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On SU(2) Chiral Perturbation Theory for $K \to \pi\pi$ Decay Amplitudes

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Abstract

We study the CP conserving decays $K \to \pi\pi$ using $SU(2)_L \times SU(2)_R$ Chiral Perturbation Theory (SU(2) ChPT) at next-to-leading order. The external pions in $K \to \pi\pi$ are hard, thus complicating the power counting of the theory. However we provide arguments about the possibility of calculating the pion chiral logarithms, i.e. the corrections of $O(M^2 \log(M^2))$, for the nonleptonic decay $K \to \pi\pi$. A comparison with SU(3) ChPT is done to check our results.

Master of Science Thesis

Contents

1 Introduction

The nonleptonic decay of hadrons is not very well understood yet. The $\Delta I = 1/2$ rule and the value of ϵ'/ϵ are some of the main problems in this field. The purpose of this thesis is to calculate the pion chiral logarithms, i.e. the corrections of $O(M^2\log(M^2))$, of the nonleptonic decay of the kaon, $K \to \pi\pi$, within the framework of Chiral Perturbation Theory. Chiral Perturbation Theory (ChPT), introduced around 30 years ago [1], is a low energy effective field theory of Quantum Chromo Dynamics(QCD), the theory of the strong interaction. The essence of an effective theory is to consider the relevant degrees of freedom for the problem at hand. In our case, the degrees of freedom are no longer quarks and gluons but hadrons (at low energies quarks form bound states). The first application of ChPT at one loop to nonleptonic weak decays was made by Kambor, Missimer and Wyler [2, 3].

Currently, lattice QCD simulations use values of $m_\pi \sim (300 - 800)$ MeV . This means that the results obtained have to be extrapolated to the physical masses [4]. Chiral Perturbation Theory is used to perform this extrapolation. Recently, Flynn and Sachrajda studied the semileptonic decay of the kaon $K_{\ell 3}$, $K \to \pi\nu_\ell \ell$, in the framework of SU(2) ChPT with this purpose [5]. The motivation of our work is that the study of the nonleptonic decay of the kaon within SU(2) ChPT can be useful to realize the so called chiral extrapolations.

We use SU(2) ChPT with the kaon included as a heavy meson. A problem arises when taking into account processes in which the heavy particle, the kaon, is annihilated. In this case external pions are hard, meaning that $2p_K.p_\pi = M_K^2$, and the power counting scheme breaks down. Flynn and Sachrajda provide some arguments to deal with this problem in the case of $K_{\ell 3}$ decays [5], though no strict power counting scheme has been established. In the present work this analysis is extended to the nonleptonic kaon decay $K \to \pi\pi$.

Sections 2 and 3 cover the theoretical framework of SU(2) ChPT. The contribution of this thesis is presented in Sections 4, 5 and 6. Sections 4 and 5 are devoted to the calculation of the $K \to \pi\pi$ decay amplitude at next-to-leading order in SU(2) ChPT, including the construction of the weak Lagrangian and a discussion of the power counting problem. Finally, in Section 6 we make a matching of our results with the ones from chiral SU(3) to check our results.

2 QCD and Chiral Symmetry

At low energies perturbative calculations are not possible because the strong coupling constant $\alpha_s \gg 1$ and so perturbation theory breaks down. In this regime quarks form bound states (hadrons), that means that to study processes including hadrons, like $K \to \pi\pi$, a different approach is needed.

The construction of a low energy effective field theory of QCD relying on the symmetry properties of the QCD Lagrangian provides a way to deal with the low energy regime of

QCD [1]. So we start studying the symmetries of $\mathcal{L}_{QCD}$ and its implications.

2.1 Chiral Symmetry

The two flavour QCD Lagrangian is given by

$$\mathcal{L}_{QCD} = -\frac{1}{2g^2}\mathrm{tr}(G_{\mu\nu}G^{\mu\nu}) + \mathcal{L}_{ud}, \tag{2.1}$$

where

$$\mathcal{L}_{ud} = (\bar{u} \ \ \bar{d}) \begin{pmatrix} \mathcal{D} - m_u & \\ & \mathcal{D} - m_d \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}. \tag{2.2}$$

$G_{\mu\nu}$ denotes the field strength associated with the gluon field G_μ , “tr” means color trace and $\mathcal{D} = i\gamma^\mu(\partial_\mu - iG_\mu)$.

We can rewrite this Lagrangian in terms of the left-handed and right-handed fields

$$q_L = \frac{1}{2}(1-\gamma^5)q, \qquad q_R = \frac{1}{2}(1+\gamma^5)q, \tag{2.3}$$

$$\mathcal{L}_{ud} = (\bar{u}_L \ \ \bar{d}_L) \begin{pmatrix} \mathcal{D} & \\ & \mathcal{D} \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix} - (\bar{u}_L \ \ \bar{d}_L) \begin{pmatrix} m_u & \\ & m_d \end{pmatrix} \begin{pmatrix} u_R \\ d_R \end{pmatrix} + \ (L \leftrightarrow R)\,. \tag{2.4}$$

In the chiral limit of vanishing quark masses, $m_u = m_d = 0$, $\mathcal{L}_{QCD}$ is invariant under $U(1)_V \times U(1)_A \times SU(2)_L \times SU(2)_R$ transformations. The action of $(g_L, g_R) \in SU(2)_L \times SU(2)_R$ on the quark fields is defined by

$$q_L \rightarrow g_L q_L \quad , \quad q_R \rightarrow g_R q_R. \tag{2.5}$$

And the action of $U(1)_V$ and $U(1)_A$ is given respectively by

$$q \rightarrow e^{i\theta} q \tag{2.6}$$

$$q \rightarrow e^{i\gamma^5\theta} q \ , \tag{2.7}$$

where θ is a real parameter and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

2.2 Noether Theorem and Anomalies

Symmetries of the Lagrangian give rise to conserved quantities according to the Noether theorem. There is one conserved current for each continuous parameter in the symmetry group [6]. $U(1)_V \times U(1)_A \times SU(2)_L \times SU(2)_R$ has 8 continuous parameters, so we expect to have 8 conserved currents. However, in the full quantum theory the current associated with $U(1)_A$ is not conserved due to an anomaly [7, 8]. Whenever a classical symmetry is no longer a symmetry in the quantum theory, we say that there is an anomaly. $U(1)_V$ is still conserved in the quantum theory and gives rise to baryon number conservation.

2.3 Goldstone Theorem

Suppose that we have a system described by a Hamiltonian $\mathcal{H}$ invariant under the group transformations G. Denoting Q_i the generators of G, so that

$$[Q_i, \mathcal{H}] = 0 \qquad ; i = 1 \qquad , \qquad \ldots n_G. \tag{2.8}$$

Assume that the ground state is not invariant under G, so that for some generators X_i

$$X_i|0\rangle \neq 0. \tag{2.9}$$

This is known as spontaneous symmetry breaking. We can separate the generators of G into

$$\{Q_1, \ldots Q_{n_G}\} = \{H_1, \ldots, H_{n_H}, X_1, \ldots, X_{n_G - n_H}\}, \tag{2.10}$$

and

$$\begin{aligned} [Q_i, \mathcal{H}] &= 0\,, \\ H_i|0\rangle &= 0. \end{aligned} \tag{2.11}$$

From the fact that $[X_i, \mathcal{H}] = 0$ we have that the theory contains $n_G - n_H$ massless excitations $X_i|0\rangle$, i.e. $X_i|0\rangle$ are eigenstates of $\mathcal{H}$ with the same eigenvalue as the vacuum. The generators H_i form a subalgebra $[H_i, H_k]|0\rangle = 0; i = 1, \ldots, n_H$ and the X_i are generators of the coset group G/H.

Another consequence of the spontaneous symmetry breaking is that if we consider the conserved currents J^i_μ associated with the generators Q_i, then

$$\langle 0|J^\mu_i|\pi^a(p)\rangle = iF^a_i p^\mu. \tag{2.12}$$

Where the matrix F^a_i has rank $N_{GB} = n_G - n_H$

We can illustrate the Goldstone theorem by analyzing a simple model. Suppose that ϕ is a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2. \tag{2.13}$$

This Lagrangian is invariant under $U(1)$ global transformations

$$\phi \to e^{i\chi}\phi. \tag{2.14}$$

For $\mu^2 < 0$ and $\lambda > 0$ the potential is the well known Mexican hat potential. The minimum is along a circle defined by

$$\phi_1^2 + \phi_2^2 = \frac{-\mu^2}{\lambda} = v^2. \tag{2.15}$$

To study the low energy spectrum we have to choose a vacuum, for example $\phi_1 = v$, $\phi_2 = 0$ and then expand the field ϕ around it

$$\phi = \frac{(v + \eta(x) + i\rho(x))}{\sqrt{2}}. \tag{2.16}$$

η and ρ are real scalar fields. The Lagrangian then can be written as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}(\partial_\mu \eta)^2 + \mu^2\eta^2 - \lambda v(\eta\rho^2 + \eta^3) - \frac{\lambda}{2}\eta^2\rho^2 - \frac{\lambda}{4}\eta^4 - \frac{1}{4}\rho^4 + const. \tag{2.17}$$

Thus, there is a massless particle associated with the ρ field. The spontaneous breaking of the $U(1)$ symmetry causes the existence of this massless particle, as predicted by the Goldstone theorem.

Another consequence of the Goldstone theorem is that the Goldstone boson's interaction vanishes at zero momenta [9].

2.4 Symmetry Breaking in QCD

It has been argued that the QCD Lagrangian neglecting the mass term is invariant under G, but in the particle spectrum this symmetry is not observed at all, there are no parity doublets in nature.
There are several reasons to believe that the chiral symmetry group G is spontaneously broken into the isospin subgroup: $SU(2)_L \times SU(2)_R \to SU(2)_V$. It has been proven that $H = SU(2)_V$ is not spontaneously broken [10]. A sufficient condition for the spontaneous symmetry breaking of the theory is a non-vanishing quark condensate,i.e $\langle 0|q\bar{q}|0\rangle \neq 0$ [11]. Lattice QCD results as well as $K_{\ell 4}$ decays data , $K \to \pi\pi\ell\nu_\ell$, confirm that the quark condensate does not vanish [12].

The Goldstone theorem then implies the existence of 3 massless Goldstone bosons, but there are no massless mesons in nature (the lightest ones are the three pions $\pi^\pm, \pi^0$). This is due to the fact that the chiral limit is just an approximation.

The chiral symmetry G is explicitly broken by the mass term

$$\mathcal{L}_m = -\bar{u}m_u u - \bar{d}m_d d. \tag{2.18}$$

However the quark masses are small compared to the chiral symmetry breaking scale $\Lambda_\chi \sim 1$ GeV, so that we can treat the mass term as a small perturbation. We include the quark masses in the Lagrangian with external fields [13, 14].

3 Effective Field Theories

3.1 The Linear Sigma Model

The linear sigma model provides a model to study the consequences of the Goldstone theorem in a case where explicit calculations can be done. We will study the symmetry properties of the sigma model Lagrangian $\mathcal{L}_\sigma$, the spontaneous symmetry breaking of the theory, the appearance of Goldstone bosons and their scattering in the quantized theory. Afterwards, we will construct the non-linear sigma model, an effective field theory of the linear sigma model at low energies.

The Lagrangian of the linear sigma model is

$$\mathcal{L}_{l\sigma} = \frac{1}{2}\mathrm{tr}(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger) - \lambda[\mathrm{tr}(\Sigma\Sigma^\dagger - v^2)], \tag{3.19}$$

where $\Sigma = \begin{pmatrix} \frac{\sigma + \pi^3}{\sqrt{2}} & \frac{\pi^1 + i\pi^2}{\sqrt{2}} \\ \frac{\pi^1 - i\pi^2}{\sqrt{2}} & \frac{\sigma - \pi^3}{\sqrt{2}} \end{pmatrix}$, $\Sigma^\dagger = \Sigma$

$\mathcal{L}_{l\sigma}$ has an $SU(2) \times SU(2)$ symmetry

$$\Sigma \to g_R \Sigma g_L^\dagger. \tag{3.20}$$

Using the canonical choice for the vacuum

$$\Sigma = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}, \tag{3.21}$$

we see that the vacuum is invariant only under transformations with $g_R^\dagger = g_L$. The original symmetry $SU(2) \times SU(2)$ is spontaneously broken to SU(2) .

Let us look at the particle spectrum of the theory at low energies, putting $\sigma = v + \bar{\sigma}$

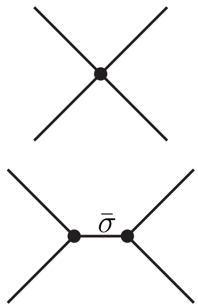
$$\mathcal{L}_{l\sigma} = \frac{1}{2}\partial_\mu\bar{\sigma}\partial^\mu\bar{\sigma} + \partial_\mu\pi^i\partial^\mu\pi^i - 4\lambda v^2\bar{\sigma}^2 - 4\lambda v\bar{\sigma}\pi^i\pi^i - \lambda\pi^i\pi^i\pi^j\pi^j - \lambda\bar{\sigma}^4 - 4\lambda v\bar{\sigma}^3 - 2\lambda\bar{\sigma}^2\pi^i\pi^i\ . \tag{3.22}$$

So we have 3 massless pions and one σ particle with mass

$$m_\sigma^2 = 8\lambda v^2\ . \tag{3.23}$$

This result was expected from the Goldstone theorem since the symmetry group $SU(2) \times SU(2)$ of 6 parameters is spontaneously broken to SU(2) with 3 parameters.

We can check that the interactions of Goldstone bosons vanishes at zero momenta by studying the scattering $\pi^1(p_1)\ \pi^1(p_2) \to \pi^2(p_3)\ \pi^2(p_4)$ at tree level.



$$= -8i\lambda$$

$$= \frac{i^3}{s - m_\sigma^2} + 16v^2\lambda^2 4$$

Figure 1: Diagrams contributing to $\pi^1(p_1)\ \pi^1(p_2) \to \pi^2(p_3)\ \pi^2(p_4)$ scattering at tree-level

We define $s = (p_1 + p_2)^2$. The tree-level amplitude for the process at $s \ll m_\sigma^2$ is

$$A = i8\lambda\big(-1 - \frac{8v^2\lambda}{s - m_\sigma^2}\big) \approx i8\lambda\big(\frac{s}{m_\sigma^2} + \frac{s^2}{m_\sigma^4} + \cdots\big)\ , \tag{3.24}$$

so that the amplitude vanishes at $s = 0$. Another approach that can be taken to study the low energy process, $s \ll m_\sigma^2$, is to construct an effective Lagrangian with only pions, the σ particle is integrated out from the diagrams shown in Figure 1 [15]

The general formalism for the effective Lagrangian in the case of spontaneously broken symmetries is known as the CCWZ formalism due to Callan, Coleman, Wess and Zumino and will be discussed in Section 3.2 [16, 17].

Let U be a $G/H = SU(2) \times SU(2)/SU(2) \simeq SU(2)$ matrix such that under chiral transformations

$$U \to g_R U g_L^\dagger\ . \tag{3.25}$$

We have to construct the more general Lagrangian invariant under chiral and Lorentz transformations. There are no invariant terms without derivatives and with two derivatives there is only

$$\mathcal{L} = \frac{F^2}{4}\mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger)\ , \tag{3.26}$$

F is the pion decay constant in the limit of massless pions and U can be written in terms of the generators of SU(2), the Pauli matrices τ^i, as [6]

$$U = e^{\frac{i}{F}(\pi^i\tau^i)}. \tag{3.27}$$

Expanding the Lagrangian one gets

$$\mathcal{L} = \frac{1}{2}\partial_\mu\pi^i\partial^\mu\pi^i + \frac{2}{24F^2}(4\pi^1\pi^2\partial_\mu\pi^1\partial^\mu\pi^2 - 2(\pi^1)^2\partial_\mu\pi^2\partial^\mu\pi^2 - 2(\pi^2)^2\partial_\mu\pi^1\partial^\mu\pi^1). \tag{3.28}$$

This gives the following amplitude for the scattering $\pi^1(p_1)\pi^1(p_2) \to \pi^2(p_3)\pi^2(p_4)$

$$A = \frac{i}{F}(s - \frac{1}{3}(p_1^2 + p_2^2 + p_3^2 + p_4^2)). \tag{3.29}$$

This result agrees on shell $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$ with the one obtained from the linear sigma model at first order in s if we identify F with v^2.

3.2 SU(2) Chiral Perturbation Theory for Kaons

We need to construct an effective Lagrangian to model the dynamics of pions and kaons at low energies. We will use the SU(2) ChPT formalism with kaons included as heavy mesons [18]

Consider again the situation of Section 2.3. A system with a Lagrangian $\mathcal{L}$ invariant under the group G, which is spontaneously broken into H. The space of Goldstone bosons field configurations M_1 is obtained by the action of G on the vacuum, i.e. $M_1 = \{g(0) \equiv g(x)|0\rangle \ , \ g \in G\}$.

We can prove that all elements of one coset $gH = \{gh \ ; g \in G \ , h \in H\}$ give rise to the same Goldstone boson field configuration when acting on the vacuum. If g' is in the same coset as g, there exist an h, such that $g = g'h$. Then

$$g(0) = (g'h)(0) = g'(h(0)) = g'(0) \ . \tag{3.30}$$

This implies that $M_1 \subset G/H$, i.e. that to each Goldstone boson field configuration we can associate at least one coset gH. We can prove the converse, let $g \neq g'h$ for $h \in H$ and e be the identity of G. Suppose that $g(0) = g'(0)$ and consider

$$e(0) = 0 = (g^{-1}g)(0) = g^{-1}(g(0)) = g^{-1}(g'(0)) = (g^{-1}g')(0). \tag{3.31}$$

Which implies that $g \in g'H$ in contradiction with our original assumption. This leads to the conclusion that M_1 is isomorphic to G/H.

Now we want to see how the Goldstone field configuration, parametrized by $\Phi(x)$, transform under the action of G. Let $\Xi \in gH$ be a representative of the coset such that $\Phi(x) = \Xi(x)\Phi_0$, where Φ_0 represents the vacuum configuration. Then

$$g(\Phi(x)) = g(\Xi(x)\Phi_0) = (g\Xi(x))\Phi_0 = (\Xi'(x))\Phi_0 = \Phi'(x), \tag{3.32}$$

so that the action of G on G/H is given by, $\Xi(x) \to g\Xi(x)$.

Now we are ready to apply these ideas to the case of QCD. Consider $G = SU(2)_L \times SU(2)_R$ and $H = SU(2)_V$. In this case the space of Goldstone boson field configurations is isomorphic to $G/H = SU(2)_L \times SU(2)_R/SU(2)_V \simeq SU(2)$. As explained earlier, the same configuration $\Phi(x)$ is obtained by elements of the same coset,

$$\Xi(x)\Phi_0 = \Xi'(x)\Phi_0, \text{ if } \Xi = \Xi' \ h. \tag{3.33}$$

The CCWZ prescription is to choose a set of broken generators X, and define

$$\Xi(x) = e^{iX^a\cdot\pi_a(x)}. \tag{3.34}$$

In our case, we have that G has 6 generators $\frac{1}{2}\sigma_L^a$ and $\frac{1}{2}\sigma_L^a$ wich act on left and right handed fields respectively. The generators of H are $\sigma^a = \frac{1}{2}\sigma_L^a + \frac{1}{2}\sigma_R^a$. The fundamental representation of σ^a are the well known Pauli matrices, we will be referring to these matrices in the following. Under the transformation G, $\Xi(x) \in gH$ transforms as $\Xi(x) \to g\Xi(x)$, so that it is no longer in the standard form (3.34).

We can write $g\ \Xi(x) = \Xi'(x)h$ for some $h \in H$ with $\Xi'(x)$ in the standard form. h is known as a compensatory field. So, we define the transformation of $\Xi(x)$ under G to be

$$\Xi(x) \to g\Xi(x)h^{-1}(g, \Xi(x))\ . \tag{3.35}$$

Where the transformation h needed to return to our standard choice given in (3.34) depends on g and $\Xi(x)$. Choosing $X^a = \frac{1}{2}\sigma_R^a - \frac{1}{2}\sigma_L^a$, we have by (3.34)

$$\Xi(x) = e^{iX^a\cdot\pi_a(x)} = \exp\ i\begin{bmatrix} -X^a\cdot\pi_a & 0 \\ 0 & X^a\cdot\pi_a \end{bmatrix} = \begin{pmatrix} u^\dagger(x) & 0 \\ 0 & u(x) \end{pmatrix}\ , \tag{3.36}$$

where

$$u(x) = e^{iX^a\cdot\pi_a(x)}. \tag{3.37}$$

In terms of the following representations of g and h[1]

$$g = \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix},$$

$$h = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}\ , \tag{3.38}$$

where $g_I \in SU(2)_I\ ; I = L, R$ and $h \in SU(2)$. The transformation of $\Xi(x)$ according to (3.35) is given by

$$\begin{pmatrix} u^\dagger(x) & 0 \\ 0 & u(x) \end{pmatrix} \to \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}\begin{pmatrix} u^\dagger(x) & 0 \\ 0 & u(x) \end{pmatrix}\begin{pmatrix} h^{-1} & 0 \\ 0 & h^{-1} \end{pmatrix}, \tag{3.39}$$

so that $u(x) \to g_R u(x) h^{-1} = h u(x) g_L^\dagger$. It is appropriate to use fields with mass dimensions, the standard choice is to use

$$u(x) = e^{iX^a\cdot\pi_a/2F}, \tag{3.40}$$

where $F \approx 93$ MeV is the pion decay constant. The product $X^a\cdot\pi_a$ expressed in terms of the physical pion fields is given by

$$X^a\cdot\pi_a \equiv \phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}. \tag{3.41}$$

[1]We use the same notation for the 4×4 representation of h and for a SU(2) matrix.

It is also useful to consider $U = u^2$, because of its transformation properties under G,

$$U \to g_R U g_L^\dagger. \tag{3.42}$$

Now that we understand the transformation properties of the pion fields, we need to add the kaon to our description. We parametrize the kaon in terms of the physical particle fields as

$$K(x) = \begin{pmatrix} K^+(x) \\ K^0(x) \end{pmatrix}. \tag{3.43}$$

Because kaons have isospin $\frac{1}{2}$, we can define the action of G on the kaon fields $K(x)$ using the compensator h

$$K \to hK\ . \tag{3.44}$$

3.2.1 The Strong Lagrangian

The strong Lagrangian was worked out by Roessl up to fourth chiral order [18] [2]. We will use the Lagrangian up to second chiral order. In order to construct all the possible terms that respect the symmetries of QCD: chiral symmetry, parity (P), charge conjugation (C) and hermiticity, it is useful to build a set of basic building blocks that transform in certain way under the chiral transformations,

$$A \to hAh^\dagger, \tag{3.45}$$

where h is the compensator field introduced earlier.

So far we have been working with global chiral transformations, but its possible to consider local transformations. In the formulation of Gasser and Leutwyler [13, 14] the chiral lagrangian of QCD is extended by including external hermitian matrix fields v_μ, a_μ, s, p .

$$\mathcal{L} = \mathcal{L}_{QCD} + \bar{q}\gamma^\mu(v_\mu + a_\mu\gamma^5)q - \bar{q}(s - ip\gamma^5)q. \tag{3.46}$$

The covariant derivative needed to maintain local invariance is

$$\begin{aligned} \nabla_\mu A &= \partial_\mu A + [\Gamma_\mu, A], \\ \nabla_\mu K &= \partial_\mu K + \Gamma_\mu K. \end{aligned} \tag{3.47}$$

Where

$$\begin{aligned} \Gamma_\mu &= \frac{1}{2}\{u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\}, \\ r_\mu &= v_\mu + a_\mu, \\ l_\mu &= v_\mu - a_\mu. \end{aligned} \tag{3.48}$$

[2] Ouellete added some terms that Roessl apparently missed starting at third chiral order [19]

operator	P	C	h.c.
u_μ	$-\varepsilon(\mu)u_\mu$	u_μ^T	u_μ
$\chi_\pm$	$\pm\chi_\pm$	$\chi_\pm^T$	$\pm\chi_\pm$

Table 1: P, C and hermiticity properties of operators contained in chiral Lagrangians. $\varepsilon(0) = -\varepsilon(\mu \neq 0) = 1$

The mass term is introduced via the external field s, which in the isospin limit is equal to $s = \hat{m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The notation used for the basic building blocks is the one used by Bijnens, Colangelo and Ecker [20] and their transformation properties are listed in Table 1. These are the basic building blocks necessary to construct the Lagrangian up to second chiral order

- $u_\mu = i\{u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\}$
- $\chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u$

where

$$\begin{aligned}
&\chi = 2B(s + ip), \\
&\phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \\
&u = e^{i\phi/2F} \approx \mathbf{1} + \big(i\phi/2F\big) + \frac{1}{2}\big(i\phi/2F\big)^2 + \frac{1}{3!}\big(i\phi/2F\big)^3 + \cdots .
\end{aligned} \tag{3.49}$$

According to Roessl [18] the effective ChPT SU(2) Lagrangian contributing to the pion-kaon scattering up to second chiral order is given by

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}^{(2)}_{\pi\pi} + \mathcal{L}^{(1)}_{\pi K} + \mathcal{L}^{2}_{\pi K}, \\
\mathcal{L}^{(2)}_{\pi\pi} &= \frac{F^2}{4}(\langle u_\mu u^\mu\rangle + \langle\chi_+\rangle), \\
\mathcal{L}^{(1)}_{\pi K} &= \nabla_\mu K^\dagger \nabla^\mu K - \bar{M}_K^2 K^\dagger K, \\
\mathcal{L}^{(2)}_{\pi K} &= A_1\langle u_\mu u^\mu\rangle K^\dagger K + A_2\langle u^\mu u^\nu\rangle \nabla_\mu K^\dagger \nabla_\nu K + A_3 K^\dagger \chi_+ K + A_4\langle\chi_+\rangle K^\dagger K.
\end{aligned} \tag{3.50}$$

The chiral order associated with each class of terms corresponds to the chiral order of the leading tree-contributions and is indicated as an upper index (i). $\langle\rangle$ corresponds to trace and $\bar{M}_K^2$ corresponds to the kaon mass in the chiral limit and $M^2 = 2B\hat{m}$ is the lowest order pion mass.

4 Construction of the Effective Weak Lagrangian

We have to construct the effective Lagrangian that describes the weak decay of the kaon. The weak decay $K \to \pi\pi$ is a strangeness changing transition with $|\Delta S| = 1$ that occurs via the exchange of a W boson. The W boson is integrated out in the same way as the heavy quarks. Like in the strong case, we have to construct the most general effective Lagrangian with the same symmetries of the weak Lagragian in terms of the meson fields.

Nonleptonic decays are difficult to analayze theoretically. Important problems remain like the $\Delta I = 1/2$ rule and the prediction of ϵ'/ϵ, related to CP violation. In the framework of the short distance expansion formalism the $\Delta S = -1$ effective non-leptonic Hamiltonian is expressed as an expansion in terms of local operators, the most important ones are given by[21].

$$\bar{s}\gamma_\mu(1-\gamma^5)u \ \bar{u}\gamma^\mu(1-\gamma^5)d \qquad , \qquad \bar{s}\gamma_\mu(1-\gamma^5)d \ \bar{u}\gamma^\mu(1-\gamma^5)u. \tag{4.51}$$

This product contains two currents, one with isospin $I = 1$ and the other with $I = 1/2$. Coupling the isospin of the two currents we get a $I = 1/2$ and a $I = 3/2$

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{\frac{1}{2}} \oplus \mathbf{\frac{1}{2}} \oplus \mathbf{\frac{3}{2}}. \tag{4.52}$$

Thus the nonleptonic effective Lagrangian should transform under separate left-handed and right-handed chiral rotations as $(2_L, 1_R)$ and $(4_L, 1_R)$, i.e. we need an isospin 1/2 and a isospin 3/2 part.

We have to construct an effective Lagrangian with this symmetry properties using

$$\begin{aligned} u &\to h u g_L^\dagger, \\ K &\to hK, \end{aligned} \tag{4.53}$$

and the basic building blocks introduced in Section 3.2.1, $\chi_\pm$ and u_μ

4.1 Isospin 1/2 terms

To obtain the isospin 1/2 terms we need to add a spurion field $t^j_{1/2}$. Its called spurion because it does not contain any physical fields and is only included to obtain chiral invariant terms. Under chiral transformations the spurion should transform as $t^{j'}_{1/2} \to t^j_{1/2}(g_L^\dagger)_j^{\ j'}$, where g_L is an SU(2) matrix. Explicit values for $t^j_{1/2}$ are $t = (0 \ \ 1)$.

The reason for the upper and lower indices is that $\bar{u}$ and $\bar{d}$ are in the conjugate representation of SU(2), while u and d are in the fundamental representation of SU(2). The fundamental representation of SU(2) is a doublet

$$\begin{pmatrix} u \\ d \end{pmatrix}, \tag{4.54}$$

that transforms under SU(2) according to

$$\begin{pmatrix} u\,' \\ d\,' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}. \tag{4.55}$$

Taking the conjugate of this expression $u \to \bar{u}$, $d \to \bar{d}$, we can see that the transformation of the conjugate representation under SU(2) is,

$$\begin{pmatrix} \bar{u}\,' & \bar{d}\,' \end{pmatrix} = \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.56}$$

This implies that we have to distinguish between the transformation of the quark fields q in the fundamental representation from the one of fields on the conjugate representation.

$$\begin{aligned} {q_i}' &= q_j {(g_L)_i}^j, \\ \bar{q}\,'^{\,i} &= \bar{q}^j {(g_L^\dagger)_j}^i. \end{aligned} \tag{4.57}$$

The transformation of our basic elements is given by

$$\begin{aligned} {u_i}'^{\,j} &= {(h)_i}^s {u_s}^k {(g_L^\dagger)_k}^j, \\ {K_i}' &= {(h)_i}^j K_j, \\ {(u_\mu)_i}'^{\,j} &= {(h)_i}^m (u_\mu)_m^s {(h^\dagger)_s}^j, \\ (\chi_\pm)_i'^{\,j} &= {(h)_i}^m (\chi_\pm)_m^s {(h^\dagger)_s}^j. \end{aligned} \tag{4.58}$$

The isospin 1/2 terms up to second order in derivatives or first order in χ are

- $t_{1/2} u^\dagger K$
- $t_{1/2} u^\dagger u_\mu u^\mu K$
- $\langle u_\mu u^\mu \rangle t_{1/2} u^\dagger K$
- $t_{1/2} u^\dagger u^\mu \nabla_\mu K$
- $t_{1/2} u^\dagger \chi_+ K$
- $t_{1/2} u^\dagger \chi_- K$
- $\langle \chi_+ \rangle t_{1/2} u^\dagger K$
- $\langle \chi_- \rangle t_{1/2} u^\dagger K$

Some of this terms can be proven to be proportional to others using the Cayley-Hamilton theorem for square 2×2 matrices

$$\{A, B\} = A\langle B\rangle + B\langle A\rangle + \langle AB\rangle \mathbf{1} - \langle A\rangle\langle B\rangle \mathbf{1}. \tag{4.59}$$

For example $\langle \chi_+\rangle t_{1/2}u^\dagger K = 2t_{1/2}u^\dagger \chi_+ K$ and $t_{1/2}u^\dagger u^\mu \nabla_\mu K = 2\langle u_\mu u^\mu\rangle t_{1/2}u^\dagger K$ because $\langle \chi_+\rangle = 2\chi_+$ and $\langle u_\mu u^\mu\rangle = 2u_\mu u^\mu$. The term $\langle \chi_-\rangle t_{1/2}u^\dagger K$ vanishes because χ_- is a traceless matrix. As an example, lets prove that $\langle \chi_+\rangle = 2\chi_+$. In the isospin limit χ is diagonal, so that $\chi_+ = 2B\hat{m}({u^\dagger}^2 + u^2)$. Using (4.59) we can write

$$2\chi_+ = \langle \chi_+\rangle + 2B\hat{m}(2u\langle u\rangle + 2u^\dagger\langle u^\dagger\rangle - \langle u\rangle^2 - \langle u^\dagger\rangle^2). \tag{4.60}$$

Now, taking into account that $\langle u^\dagger\rangle = \langle u\rangle$ we obtain

$$2\chi_+ = \langle \chi_+\rangle + 4B\hat{m}\langle u\rangle(u + u^\dagger - \langle u\rangle). \tag{4.61}$$

Finally, by the definition of u, we have that $\langle u\rangle = u + u^\dagger$, which implies $2\chi_+ = \langle \chi_+\rangle$.

The $\Delta I = 1/2$ Lagrangian is then

$$\begin{aligned}\mathcal{L}_{1/2} =& iE_1 t_{1/2}u^\dagger K + E_2 t_{1/2}u^\dagger u^\mu \nabla_\mu K + iE_3\langle u_\mu u^\mu\rangle t_{1/2}u^\dagger K + iE_4 t_{1/2}u^\dagger \chi_+ K \\ &+ iE_5\langle \chi_+\rangle t_{1/2}u^\dagger K + E_6 t u^\dagger \chi_- K + E_7\langle \chi_-\rangle t_{1/2}u^\dagger K + \ldots + h.c.\end{aligned} \tag{4.62}$$

The i factors are chosen so that a real low energy constant E_i corresponds to a CP conserving term.

4.2 Isospin 3/2 terms

Coupling $u, \bar{u}$ and d to form an isospin $I = 3/2$ term with $I_z = 1/2$, we find

$$\frac{1}{\sqrt{2}}(d\bar{u}u + u\bar{u}d - d\bar{d}d), \tag{4.63}$$

so that we choose explicit components for the spurion t_z^{ij} to be $t_1^{21} = t_2^{12} = -t_2^{22} = 1$. To keep the invariance under chiral transformations, the spurion must transform according to ${t_k}^{ij} \to {t_{k'}}^{i'j'}(g_L^\dagger)_i{}^{i'}(g_L^\dagger)_j{}^{j'}(g_L)_{k'}^k$.

The isospin 3/2 terms up to second order in derivatives or first order in χ are

- ${\tau_i}^{jk}(u^\mu u_\mu)_k^i K_j$
- ${\tau_i}^{jk}(u_\mu)_j{}^i(u_\mu)_k^l K_l$
- ${\tau_i}^{jk}\nabla^\mu(u_\mu)_k{}^i K_j$
- ${\tau_i}^{jk}(\chi_+)_k{}^i K_j$
- ${\tau_i}^{jk}(\chi_-)_k{}^i K_j$

Here $\tau_i^{jk} = t_\beta^{\ \gamma\alpha}(u^\dagger)_\gamma^{\ j}(u^\dagger)_\alpha^{\ k}(u)_i^{\ \beta}$. A term like ${\tau_i}^{jk}(u^\mu u_\mu)^i_k K_j$ never contributes and ${\tau_i}^{jk}(\chi_+)_k^{\ i}K_j$ also vanishes in the isospin limit.

The $\Delta I = 3/2$ Lagrangian to second order in derivatives or first order in χ is then

$$\begin{aligned}\mathcal{L}_{3/2} =& iD_1{\tau_i}^{jk}(u_\mu)_j^{\ i}(u_\mu)^l_k K_l + D_2{\tau_i}^{jk}\nabla^\mu(u_\mu)_k^{\ i}K_j \\ &+ D_3{\tau_i}^{jk}(\chi_-)_k^{\ i}K_j + \ldots + h.c.\end{aligned} \tag{4.64}$$

4.3 On the Weak Lagrangian Ordering

Consider the momenta p_π of an external pion. Then $2\ p_K \cdot p_\pi = M_K^2$, so that the energy of the pion in the rest frame of the kaon $E_\pi \approx \frac{M_K^2}{2}$ is not small for $M_K^2 \gg M^2$. This implies that we cannot neglect operators with an arbitrary number of derivatives on external pion fields, thus complicating the whole power counting. Flynn and Sachrajda found a way to circumvent this problem in the study of K_{l3} decays based on the fact that chiral logarithms are caused by internal soft pion propagators, i.e. the momenta of internal pions is of $O(M)$. They argued that it is possible to calculate the chiral logarithms, i.e. the corrections of $O(M^2\log(M^2))$, via a redefinition of low energy constants [5]. An extension of that analysis is presented here for the nonleptonic kaon decay $K \to \pi\pi$.

Consider the case of $\mathcal{L}_{1/2}$ given in (4.62). We will show that all the matrix elements $\langle\pi(p_1)\pi(p_2)|O|K(p_K)\rangle$, where O is an operator of $\mathcal{L}_{1/2}$ are proportional to the lowest order ones up to terms of order M^2. The ordering is made in terms of an expansion in powers of M^2. The lowest order is 1, then M^2(plus logarithms), M^4, ...

A term with χ will not contribute to order 1 or $M^2\log(M^2)$ because it contains the factor $2B\hat{m} = M^2$. Thus, we have to examine terms with derivatives ∇_μ or u_μ, which always have to be in pairs by the Lorentz invariance condition on the Lagrangian. They can hit either a soft pion or a hard particle (kaon or an external pion). In the case where all derivatives are of the type ∇_μ, if one derivative hits a soft pion, the underlying soft part of the loop integrals is of the form $\int d^dp\ p_\mu/(p^2 - M^2)$, wich contributes no terms of order $M^2\log(M^2)$. The same occurs for u_μ, so that we only need to consider the case where the derivatives are acting on the hard particles. If both derivatives hit the same hard particle, they produce the mass of the particle which does not contain chiral logarithms. If they hit both external pions, we can perform a partial integration so that one derivative hits a hard pion and the other the kaon. However, because of the symmetry of $K(p_K) \to \pi(p_1)\pi(p_2)$ under the interchange of the pions, the pion momenta coming from the derivative on the hard pion must come in the form $p_1 + p_2 = p_K$, resulting in a kaon mass when contracted with the other derivative. We have shown that all terms with extra ∇_μ derivatives gives no contribution to the chiral logarithms.

We need to consider also the case of u_μ derivatives acting on the hard pions. We can prove that terms like $t_{1/2}u^\dagger u^\mu\nabla_\mu K$ and $\langle u_\mu u^\mu\rangle t_{1/2}u^\dagger K$ are proportional to the lowest order one $t_{1/2}u^\dagger K$ by using the identity 4.65, valid for terms that transform under the action of the chiral group like $\tilde{K} \to h\tilde{K}$.

$$\partial_\mu(t_{1/2}u^\dagger \tilde{K}) = \frac{1}{2}t_{1/2}u^\dagger u_\mu \tilde{K} + t_{1/2}u^\dagger \nabla_\mu \tilde{K}. \tag{4.65}$$

Using the fact that matrix elements of a total derivative vanish because of the conservation of momentum, we have that, putting $\tilde{K} = u^\mu K$ and $\tilde{K} = \nabla^\mu K$,

$$\begin{aligned} 0 &= \frac{1}{2}t_{1/2}u^\dagger u_\mu u^\mu K + t_{1/2}u^\dagger \nabla_\mu u^\mu K + t_{1/2}u^\dagger u_\mu \nabla^\mu K, \\ 0 &= \frac{1}{2}t_{1/2}u^\dagger u_\mu \nabla^\mu K + t_{1/2}u^\dagger \nabla_\mu \nabla^\mu K \end{aligned} \tag{4.66}$$

This shows that $t_{1/2}u^\dagger u^\mu \nabla_\mu K$ and $\langle u_\mu u^\mu \rangle t_{1/2}u^\dagger K$ are proportional to the lowest order one $t_{1/2}u^\dagger K$ plus terms of $O(M^2)$. Terms with more derivatives, like $\langle u_\mu u_\nu \rangle t_{1/2}u^\dagger \nabla^\mu \nabla^\nu K$, can also be removed via a partial integration. Performing a partial integration of the derivative ∇^μ we produce a term like $\langle \nabla_\mu u^\mu u_\nu \rangle t_{1/2}u^\dagger \nabla^\nu K$, which is of order M^2, or a term like $\langle u_\mu \nabla^\mu u_\nu \rangle t_{1/2}u^\dagger \nabla^\nu K$. By using the identity $\nabla_\mu u_\nu = \nabla_\nu u_\mu$, valid for zero external fields [20],we can show that this term is proportional to the lowest order one up to $O(M^2)$. We have shown that matrix elements of higher order operators are proportional to the lowest order operator up to terms of order M^2. The same type of arguments can be applied to the $\Delta I = 3/2$ operators.

This analysis is also expected to work for other processes involving hard momenta particles like B and D decays.

5 Amplitudes at Next-to-Leading Order

In this Section we present the results obtained for the decay amplitude of the CP conserving decays $K \to \pi\pi$ in the isospin limit at next-to-leading order in SU(2) ChPT. The CP conserving decays are

- $K_S \to \pi^0\pi^0$
- $K_S \to \pi^+\pi^-$
- $K^+ \to \pi^0\pi^+$

Where $K_S \simeq K_1^0 + \varepsilon K_2^0$, $K_{1(2)}^0 \equiv (K^0 - (+)\overline{K^0})/\sqrt{2}$, and CP $K_{1(2)}^0 = +(-)K_{1(2)}^0$. Since CP violation is small we set $\varepsilon = 0$. We can decompose the amplitudes for this decays in terms of the isospin invariant amplitudes A_0, A_2.

$$\begin{aligned} A[K_S \to \pi^0\pi^0] &= \sqrt{\frac{2}{3}}A_0 - \frac{2}{\sqrt{3}}A_2 \\ A[K_S \to \pi^+\pi^-] &= \sqrt{\frac{2}{3}}A_0 + \frac{2}{\sqrt{3}}A_2 \\ A[K^+ \to \pi^0\pi^+] &= \frac{\sqrt{3}}{2}A_2 \end{aligned} \tag{5.67}$$

5.1 Tree Level Contribution

The diagrams contributing at tree-level order are shown.

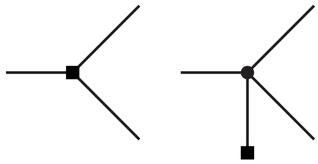


Figure 2: Diagrams contributing to $K \to \pi\pi$ at tree level. A black box indicates a vertex from the weak Lagrangian and a black circle represent a vertex from the strong Lagrangian

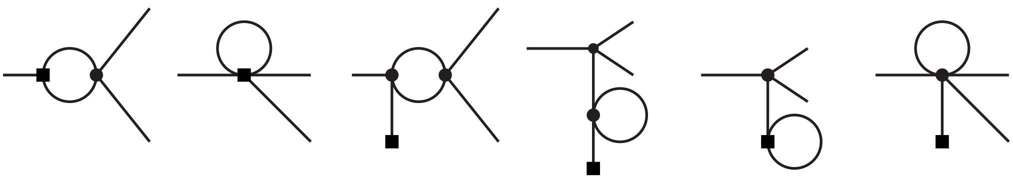
The leading order (LO) contribution to the decay amplitude, neglecting terms proportional to M^2, is given by

$$
\begin{aligned}
A_0^{LO} &= i\frac{\sqrt{3}}{2F^2}(-\frac{1}{2}E_1 + E_2\bar{M}_K^2 - 4E_3\bar{M}_K^2 + A_1E_1), \\
A_2^{LO} &= i\frac{\sqrt{3}}{\sqrt{2}F^2}(D_2 - 2D_1)\bar{M}_K^2. \qquad (5.68)
\end{aligned}
$$

The low energy constants E_i are defined in 4.62, the D_i are defined in 4.64 and A_1 in 3.50.

5.2 One-Loop Contribution

In this section we present the one-loop contribution to the decay $K \to \pi\pi$. Only pion loops are present, the role of closed kaon loops is examined by Roessl [18]. Kaon loops are taken into account by a redefinition of the low energy constants.



(a) (b) (c) (d) (e) (f)

Figure 3: Diagrams contributing to $K \to \pi\pi$ at one loop. A black box indicates a vertex from the weak Lagrangian and a black circle represent a vertex from the strong Lagrangian

The contribution of each diagram in Figure 3 is listed in Table 5.2. Diagrams (a) and (c) have $\pi\pi$ and $K\pi$ intermediate states. The contribution from the $K\pi$ intermediate states vanishes for both. Diagrams (c) and (d) have no contribution of order $M^2\log(M^2)$. Z denotes the contribution from the wave-function renormalization of the pion. There is no contribution to $O(M^2\log(M^2))$ from the kaon wave-function renormalization.

Diagram	A_0	A_2
Z	$-\frac{2F^2}{3}A_0^{LO}$	$-\frac{2F^2}{3}A_2^{LO}$
(a)	$\sqrt{3}i\left(-\frac{1}{3}E_1+\frac{2}{3}E_2\bar{M}_K^2\right)$	$\sqrt{\frac{3}{2}}i\left(-\frac{2}{3}D_2\bar{M}_K^2\right)$
(b)	$\sqrt{3}i\left(-\frac{5}{96}E_1-\frac{7}{48}E_2\bar{M}_K^2-\frac{25}{12}E_3\bar{M}_K^2\right)$	$\sqrt{\frac{3}{2}}i\left(-\frac{61}{12}D_1+\frac{77}{24}D_2\right)\bar{M}_K^2$
(e)	$\sqrt{3}i\frac{3}{16}A_1E_1$	
(f)	$\sqrt{3}i\left(\frac{1}{8}E_1+\frac{1}{3}A_1E_1\right)$	

Table 2: Coefficients of $\bar{A}(M^2)/F^4$ in the contributions to the isospin amplitudes A_0 and A_2. Z denotes the wave-function renormalization

The total amplitude at next-to-leading order (NLO) is given by

$$\begin{aligned}
A_0^{NLO} &= A_0^{LO}+\frac{\sqrt{3}i}{16\pi^2F^4}\bar{M}_K^2M^2\log(M^2)\left(\frac{3}{16}\mathrm{E}_2-\frac{3}{4}\mathrm{E}_3\right)\\
&+ \frac{\sqrt{3}i}{16\pi^2F^4}M^2\log(M^2)\left(-\frac{3}{32}\mathrm{E}_1+\frac{3}{16}\mathrm{E}_1\mathrm{A}_1\right)+c_1M^2+O(M^4),\\
A_2^{NLO} &= A_2^{LO}+\frac{\sqrt{3}i}{16\sqrt{2}\pi^2F^4}M^2\bar{M}_K^2\log(M^2)\left(\frac{15}{8}\mathrm{D}_2-\frac{15}{4}\mathrm{D}_1\right)+c_2M^2+O(M^4).
\end{aligned} \tag{5.69}$$

Where c_1 and c_2 are unknown constants. The arguments of Section 4.3 can be directly checked in 5.69, higher order operators are proportional to the lowest order operators up to terms of $O(M^2)$. We can rewrite 5.69, making explicit that the chiral logarithms are unique

$$\begin{aligned}
A_0^{NLO} &= A_0^{LO}\left(1+\frac{3}{8F^2}\bar{A}(M^2)\right)+c_1M^2+O(M^4),\\
A_2^{NLO} &= A_2^{LO}\left(1+\frac{15}{8F^2}\bar{A}(M^2)\right)+c_2M^2+O(M^4).
\end{aligned} \tag{5.70}$$

The function $\bar{A}(M^2)$ is given in (A.8). (5.69) includes the wave-function renormalization. As mentioned in [5], only the pion wave-function renormalization contributes to order $M^2\log(M^2)$.

6 Matching with Chiral SU(3)

The chiral logarithms have been calculated in the SU(3) theory in Refs.[2, 3, 22] . We can check our calculations by converting the SU(3) results to SU(2). For this purpose we compare the amplitudes from SU(3) and SU(2) ChPT in the limit $M^2 \to 0$. We work with the SU(3) expressions for the isospin amplitudes A_0 and A_2 given in [22]. The lowest order SU(3) result in the limit $M^2 \to 0$ is given by

$$\begin{aligned} A_0^{LO} &= -\frac{i\sqrt{6}CF_0^4}{\bar{F}_K F^2}\left(G_8 + \frac{1}{9}G_{27}\right)\bar{M}_K^2, \\ A_2^{LO} &= -\frac{i10\sqrt{3}CF_0^4}{9\bar{F}_K F^2}G_{27}\bar{M}_K^2. \end{aligned} \tag{6.71}$$

By comparing 6.71 with the SU(2) leading order result 5.68, we can obtain the SU(3) LECs in terms of the SU(2) LECs. We confirmed that the total one-loop chiral logarithms in (5.69) are indeed correct by rewriting the SU(3) low energy constants in the SU(3) one-loop result in terms of the SU(2) low energy constants. For this we have to take into account that the pion and kaon decay constants F_π and F_K contribute to the chiral logarithms

$$\frac{1}{F_K F_\pi^2} = \frac{1}{\bar{F}_K F^2}\left(1 + \frac{19}{4}\mu_\pi + O(M^2)\right). \tag{6.72}$$

$\bar{F}_K$ and F correspond to the respective quantities in the SU(2) chiral limit, $m_u = m_d = 0$, and $\mu_\pi = \frac{1}{32\pi^2}\frac{M^2}{F_0^2}\log(\frac{M^2}{\mu^2})$

7 Conclusions and Prospects

This work extends the treatment of the semileptonic decay K_{l3} given by Flynn and Sachrajda [5] to the nonleptonic decay $K \to \pi\pi$. Regardless of the complications in the power counting introduced by the presence of hard pions, we provide arguments about the possibility of calculating the pion chiral logarithms. The SU(2) chiral logarithms were calculated and then checked by comparing them with the SU(3) ChPT results. The main result of the thesis is 5.70. All the calculations were performed in the isospin limit, including isospin breaking effects is a direct extension of the present work. In a more general perspective, it should be possible to realize similar analyzes to B and D decays.

8 Acknowledgments

I would like to thank Johan Bijnens because working under his supervision I learned a lot, not only about physics but also about being a physicist. I also want to thank the people from the Department of Theoretical Physics of Lund University for the hospitality, especially Ilaria Jemos and Lisa Carloni for helpful discussions and Torbjörn Sjöstrand for many corrections to the original version.

A Loop Integrals

In this Appendix we collect for completeness some familiar formulas for the one-loop integrals. The method used to reduce the integrals is the one form Passarino and Veltman [23]. In the remainder we use $d = 4 - 2\epsilon$.

$$
\begin{aligned}
A(m_1^2) &= \frac{1}{i}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2 - m_1^2}\,, \\
B(m_1^2, m_2^2, p^2) &= \frac{1}{i}\int \frac{d^dq}{(2\pi)^d}\frac{1}{(q^2 - m_1^2)((q-p)^2 - m_2^2)}\,, \\
B_\mu(m_1^2, m_2^2, p^2) &= \frac{1}{i}\int \frac{d^dq}{(2\pi)^d}\frac{q_\mu}{(q^2 - m_1^2)((q-p)^2 - m_2^2)} \\
&= p_\mu B_1(m_1^2, m_2^2, p^2)\,, \\
B_{\mu\nu}(m_1^2, m_2^2, p^2) &= \frac{1}{i}\int \frac{d^dq}{(2\pi)^d}\frac{q_\mu q_\nu}{(q^2 - m_1^2)((q-p)^2 - m_2^2)} \\
&= p_\mu p_\nu B_{21}(m_1^2, m_2^2, p^2) + g_{\mu\nu} B_{22}(m_1^2, m_2^2, p^2)\,, \\
B_{\mu\nu\alpha}(m_1^2, m_2^2, p^2) &= \frac{1}{i}\int \frac{d^dq}{(2\pi)^d}\frac{q_\mu q_\nu q_\alpha}{(q^2 - m_1^2)((q-p)^2 - m_2^2)} \\
&= p_\mu p_\nu p_\alpha B_{31}(m_1^2, m_2^2, p^2) + (p_\mu g_{\nu\alpha} + p_\nu g_{\mu\alpha} + p_\alpha g_{\mu\nu}) B_{32}(m_1^2, m_2^2, p^2)\,.
\end{aligned}
\tag{A.1}
$$

An expansion in ϵ leads to the following series

$$
\begin{aligned}
A(m_1^2) &= \frac{m_1^2}{16\pi^2}\lambda_0 + \overline{A}(m_1^2) + \epsilon \overline{A}^\epsilon(m_1^2) + \dots\,, \\
B_{ij}(m_1^2, m_2^2, p^2) &= \frac{1}{16\pi^2}\mathrm{pole}_{ij} + \overline{B}_{ij}(m_1^2, m_2^2, p^2) + \epsilon \overline{B}^\epsilon_{ij}(m_1^2, m_2^2, p^2) + \dots\,,
\end{aligned}
\tag{A.2}
$$

with $\overline{A}$, $\overline{B}_{ij}$ defining finite quantities and where "pole_{ij}" denotes the singular part of each of the B_{ij} functions,

$$
\begin{aligned}
&\mathrm{pole} = \lambda_0\,, \quad \mathrm{pole}_1 = \frac{\lambda_0}{2}\,, \quad \mathrm{pole}_{21} = \frac{\lambda_0}{3}\,, \quad \mathrm{pole}_{22} = \frac{\lambda_0}{4}(m_1^2 + m_2^2 - \frac{p^2}{3})\,, \\
&\mathrm{pole}_{31} = \frac{\lambda_0}{4}\,, \quad \mathrm{pole}_{32} = \frac{\lambda_0}{24}(2m_1 + 4m_2^2 - p^2)\,,
\end{aligned}
\tag{A.3}
$$

with

$$
\lambda_0 = \frac{1}{\epsilon} + \ln(4\pi) + 1 - \gamma\,. \tag{A.4}
$$

The functions defined in (A.1) can be related to the basic integrals $A(m_1^2)$ and

$B_1(m_1^2, m_2^2, p^2)$ through the identities

$$
\begin{aligned}
B_{31}(m_1^2, m_2^2, p^2) &= \frac{1}{2p^2}\Big(A(m_2^2) - (m_2^2 - m_1^2 - p^2)B_{21}(m_1^2, m_2^2, p^2) - 4B_{32}(m_1^2, m_2^2, p^2)\Big)\,, \\
B_{32}(m_1^2, m_2^2, p^2) &= \frac{1}{2p^2}\Big(-\frac{m_1^2}{d}A(m_1^2) + \frac{m_2^2}{d}A(m_2^2) - (m_2^2 - m_1^2 - p^2)B_{22}(m_1^2, m_2^2, p^2)\Big)\,, \\
B_{21}(m_1^2, m_2^2, p^2) &= \frac{1}{p^2}\Big(A(m_2^2) + m_1^2 B(m_1^2, m_2^2, p^2) - dB_{22}(m_1^2, m_2^2, p^2)\Big)\,, \\
B_{22}(m_1^2, m_2^2, p^2) &= \frac{1}{2(d-1)}\Big(A(m_2^2) + 2m_1^2 B(m_1^2, m_2^2, p^2) \\
&\quad -(p^2 + m_1^2 - m_2^2)B_1(m_1^2, m_2^2, p^2)\Big)\,, \qquad \text{(A.5)} \\
B_1(m_1^2, m_2^2, p^2) &= \frac{1}{2p^2}\Big(A(m_2^2) - A(m_1^2) + (m_1^2 - m_2^2 + p^2)B(m_1^2, m_2^2, p^2)\Big). \qquad \text{(A.6)}
\end{aligned}
$$

The finite part of the function B, $\bar{B}$, is expanded in powers of $\frac{M^2}{\bar{M}_K^2}$ according to

$$
\begin{aligned}
\bar{B}(M^2, M^2, \bar{M}_K^2) &= \frac{1}{16\pi^2}\Big[1 + i\pi - \log(\frac{\bar{M}_K^2}{\mu^2}) + \frac{M^2}{\bar{M}_K^2}\left(2 - 2i\pi - 2\log(\frac{M^2}{\bar{M}_K^2})\right) + \\
&+ \frac{M^4}{\bar{M}_K^4}\left(-1 - 2i\pi - 2\log(\frac{M^2}{\bar{M}_K^2})\right)\Big]\,, \\
\bar{B}(M^2, \bar{M}_K^2, M^2) &= \frac{1}{16\pi^2}\Big[-\log(\frac{\bar{M}_K^2}{\mu^2}) + \frac{M^2}{\bar{M}_K^2}\left(\frac{1}{2} + \log(\frac{M^2}{\bar{M}_K^2})\right) + \\
&+ \frac{M^4}{\bar{M}_K^4}\left(\frac{5}{3} + 2\log(\frac{M^2}{\bar{M}_K^2})\right) + \frac{M^6}{\bar{M}_K^6}\left(\frac{59}{12} + 5\log(\frac{M^2}{\bar{M}_K^2})\right)\Big]. \qquad \text{(A.7)}
\end{aligned}
$$

While the finite part of A is given by

$$
\bar{A}(m^2) = -\frac{m^2}{16\pi^2}\log(\frac{m^2}{\mu^2}) \qquad \text{(A.8)}
$$

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