-ph/0007095].

[49] J. Bijnens and P. Talavera, *J. High Energy Phys.* **0203** (2002) 046 [hep-ph/0203049].

[50] C. Q. Geng, I. L. Ho and T. H. Wu, *Nucl. Phys.* B **684** (2004) 281 [hep-ph/0306165].

[51] J. Bijnens and P. Talavera, *Nucl. Phys.* B **669** (2003) 341 [hep-ph/0303103].

[52] P. Post and K. Schilcher, *Eur. Phys. J.* C **25** (2002) 427 [hep-ph/0112352].

[53] J. Bijnens and K. Ghorbani, arXiv:0711.0148 [hep-ph].

[54] J. Bijnens and P. Dhonte, *J. High Energy Phys.* **0310** (2003) 061 [hep-ph/0307044].

[55] J. Bijnens, P. Dhonte and P. Talavera, *J. High Energy Phys.* **0405** (2004) 036 [hep-ph/0404150].

[56] J. Gasser *et al.*, Phys. Lett. B **652** (2007) 21 [arXiv:0706.0955 [hep-ph]], arXiv:0903.0801 [hep-ph].

[57] R. Kaiser, *J. High Energy Phys.* **0709** (2007) 065 [arXiv:0707.2277 [hep-ph]].

[58] R. Kaiser and J. Schweizer, *J. High Energy Phys.* **0606** (2006) 009 [hep-ph/0603153].

[59] J. Bijnens and K. Ghorbani, *J. High Energy Phys.* **0711** (2007) 030 [arXiv:0709.0230 [hep-ph]].

[60] S. Pislak *et al.* [BNL-E865], *Phys. Rev. Lett.* **87** (2001) 221801 [hep-ex/0106071]; *Phys. Rev.* D **67** (2003) 072004 [hep-ex/0301040].

[61] J. Bijnens, N. Danielsson and T. A. Lahde, Acta Phys. Polon. B **38** (2007) 2777 [hep-ph/0701267].

[62] P. Buettiker, S. Descotes-Genon and B. Moussallam, *Eur. Phys. J.* C **33** (2004) 409 [hep-ph/0310283].

[63] M. Knecht and A. Nyffeler, *Eur. Phys. J.* C **21** (2001) 659 [hep-ph/0106034]; S. Peris *et al.*, *J. High Energy Phys.* **9805** (1998) 011 [hep-ph/9805442]; J. Bijnens *et al.*, *J. High Energy Phys.* **0304** (2003) 055 [hep-ph/0304222]. V. Cirigliano *et al.*, *Nucl. Phys.* B **753** (2006) 139 [hep-ph/0603205].

[64] I. Jemos, talk at this conference; J. Bijnens and I. Jemos, work in progress.

[65] K. Kampf and B. Moussallam, Eur. Phys. J. C **47** (2006) 723 [hep-ph/0604125].

[66] D. G. Sutherland *Phys. Lett.* **23**(1966)384; J. S. Bell and D. G. Sutherland, *Nucl. Phys.* **B4**(1968)315.

[67] J. A. Cronin, *Phys. Rev.* **161** (1967) 1483; H. Osborn and D. J. Wallace, *Nucl. Phys.* **B20** (1970) 23.

[68] J. Gasser and H. Leutwyler, *Nucl. Phys.* **B250** (1985) 539.

[69] W. M. Yao *et al.* [Particle Data Group], *J. Phys.* **G33** (2006) 1.

[70] J. Kambor *et al.*, *Nucl. Phys.* **B465** (1996) 215 [hep-ph/9509374].

[71] A. V. Anisovich and H. Leutwyler, *Phys. Lett.* **B375** (1996) 335 [hep-ph/9601237].

[72] J. Bijnens, *Acta Phys. Slov.* **56** (2005) 305 [hep-ph/0511076].

[73] J. Bijnens *et al.*, *Nucl. Phys.* **B427** (1994) 427 [hep-ph/9403390].

[74] F. Ambrosino *et al.* [KLOE], *J. High Energy Phys.* **0805** (2008) 006 [arXiv:0801.2642 [hep-ex]]; A. Abele *et al. Phys. Lett.* **B417** (1998) 197; J. G. Layter *et al.*, *Phys. Rev.* **D7** (1973) 2565; M. Gormley *et al.*, *Phys. Rev.* **D2** (1970) 501.

[75] B. Borasoy and R. Nissler, *Eur. Phys. J.* **A26** (2005) 383 [hep-ph/0510384].

[76] S. Prakhov *et al.*, Phys. Rev. C **79** (2009) 035204 [arXiv:0812.1999 [hep-ex]]; M. Unverzagt *et al.*, Eur. Phys. J. A **39** (2009) 169 [arXiv:0812.3324 [hep-ex]]; C. Adolph *et al.* [WASA-at-COSY Collaboration], arXiv:0811.2763 [nucl-ex]. F. Ambrosino *et al.*arXiv:0707.4137 [hep-ex]; W. B. Tippens *et al. Phys. Rev. Lett.* **87** (2001) 192001; M. Bashkanov *et al.*, Phys. Rev. C **76** (2007) 048201 [arXiv:0708.2014 [nucl-ex]]. A. Abele *et al.*, Phys. Lett. B **417** (1998) 193. D. Alde *et al.*, Z. Phys. C **25** (1984) 225 [Yad. Fiz. **40** (1984) 1447]. M. N. Achasov *et al.*, JETP Lett. **73** (2001) 451.

[77] J. Bijnens and J. Gasser, *Phys. Scripta* **T99** (2002) 24 [hep-ph/0202242].

[78] S. R. Sharpe, hep-lat/0607016.

[79] J. Bijnens and T. A. Lahde, *Phys. Rev.* D **71** (2005) 094502 [hep-lat/0501014].

[80] J. Bijnens and T. A. Lahde, *Phys. Rev.* D **72** (2005) 074502 [hep-lat/0506004].

[81] J. Bijnens and N. Danielsson, *Phys. Rev.* D **74** (2006) 054503 [hep-lat/0606017].

[82] G. Colangelo, Phys. Lett. B **350** (1995) 85 [Erratum-ibid. B **361** (1995) 234] [hep-ph/9502285].

[83] J. Bijnens, G. Colangelo and G. Ecker, Phys. Lett. B **441** (1998) 437 [hep-ph/9808421].

[84] M. Büchler and G. Colangelo, Eur. Phys. J. C **32** (2003) 427 [hep-ph/0309049].

[85] M. Bissegger and A. Fuhrer, Phys. Lett. B **646** (2007) 72 [hep-ph/0612096].

[86] N. Kivel *et al.*, Phys. Rev. Lett. **101** (2008) 262001 [arXiv:0809.3236 [hep-ph]]; arXiv:0904.3008 [hep-ph].