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Relations at Order p^6 in Chiral Perturbation Theory

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Abstract

We report on a search of relations valid at order p^6 in Chiral Perturbation Theory. We have found relations between $\pi\pi$, πK scattering, $K_{\ell 4}$ decays, masses and decay constants and scalar and vector form factors. In this paper we give the relations and a first numerical check of them.

PACS: 12.39.Fe Chiral Lagrangians, 11.30.Rd Chiral symmetries , 14.40.Aq π, K, and η mesons , 12.38.Lg Other nonperturbative calculations,

1 Introduction

Chiral Perturbation Theory (ChPT) [1, 2, 3] is the effective field theory for the strong interaction at low energies. Some recent reviews are [4, 5, 6]. In the mesonic sector many calculations have now been performed to two-loop or next-to-next-to-leading order (NNLO), see the review [5]. Since in an effective field theory like ChPT there appear new Lagrangians at every order, tests of ChPT at NNLO are difficult to perform since for most processes new combinations of these parameters, called low-energy constants (LECs), appear.

One way to test ChPT at NNLO order is to find observables where the same combinations of LECs appear. Many of these pairs of observables were found in the explicit calculations but no systematic study had been done. That is the purpose of this work. We take systematically all observables that can contain a dependence on the NNLO LECs in $\pi\pi$ and πK scattering, the masses and decay constants, $\eta \to 3\pi$, $K_{\ell 4}$ and the scalar and vector formfactors and determine how many of these contain the same combinations of NNLO LECs. Of the 76 observables we include, we find 35 such combinations. These are discussed in Sects. 3 to 12. These allow in principle to test the validity of three flavour ChPT at NNLO. However, many relations involve poorly known quantities from the scalar formfactors so we have restricted the numerical discussion to the $\pi\pi$, πK and $K_{\ell 4}$ sector. The tests in the vector formfactors were already discussed extensively in the earlier work, so we do not present numerical results for those either. We find a mixed picture. Three flavour ChPT mostly works but there are problems. Some preliminary results were presented in [7].

We first discuss the $\pi\pi$ threshold parameters relations in both two and three flavour ChPT in Sect. 3. After that we restrict ourselves to three flavour ChPT, first πK threshold parameters relations in Sect. 4 and the relations between both sectors in Sect. 5. Sect. 6 discusses the relation between $K_{\ell 4}$ and πK scattering. For $\eta \to 3\pi$ we find relations involving the cubic dependence of the Dalitz plot in Sect. 7. For the scalar formfactors we find the known relations and one new one, Sect. 8, but when relating the scalar sector to other sectors we find several new relations as discussed in Sects. 9, 10, 11 and 12. We shortly recapitulate our conclusions in Sect. 13. For completeness, we have added in an appendix the correspondence between the subthreshold and threshold parameters for $\pi\pi$ and πK scattering.

2 Notation

The Lagrangian at NNLO contains 90 LECs, called the C_i in [8, 9]. Since in this work we check whether the same combinations of LECs appear we use the notation

$$[A]_{C_i} = C_i^r\text{-dependent part of } A\,. \tag{1}$$

We also use the notation B_0 and F_0, the chiral limit of F_π, the two constants that appear in the lowest order Lagrangian [3]. For the physical observables we use for each case the established notation.

We always express dimensionful quantities in the appropriate units of m_{π^+}, which is in any case standard practice for many of the quantities we consider. We use the symbol $\rho = m_K/m_\pi$ to indicate the kaon mass. This way the relations are easier to write down.

3 $\pi\pi$ scattering

The $\pi\pi$ scattering amplitude can be written as a function $A(s,t,u)$ which is symmetric in the last two arguments:

$$A(\pi^a\pi^b \to \pi^c\pi^d) = \delta^{a,b}\delta^{c,d}A(s,t,u) + \delta^{c,d}\delta^{b,d}A(t,u,s) + \delta^{a,d}\delta^{b,c}A(u,t,s)\,, \qquad (2)$$

where s,t,u are the usual Mandelstam variables. The isospin amplitudes $T^I(s,t)$ ($I = 0,1,2$) are

$$\begin{aligned} T^0(s,t) &= 3A(s,t,u) + A(t,u,s) + A(u,s,t)\,, \\ T^1(s,t) &= A(s,t,u) - A(u,s,t)\,, \\ T^2(s,t) &= A(t,u,s) + A(u,s,t)\,, \end{aligned} \qquad (3)$$

and can be expanded in partial waves

$$T^I(s,t) = 32\pi \sum_{\ell=0}^{+\infty} (2\ell+1) P_\ell(\cos\theta) t^I_\ell(s), \qquad (4)$$

where t and u have been written as $t = -\frac{1}{2}(s-4m_\pi^2)(1-\cos\theta)$, $u = -\frac{1}{2}(s-4m_\pi^2)(1+\cos\theta)$. Near threshold the t^I_ℓ are further expanded in terms of the threshold parameters

$$t^I_\ell(s) = q^{2\ell}\left(a^I_\ell + b^I_\ell q^2 + c^I_\ell q^4 + d^I_\ell q^6 + \mathcal{O}(q^8)\right) \quad q^2 = \frac{1}{4}(s-4m_\pi^2), \qquad (5)$$

where $a^I_\ell, b^I_\ell \dots$ are the scattering lengths, slopes,... and q is the magnitude of the pion three momenta in the center of mass frame. We studied only those observables where a dependence on the C_is shows up. Using $s+t+u = 4m_\pi^2$ we can write the amplitude to order p^6 as

$$A(s,t,u) = b_1 + b_2 s + b_3 s^2 + b_4 (t-u)^2 + b_5 s^3 + b_6 s(t-u)^2 + \text{non polynomial part} \qquad (6)$$

The tree level Feynman diagrams give polynomial contributions to $A(s,t,u)$ which must be expressible in terms of $b_1,\dots,b_6$.

The threshold parameters a^0_0, b^0_0, c^0_0, d^0_0, a^2_0, b^2_0, c^2_0, d^2_0, a^1_1, b^1_1, c^1_1, a^0_2, b^0_2, a^2_2, b^2_2, a^1_3 are all those that can receive contributions from tree level LECs up to order p^6, but results [10] have only been presented for a^0_0, b^0_0, a^2_0, b^2_0, a^1_1, b^1_1, a^0_2, b^0_2, a^2_2, b^2_2 and a^1_3. At present we thus can only use those 11 to test ChPT. We do not consider b^1_3 for which numerical results are also given in [10] since it does not depend at tree level on any LECs to order p^6. For those 11 we obtain the following five relations:

$$\begin{aligned}
\left[5b_0^2 - 2b_0^0 - 27a_1^1 - 15a_0^2 + 6a_0^0\right]_{C_i} &= -18\left[b_1^1\right]_{C_i}, &(7)\\
\left[3a_1^1 + b_0^2\right]_{C_i} &= 20\left[b_2^2 - b_2^0 - a_2^2 + a_2^0\right]_{C_i}, &(8)\\
\left[b_0^0 + 5b_0^2 + 9a_1^1\right]_{C_i} &= 90\left[a_2^0 - b_2^0\right]_{C_i}, &(9)\\
\left[3b_1^1 + 25a_2^2\right]_{C_i} &= 10\left[a_2^0\right]_{C_i}, &(10)\\
\left[-5b_2^2 + 2b_2^0\right]_{C_i} &= 21\left[a_3^1\right]_{C_i}, &(11)
\end{aligned}$$

All quantities are expressed in units of $m_{\pi^+}^2$. In fact, since these relations hold for every contribution to the polynomial part, they are valid for the NLO tree level contribution as well and for two- and three-flavour ChPT. Therefore they do not get contributions from the L_is at NLO, but only at NNLO via the non polynomial part of Eq. (6).

The first three involve quantities that already have tree level contributions at lowest order, the fourth starts with tree level at NLO and the last only has tree level contributions starting at NNLO. The terms in the first three are arranged such that the quantities starting at lowest order are all on the left-hand-side.

Let us now look at the numerical results. As experimental input we use the Roy equation analysis together with input from ChPT and the pion scalar form-factor done in [10]. In Tab. 1 we quote the left-hand-side (LHS) and right-hand-side (RHS) of each of the relations with the threshold parameters as quoted in [10]. We have added the errors for the several quantities quadratically which probably results in an underestimate of the error. The results are quoted in the second column of Tab. 1. The next columns give the contribution from pure one-loop at NLO, the tree level NLO contribution at one-loop using the fitted values of fit 10 in [11], the pure two-loop contribution, and the L_i dependent part at NNLO (called NNLO 1-loop) using again fit 10 of [11]. Of these the tree level NLO contribution must satisfy the relations, the others need not. The numerical results have been calculated using the formulas of [12]. The column labeled remainder is the result of [10] minus the three-flavour ChPT prediction. This is thus the contribution of the NNLO LECs and from higher orders.

The theoretical errors are more difficult to estimate. The error shown in the sixth column in brackets in Tab. 1 is obtained by varying all the L_i^r around the central values of fit 10 of [11] exploring the region with $\chi^2/dof \approx 1$ using the full covariance matrix as obtained for that fit by the authors of [11]. The error is then estimated as the maximum deviation observed. The error for the L_i^r contribution at NLO is not shown since it drops out of the relations.

As we see, the first three relations are very well satisfied. The last two work at a level around two sigma. Uncertainties on the theoretical results are mostly on the last quoted digit, no uncertainty due to fit 10 is included. Note that the $\pi\pi$ threshold parameters were not used as input in fit 10.

We can also check how the two-flavour predictions hold up. Here the expansion parameter is different. The corrections are in powers of m_π^2 rather than in powers of m_K^2.

	[10]	NLO 1-loop	NLO LECs	NNLO 2-loop	NNLO 1-loop	remainder
LHS (7)	0.009 ± 0.039	0.054	-0.044	-0.041	$-0.002(3)$	0.041 ± 0.039
RHS (7)	-0.102 ± 0.002	-0.009	-0.044	-0.060	$-0.008(6)$	0.018 ± 0.002
10 LHS (8)	0.334 ± 0.019	0.209	0.097	0.103	0.029(11)	-0.105 ± 0.019
10 RHS (8)	0.322 ± 0.008	0.177	0.097	0.120	0.034(13)	-0.107 ± 0.008
LHS (9)	0.216 ± 0.010	0.166	0.029	0.053	0.016(6)	-0.047 ± 0.010
RHS (9)	0.189 ± 0.003	0.145	0.029	0.049	0.020(7)	-0.054 ± 0.003
10 LHS (10)	0.213 ± 0.005	0.137	0.032	0.053	0.035(12)	-0.043 ± 0.005
10 RHS (10)	0.175 ± 0.003	0.121	0.032	0.050	0.029(10)	-0.057 ± 0.003
10^3 LHS (11)	0.92 ± 0.07	0.36	0.00	0.56	$-0.01(13)$	0.00 ± 0.07
10^3 RHS (11)	1.18 ± 0.04	0.42	0.00	0.57	0.03(13)	0.15 ± 0.04

Table 1: The relations found in the $\pi\pi$-scattering. The lowest order contribution is always zero by construction. The NLO LEC part satisfies the relation. Notice the extra factors of ten for some of them. All quantities are in the units of powers of m_{π^+}. See text for a longer discussion.

The expansion should thus converge better and the conclusion was drawn in [10] that two-flavour ChPT works for $\pi\pi$-scattering at threshold (and even better where they performed their subtractions). We do not use the numbers quoted in [13, 21] since the LECs used there have been superceded by those of [10] and [11] respectively. Testing our relations for two-flavour ChPT thus gives a good indication of the best results we can expect for the three-flavour case. We use the threshold parameters as quoted in [10] for their best fit of the NLO LECs and using the formulas of [13]. The result is shown in Tab. 2. We see the same pattern as for the three flavour case. The first three relations are very well satisfied while the last two are somewhat worse but here below two sigma.

An alternative way to look at the results is to directly test the relations. In the previous tables we have presented results separately for the LHS and RHS in order to show how well the combinations of the NNLO LECs would be the same if determined in the two different ways. We can also instead show LHS minus RHS for our relations which directly tests the loop content of ChPT. For the $\pi\pi$ and πK case this is equivalent to comparing the exact results for the dispersive part with the ChPT result for the dispersive part since the subtraction constants used in [10] drop out in the relations we consider[1]. This is shown in Tab. 3. We see here also good agreement for the first three and about two sigma for the last two relations. The results given in Tab. 3 are depicted graphically in Fig. 1. Keep in mind here that the errors for the dispersive result might be underestimated since we combined them quadratically.

[1]We thank the referee for pointing this out.

	[10]	two-flavour [10]	remainder
LHS (7)	0.009 ± 0.039	-0.003	0.007 ± 0.039
RHS (7)	-0.102 ± 0.002	-0.097	-0.005 ± 0.002
10 LHS (8)	0.334 ± 0.019	0.332	0.002 ± 0.019
10 RHS (8)	0.322 ± 0.008	0.318	0.004 ± 0.075
LHS (9)	0.216 ± 0.010	0.206	0.010 ± 0.010
RHS (9)	0.189 ± 0.003	0.189	0.000 ± 0.003
10 LHS (10)	0.213 ± 0.005	0.204	0.009 ± 0.005
10 RHS (10)	0.175 ± 0.003	0.176	-0.001 ± 0.003
10^3 LHS (11)	0.92 ± 0.07	1.00	-0.08 ± 0.07
10^3 RHS (11)	1.18 ± 0.04	1.15	0.04 ± 0.04

Table 2: The relations found in the $\pi\pi$-scattering evaluated in two-flavour ChPT. In the second column we have used the NNLO results quoted in [10]. Notice the extra factors of ten for some of them. See text for a longer discussion.

	disp/exp	NLO	NLO+NNLO	NLO+NNLO(2)
LR (7)	0.111 ± 0.039	0.062	0.087(3)	0.094
10 LR (8)	0.012 ± 0.021	0.031	0.010(2)	0.014
LR (9)	0.026 ± 0.011	0.021	0.020(3)	0.017
10 LR (10)	0.038 ± 0.006	0.016	0.024(2)	0.028
10^3 LR (11)	-0.26 ± 0.08	-0.06	$-0.11(2)$	-0.14
LR (23)	-1.5 ± 0.7	-0.26	$-0.34(7)$	-
10 LR (21)	-0.05 ± 0.02	0.02	0.03(5)	-
100 LR (24)	0.36 ± 0.60	0.06	$-0.13(13)$	-
100 LR (22)	0.12 ± 0.01	0.03	0.06(1)	-
10^3 LR (26)	-0.03 ± 0.08	0.07	0.03(2)	-
10^3 LR (28)	-0.04 ± 0.03	0.00	0.08(5)	-
10 LR (29)	-0.04 ± 0.02	-0.06	$-0.07(2)$	-
LR (33)	-1.24 ± 0.11	-0.41	$-0.74(10)$	-

Table 3: Tests of the relations as seen as a test of the loop contributions. disp/exp are the dispersive and experimental inputs used as described in the text. LR stands for LHS-RHS. All quantities are in units of m_{π^+}. Results are shown for the relations for $\pi\pi$, πK, $\pi\pi$ vs πK and $K_{\ell 4}$ vs πK.

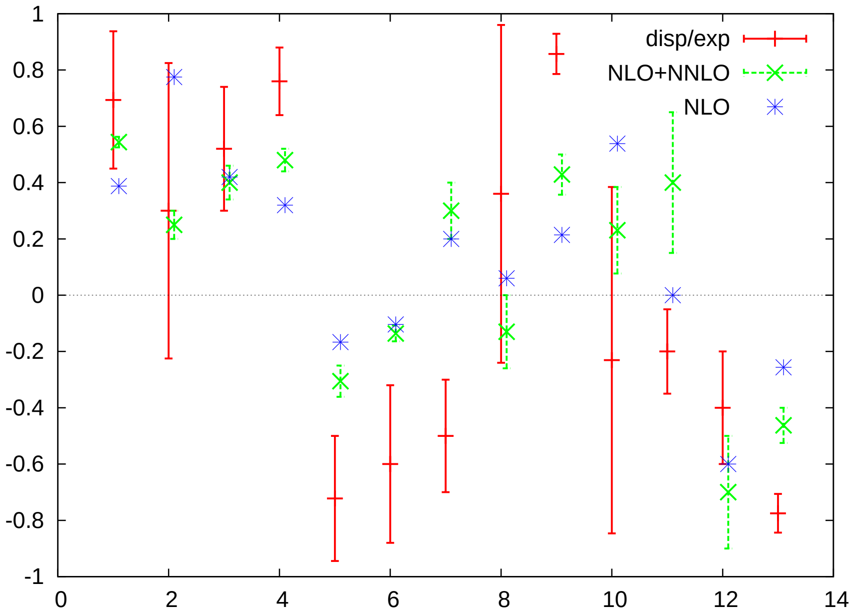

Figure 1: The relations with the dispersive/experimental results shown as full lines with errors, the NLO result as stars and the sum of NLO+NNLO as crosses with the errors indicated as dashed lines. The scale is arbitrary. The relations appear in the order given in Tab. 3. 1-5 $\pi\pi$, 6-10 πK, 11-12 $\pi\pi$ vs πK and 13 $K_{\ell 4}$ vs πK.

4 πK scattering

The πK scattering amplitude has amplitudes $T^I(s,t,u)$ in the isospin channels $I = 1/2, 3/2$. As for $\pi\pi$ scattering, it is possible to define scattering lengths a^I_ℓ, b^I_ℓ. So we introduce the partial wave expansion of the isospin amplitudes

$$T^I(s,t,u) = 16\pi \sum_{\ell=0}^{+\infty} (2\ell+1) P_\ell(\cos\theta) t^I_\ell(s), \tag{12}$$

and we expand the $t^I_\ell(s)$ near threshold:

$$t^I_\ell(s) = \frac{1}{2}\sqrt{s} q^{2\ell}_{\pi K} \left(a^I_\ell + b^I_\ell q^2_{\pi K} + c^I_\ell q^4_{\pi K} + \mathcal{O}(q^6_{\pi K}) \right), \tag{13}$$

where

$$q^2_{\pi K} = \frac{s}{4}\left(1 - \frac{(m_K + m_\pi)^2}{s}\right)\left(1 - \frac{(m_K - m_\pi)^2}{s}\right), \tag{14}$$

is the magnitude of the three-momentum in the center of mass system. The Mandelstam variables are in terms of the scattering angle given by

$$t = -2q^2_{\pi K}(1 - \cos\theta), \quad u = -s - t + 2m^2_K + 2m^2_\pi \,. \tag{15}$$

Again we studied only those observables where a dependence on the C_is shows up.

It is also customary to introduce the crossing symmetric and antisymmetric amplitudes $T^\pm(s,t,u)$ which can be expanded around $t = 0$, $s = u$ using $\nu = (s-u)/(4m_K)$, called the subthreshold expansion:

$$T^+(s,t,u) = \sum_{i,j=0}^{\infty} c^+_{ij} t^i \nu^{2j}, \qquad T^-(s,t,u) = \sum_{i,j=0}^{\infty} c^-_{ij} t^i \nu^{2j+1}. \tag{16}$$

There are ten subthreshold parameters that have tree level contributions from the NNLO LECs. In c^-_{01} and c^-_{20} the same combination $-C_1 + 2C_3 + 2C_4$ appears [12], thus

$$16\rho^2 \left[c^-_{20}\right]_{C_i} = 3\left[c^-_{01}\right]_{C_i} \,. \tag{17}$$

Eq. (17) leads to one relation between the subthreshold parameters.

If we look at the a^I_ℓ and b^I_ℓ that get contributions from the NNLO LECs there are 14 such. 7 for each isospin channel. The isospin odd channel only involves T^-:

$$T^{1/2}(s,t,u) - T^{3/2}(s,t,u) = 3T^-(s,t,u)\,. \tag{18}$$

This combination has only three subtreshold parameters that get independent contributions from the NNLO LECs. So for 7 differences of a^I_ℓ and b^I_ℓ and three parameters we expect four relations. The threshold parameters are expressed in units of m_{π^+} and we use the symbol $\rho = m_K/m_\pi$. We use the notation $a_\ell^- = a_\ell^{1/2} - a_\ell^{3/2}$ and $b_\ell^- = b_\ell^{1/2} - b_\ell^{3/2}$

$$
\begin{aligned}
70\rho^3(\rho+1)^2\left[a_3^-\right]_{C_i} &= -\left(\rho^2+\rho+1\right)\left[a_0^-\right]_{C_i} + 2\rho^2\left[b_0^-\right]_{C_i} + 6\rho^2\left[a_1^-\right]_{C_i}\,, &(19)\\
140\rho^3\left(\rho^2+1\right)\left[a_3^-\right]_{C_i} &= \left(\rho^2+1\right)\left[a_0^-\right]_{C_i} + 6\left(-\rho^2+\rho-1\right)\rho\left[a_1^-\right]_{C_i} + 12\rho^3\left[b_1^-\right]_{C_i}\,, &(20)\\
5\left(\rho^2+1\right)\left[a_2^-\right]_{C_i} &= \left[a_1^-\right]_{C_i} + 2\rho\left[b_1^-\right]_{C_i}\,, &(21)\\
7\left(\rho^2+1\right)\left[a_3^-\right]_{C_i} &= \left[a_2^-\right]_{C_i} + 2\rho\left[b_2^-\right]_{C_i}\,. &(22)
\end{aligned}
$$

We can eliminate $\left[a_3^-\right]_{C_i}$ from (19) and (20) to obtain a relation involving only $\ell = 0, 1$ threshold parameters:

$$
\begin{aligned}
\left(\rho^4+3\rho^3+3\rho+1\right)\left[a_1^-\right]_{C_i} &= 2\rho^2(\rho+1)^2\left[b_1^-\right]_{C_i} - \frac{2}{3}\rho\left(\rho^2+1\right)\left[b_0^-\right]_{C_i}\\
&\quad + \frac{1}{2\rho}\left(\rho^2+\frac{4}{3}\rho+1\right)\left(\rho^2+1\right)\left[a_0^-\right]_{C_i}\,. \qquad (23)
\end{aligned}
$$

We prefer to express the other relation in one involving b_2^-

$$
5(\rho+1)^2\left[b_2^-\right]_{C_i} = \frac{(\rho-1)^2}{\rho^2}\left[a_1^-\right]_{C_i} - \frac{\rho^4+\frac{2}{3}\rho^2+1}{4\rho^4}\left[a_0^-\right]_{C_i} + \frac{\rho^2-\frac{2}{3}\rho+1}{2\rho^2}\left[b_0^-\right]_{C_i}\,. \qquad (24)
$$

The combination that involves only T^+ is

$$
T^{1/2}(s,t,u) + 2T^{3/2}(s,t,u) = 3T^+(s,t,u)\,. \qquad (25)
$$

This brings in 7 more threshold parameters, but there are 6 fully independent subtreshold parameters so we expect only one more relation. Using the notation $a_\ell^+ = a_\ell^{1/2} + 2a_\ell^{3/2}$ and $b_\ell^+ = b_\ell^{1/2} + 2b_\ell^{3/2}$, we find:

$$
7\left[a_3^+\right]_{C_i} = \frac{1}{2\rho}\left[a_2^+\right]_{C_i} - \left[b_2^+\right]_{C_i} + \frac{1}{5\rho}\left[b_1^+\right]_{C_i} - \frac{1}{60\rho^3}\left[a_0^+\right]_{C_i} - \frac{1}{30\rho^2}\left[b_0^+\right]_{C_i}\,. \qquad (26)
$$

These relations hold for all tree-level contributions up to NNLO[2]. In particular, the lowest order contributions satisfy them.

Note that because of the nonlinearity in s present in (14) the higher order threshold parameters are already nonzero at lowest order. This makes fitting the threshold-expansion numerically more unstable since we need to use a fitting polynomial to higher order in $q_{\pi K}^2$ compared to what was needed for the $\pi\pi$ case.

The column labeled [14] uses the results of the Roy-Steiner analysis of [14] of πK scattering. We have combined errors quadratically which due to the presence of correlations can lead to a serious underestimate of the errors on the combinations.

The numerical results for the theory are calculated with the formulas of [15] where the NLO LECs we use are those of fit 10 of [11]. The columns in Tab. 4 have the same meaning

[2] This was written wrong in the preliminary report [7].

	[14]	NLO 1-loop	NLO LECs	NNLO 2-loop	NNLO 1-loop	remainder
LHS (23)	5.4 ± 0.3	0.16	0.97	0.77	$-0.11(11)$	0.6 ± 0.3
RHS (23)	6.9 ± 0.6	0.42	0.97	0.77	$-0.03(7)$	1.8 ± 0.6
10 LHS (21)	0.32 ± 0.01	0.03	0.12	0.11	$0.00(2)$	0.07 ± 0.01
10 RHS (21)	0.37 ± 0.01	0.02	0.12	0.10	$-0.01(2)$	0.14 ± 0.01
100 LHS (24)	-0.49 ± 0.02	0.08	-0.25	-0.17	$0.05(3)$	-0.21 ± 0.02
100 RHS (24)	-0.85 ± 0.60	0.03	-0.25	0.11	$-0.03(13)$	-0.71 ± 0.60
100 LHS (22)	0.13 ± 0.01	0.04	0.00	0.01	$0.03(1)$	0.05 ± 0.01
100 RHS (22)	0.01 ± 0.01	0.01	0.00	0.00	$0.00(1)$	-0.01 ± 0.01
10^3 LHS (26)	0.29 ± 0.03	0.09	0.00	0.06	$0.01(2)$	0.13 ± 0.03
10^3 RHS (26)	0.31 ± 0.07	0.03	0.00	0.06	$0.05(3)$	0.17 ± 0.07

Table 4: The relations found in the πK-scattering. The tree level contribution to the LHS and RHS of relation 1 is 3.01 and vanishes for the others. The NLO LECs part satisfies the relation. Notice the extra factors of ten for some of them. See text for a longer discussion. All quantities are in the units of powers of m_{π^+}.

as in Tab. 1 and the errors on the ChPT part have been evaluated as discussed for the $\pi\pi$ case. The first relation is reasonably well satisfied, somewhat below two sigma. The second relation has a large discrepancy in view of the experimental error but if we assume a theory error of about half the NNLO contribution it seems reasonable given The third relation is well satisfied but the RHS has a rather large experimental error. The fourth relation does not work well, mainly due to the fact that we seem to underestimate the value for a_3^-. The last relation again works reasonably well. The same relations but now LHS-RHS are shown in Tab. 3 and depicted graphically in Fig. 1. The conclusions are the same.

5 $\pi\pi$ and πK scattering

If we consider the $\pi\pi$ and πK system together we get two more relations due to the identities

$$[b_5]_{C_i} = \left[c_{30}^+\right]_{C_i} + \frac{3}{\rho}\left[c_{20}^-\right]_{C_i}\,, \qquad [b_6]_{C_i} = \frac{1}{4\rho}\left[c_{20}^-\right]_{C_i} + \frac{1}{16\rho^2}\left[c_{11}^+\right]_{C_i}\,, \tag{27}$$

where c_{ij}^- (c_{ij}^+) are expressed in units of $m_\pi^{2i+2j+1}(m_\pi^{2i+2j})$. We can express these relations in terms of the threshold parameters:

$$6\left[a_3^1\right]_{C_i} = (1+\rho)\left[a_3^+ + 3a_3^-\right]_{C_i}\,, \tag{28}$$

$$3\left[(1+\rho)^2\left[b_2^2\right]_{C_i} + 7(1-\rho)^2\left[a_3^1\right]_{C_i}\right] = (1+\rho)\left[7\left(1-4\rho+\rho^2\right)\left[a_3^-\right]_{C_i} + \left[a_2^+ + 2\rho b_2^+\right]_{C_i}\right]. \tag{29}$$

	[10],[14] [17],[18]	NLO 1-loop	NLO LECs	NNLO 2-loop	NNLO 1-loop	remainder
10^3 LHS (28)	0.34 ± 0.01	0.12	0.00	0.16	0.00(4)	0.05 ± 0.01
10^3 RHS (28)	0.38 ± 0.03	0.12	0.00	0.05	0.04(2)	0.16 ± 0.03
10 LHS (29)	-0.13 ± 0.01	-0.12	0.00	-0.05	0.02(2)	0.01 ± 0.01
10 RHS (29)	-0.09 ± 0.02	-0.05	0.00	-0.02	$-0.01(1)$	-0.01 ± 0.02
LHS (33)	-0.73 ± 0.10	-0.23	0.00	-0.15	$-0.05(6)$	-0.29 ± 0.10
RHS (33)	0.50 ± 0.07	0.19	0.00	0.10	0.03(4)	0.18 ± 0.07

Table 5: The relations found between $\pi\pi$ and πK-scattering lengths and between the curvature in F in $K_{\ell 4}$ and πK scattering. See text for a longer discussion. All quantities are in units of powers of m_{π^+}.

Here all the quantities are expressed in powers of m_{π^+}.

The numerical results are quoted in Tab. 5. The first relation does not work but the second is well satisfied. If we look in the numerical results we see that a_3^- plays a minor role in the RHS of the second relation but is important in the first, so this could be the same problem that appeared for relation (22). The same relations but now LHS-RHS are shown in Tab. 3 and depicted graphically in Fig. 1. The conclusions are the same. A related analysis can be found in [16].

6 $K_{\ell 4}$

The decay $K^+(p) \to \pi^+(p_1)\pi^-(p_2)e^+(p_\ell)\nu(p_\nu)$ is given by the amplitude [19]

$$T = \frac{G_F}{\sqrt{2}} V_{us}^\star \bar{u}(p_\nu)\gamma_\mu(1-\gamma_5)v(p_\ell)(V^\mu - A^\mu) \tag{30}$$

where V^μ and A^μ are parametrized in terms of four form factors: F, G, H and R (but the R-form factor is negligible in decays with an electron in the final state). Using partial wave expansion and neglecting d wave terms one obtains [20]:

$$\begin{aligned} F &= f_s + f_s' q^2 + f_s'' q^4 + f_e' s_e/4m_\pi^2 + f_t \sigma_\pi X \cos\theta + \dots\,, \\ G_p &= g_p + g_p' q^2 + g_g'' q^4 + g_e' s_e/4m_\pi^2 + g_t \sigma_\pi X \cos\theta + \dots \end{aligned} \tag{31}$$

Here $s_\pi(s_e)$ is the invariant mass of dipion (dilepton) system, and $q^2 = s_\pi/(4m_\pi^2) - 1$. θ is the angle of the pion in their restframe w.r.t. the kaon momentum and $t-u = -2\sigma_\pi X \cos\theta$. We found one relation between the quantities defined in (31) and πK scattering:

$$\sqrt{2}\,[f_s'']_{C_i} = 64\rho F_\pi \left[c_{30}^+\right]_{C_i}\,. \tag{32}$$

This translates into a relation between πK threshold parameters and f_s'' which, with all quantities expressed in units of m_{π^+}, reads:

$$\sqrt{2}\,[f_s'']_{C_i} = 32\pi \frac{\rho}{1+\rho} F_\pi \left[\frac{35}{6}\left(2+\rho+2\rho^2\right)\left[a_3^+\right]_{C_i} - \frac{5}{4}\left[a_2^+ + 2\rho b_2^+\right]_{C_i}\right]\,. \tag{33}$$

There is no more relation involving the quantities discussed so far, $\pi\pi$ and πK scattering, and $K_{\ell 4}$.

Numerical results for (33) are shown in Tab. 5. The experimental results is taken from [18] for f_s''/f_s and from [17] for f_s. This should be an acceptable combination since the central value for f_s'/f_s and f_s''/f_s from [17] are within 10% of those of [18]. The theoretical results are using the formulas of [21] and fit 10 of [11]. This relation has problems. The sign is even different on both sides. In both cases we also see that the ChPT series has a large NNLO contribution. For completeness, LHS-RHS is given in Tab. 3 and Fig. 1.

There have been indications from dispersive methods that ChPT might underestimate the curvature f_s''. Dispersion relations were used in [22] for $K_{\ell 4}$. If one looks at Fig. 7 in [11], one can see that the dispersive result of [22] has a larger curvature than the two-loop result. For this reason, we do not consider this discrepancy a major problem for ChPT.

7 $\eta \to 3\pi$

The amplitude for the decay $\eta(p_\eta) \to \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)$ can be written as

$$A(\eta \to \pi^+\pi^-\pi^0) = \sin\epsilon M(s,t,u)\,. \tag{34}$$

Here we used the Mandelstam variables

$$\begin{aligned} s &= (p_+ + p_-)^2 = (p_\eta - p_0)^2\,, \\ t &= (p_+ + p_0)^2 = (p_\eta - p_-)^2\,, \\ u &= (p_- + p_0)^2 = (p_\eta - p_+)^2,\,. \end{aligned} \tag{35}$$

which are linearly dependent $s + t + u = m^2_{\pi^o} + m^2_{\pi^-} + m^2_{\pi^+} + m^2_\eta \equiv 3s_0\,..$ G-parity requires the amplitude to vanish at the limit $m_u = m_d$ and therefore it must inevitably be accompanied by an overall factor of $m_u - m_d$ which we have chosen to be in the form of $\sin(\epsilon) \approx (\sqrt{3}/4)(m_d - m_u)/(m_s - \hat{m})$. Since the amplitude is invariant under charge conjugation we have $M(s,t,u) = M(s,u,t)$. Similar to the $\pi\pi$ scattering, we can write the amplitude as

$$M(s,t,u) = \eta_1 + \eta_2 s + \eta_3 s^2 + \eta_4 (t-u)^2 + \eta_5 s^3 + \eta_6 s(t-u)^2 + \text{non polynomial part} \tag{36}$$

to NNLO in ChPT. Using the results of [23] we then obtain two relations

$$[\eta_5]_{C_i} = 3\,[\eta_6]_{C_i}\,, \tag{37}$$

$$[\eta_5]_{C_i} = -768\rho^3\left[c^-_{01}\right]_{C_i} = -\pi\,(1+\rho)\,\frac{35}{2}\left[a^-_3\right]_{C_i}\,. \tag{38}$$

Since η_5 is not unambiguously determined from the measured Dalitz-plot parameters and η_6 is not measured at all we do not present numerical results for this. The overall factor $\sin\epsilon$ itself is part of the uncertainty involved. Unfortunately, no relations are present for $\eta_1, \ldots, \eta_4$ which would have helped in the numerical prediction for $\eta \to 3\pi$ using the results of [23].

8 Scalar formfactors

The scalar form factors for the pions and the kaons are defined as

$$F_{ij}^{M_1M_2}(t) = \langle M_2(p)|\bar{q}_i q_j|M_1(q)\rangle, \tag{39}$$

where $t = p - q$, $i, j = u, d, s$ are flavour indices and M_i denotes a meson state with the indicated momentum. Due to isospin symmetry not all of them are independent, therefore we consider only

$$\begin{aligned} F_S^\pi &= 2F_{uu}^{\pi^0\pi^0} \qquad F_{Ss}^\pi = F_{ss}^{\pi^0\pi^0}, \qquad F_{Ss}^K = F_{ss}^{K^0K^0}, \\ F_S^K &= F_{Su}^K + F_{Sd}^K = F_{uu}^{K^0K^0} + F_{dd}^{K^0K^0}, \qquad F_S^{\pi K} = F_{su}^{K^+\pi^0}. \end{aligned} \tag{40}$$

Near $t = 0$ these are expanded via

$$F_S(t) = F_S(0) + F_S' t + F_S'' t^2 + \dots . \tag{41}$$

The NNLO ChPT calculation for these quantities was performed in [24] where it was found that the curvatures F_S'' only depend on two of the NNLO LECs. As a consequence there are four relations

$$\begin{aligned} [F_S^{\pi\prime\prime}]_{C_i} &= 2\left[F_{Su}^{K\prime\prime}\right]_{C_i} = 2\left[F_{Ss}^{K\prime\prime}\right]_{C_i}, \\ [F_{Ss}^{\pi\prime\prime}]_{C_i} &= \left[F_{Sd}^{K\prime\prime}\right]_{C_i}, \\ 2\left[F_S^{K\pi\prime\prime}\right]_{C_i} &= [F_S^{\pi\prime\prime}]_{C_i} - 2\,[F_{Ss}^{\pi\prime\prime}]_{C_i}. \end{aligned} \tag{42}$$

There is also a relation involving the slopes

$$[F_S^{\pi\prime}]_{C_i} - 2\,[F_{Ss}^{\pi\prime}]_{C_i} - 2\left[F_{Sd}^{K\prime}\right]_{C_i} + 2\left[F_{Ss}^{K\prime}\right]_{C_i} - 4\left[F_S^{K\pi\prime}\right]_{C_i} = 0\,. \tag{43}$$

This is a consequence of the "scalar Sirlin" relation derived in general in [24].

In addition to those already known we found a relation between the values at $t = 0$ which with $\rho = m_K/m_\pi$ reads

$$\begin{aligned} 2\rho^6\,[F_S^\pi(0)]_{C_i} &= \rho^4\left(2\rho^4 - \rho^2 - 3\right)[F_{Ss}^\pi(0)]_{C_i} + \left(3\rho^2 - 1\right)\left[F_{Su}^K(0)\right]_{C_i} \\ &\quad + \left(6\rho^4 - 3\rho^2 - 1\right)\left[F_{Sd}^K(0)\right]_{C_i} + \left(\rho^2 + 1\right)\left[F_{Ss}^K(0)\right]_{C_i}. \end{aligned} \tag{44}$$

The scalar formfactors had a significant dependence on what was used as input for L_4^r and L_6^r [24]. The curvature relations were studied there and found to sometimes work and sometimes not, see Tab. 2 and Sect. 5.5 in [24]. We intend to come back to these relations when constraints on L_4^r and L_6^r have been included in a general fit.

9 Scalar formfactors, masses and decay constants

The three masses M_π^2, M_K^2, M_η^2 and decay constants F_π, F_K, F_η are not related, they all have a different dependence on the NNLO LECs. We do find some relations however when we combine them with the scalar formfactors. The two-loop calculations for masses and decay constants can be found in [25, 26] for π and η and in [25] for the kaon.

There are two relations between the $F_S(0)$ and the ChPT expansion of the masses M_π^2, M_K^2:

$$
\begin{aligned}
2B_0\left[M_\pi^2\right]_{C_i} &= \frac{1}{3}\left\{(2\rho^2-1)\left[F_{Ss}^\pi(0)\right]_{C_i}+\left[F_S^\pi(0)\right]_{C_i}\right\} \\
2B_0\left[M_K^2\right]_{C_i} &= \frac{1}{3}\left\{(2\rho^2-1)\left[F_{Ss}^K(0)\right]_{C_i}+\left[F_S^K(0)\right]_{C_i}\right\}.
\end{aligned}
\tag{45}
$$

Remember we express everything in units powers of m_π. One could arrive to the same conclusion using the Feynman-Hellmann Theorem (see e.g. [27] or [24]) which implies for $q = u, d, s$ and $M = \pi, K$

$$
F_{Sq}^M(t=0) \;=\; \langle M|\bar{u}u|M\rangle = \frac{\partial m_M^2}{\partial m_q}\,. \tag{46}
$$

On the other hand the ChPT expansion leads to

$$
\left[M_\pi^2\right]_{C_i} = \sum_i C_i (m_q)^3 = f(m_u, m_d, m_s), \tag{47}
$$

that is an homogeneous function of order three. Thanks to the Euler Theorem, $[M_\pi^2]_{C_i}$ can be written in terms of its derivatives ($f(\mathbf{x}) = \frac{1}{3}\sum_{i=1}^n \frac{\partial f}{\partial x_i} x_i \quad \mathbf{x}\in\mathbb{R}^n$). These are exactly the relations in Eq. (45). Something similar holds for the p^4 expression but with a factor 1/2 instead of 1/3.

There are two more relations if we also include the decay constants. The first one is

$$
\begin{aligned}
&\left(\rho^2-1\right)^2\frac{B_0}{F_0}\left[F_K-F_\pi\right]_{C_i}+\left(\rho^2+1\right)B_0\left[M_K^2-M_\pi^2\right]_{C_i} = \\
&\qquad \left(\rho^4-1\right)\left[F_S^{K\pi}(0)\right]_{C_i}+\left(\rho^2-1\right)^3\left[F_S^{K\pi\prime}\right]_{C_i}+\left(\rho^2-1\right)^3\left(\rho^2+1\right)\left[F_S^{K\pi\prime\prime}\right]_{C_i}.
\end{aligned}
\tag{48}
$$

This relation is the same as the one found in [28] for the $K_{\ell 3}$ scalar formfactor when one uses

$$
\partial^\mu \bar{s}\gamma_\mu u = (m_s - m_u)\, i\,\bar{s}u\,, \tag{49}
$$

and rewrites the quark masses into the pion and kaon mass. The second relation, in the simplest form we found, reads

$$
\begin{aligned}
&\left(4\rho^2\left[F_\pi\right]_{C_i}-4\left[F_K\right]_{C_i}\right)\frac{B_0}{F_0} = 2\rho^4\left[F_S^{\pi\prime}\right]_{C_i}+\left(2\rho^6-\rho^4+\rho^2\right)\left[F_{Ss}^{\pi\prime}\right]_{C_i} \\
&\qquad +\left(-2\rho^4+\rho^2-1\right)\left[F_{Sd}^{K\prime}\right]_{C_i}-\left(\rho^2+1\right)\left[F_{Su}^{K\prime}\right]_{C_i}+\left(-3\rho^2+1\right)\left[F_{Ss}^{K\prime}\right]_{C_i}.
\end{aligned}
\tag{50}
$$

We have not presented numerical results for the relations in this section since the assumptions underlying fit 10 of [11] were such that all the left hand sides evaluate to zero. In addition the right hand sides tend to contain poorly known quantities.

10 Vector formfactors

The vector formfactors have been discussed extensively in [29] and [28]. There three relations between the curvatures and the the Sirlin relation between the slopes [30] were found. We also find the expected relationship between the scalar formfactors and the scalar formfactor in $K_{\ell 3}$ which followed from (49). The numerical results for the relation between the slopes and curvatures were discussed extensively in [29, 28] and found to work well. So this sector had the expected total of 7 relations added to those discussed above.

11 Scalar formfactors, $\pi\pi$ and πK scattering

There are two more relations when we combine the scalar formfactors and the scattering amplitudes for $\pi\pi$ and πK scattering. All three quantities are needed. These relations are rather complicated. The first relation is:

$$
\begin{aligned}
&\rho^4\left[105a_3^1+15b_2^2-3a_1^1+3b_0^2-8a_0^2\right]_{C_i}+(1+\rho)\left(35\rho^4\left[a_3^-\right]_{C_i}-\frac{1}{3}\rho\left[a_0^-\right]_{C_i}\right)\\
&+\frac{2}{\rho+1}\rho^3\left[a_1^+\right]_{C_i}+\frac{10}{\rho+1}\rho^4(2+\rho+2\rho^2)\left[b_2^++7a_3^+\right]_{C_i}+\frac{2}{3}(\rho+1)\left(\rho^2+1\right)\left[a_0^+\right]_{C_i}\\
&-\frac{10}{\rho+1}\rho^3\left(2+3\rho+2\rho^2\right)\left[a_2^+\right]_{C_i}-\frac{4}{\rho+1}\rho^3(1+\rho+\rho^2)\left[b_1^+\right]_{C_i}=\\
&\frac{\rho^2}{8\pi B_0F_0^2\left(1-3\rho^2\right)}\Bigg[-\left(1-\rho^2\right)\left[F_{Ss}^K(0)\right]_{C_i}+2(1-3\rho^2+3\rho^4)\left[F_{Sd}^K(0)\right]_{C_i}\\
&+(1-3\rho^2+3\rho^4-5\rho^6+2\rho^8)\left[F_{Ss}^\pi(0)\right]_{C_i}+\frac{1}{2}(1-5\rho^2+8\rho^4-4\rho^6)\left[F_S^\pi(0)\right]_{C_i}\Bigg]\,.
\end{aligned}
\tag{51}
$$

The second relation involves even more quantities:

$$
\begin{aligned}
&-(1-\rho^4)8\pi B_0F_0^2\Bigg[\rho^2\left[b_0^0-12a_0^2+2b_0^2+45b_2^2-315a_3^1\right]_{C_i}\\
&\qquad\qquad+210\rho^2(1+\rho)\left[a_3^-\right]_{C_i}-2\frac{1+\rho}{\rho}\left[a_0^-\right]_{C_i}\Bigg]\\
&+8\pi B_0F_0^2(\rho-1)\Bigg[120\rho^4\left[b_2^++7a_3^+\right]_{C_i}-6\rho^2\left[a_1^++2\rho b_1^+\right]_{C_i}+2(1+\rho)^2\left[2a_0^+-15\rho^2a_2^+\right]_{C_i}\Bigg]\\
&=(1-\rho^2)\Bigg[12(1-\rho^4)(F_{Ss}^{K\prime\prime}-F_{Sd}^{K\prime\prime})+12(1-\rho^2-2\rho^4)F_S^{K\pi\prime}+6\rho^2(1+2\rho^2)F_{Ss}^{K\prime}\\
&\qquad\qquad-12\rho^4F_{Sd}^{K\prime}+6\rho^2F_{Su}^{K\prime}-12\rho^4F_{Ss}^{\pi\prime}\Bigg]\\
&\quad+(1+\rho^2)12F_S^{K\pi}(0)+12\rho^2F_{Su}^K(0)+3(-1+3\rho^2+6\rho^4)F_{Sd}^K(0)\\
&\quad+(1-\rho^2)(2-\rho^2-8\rho^4-8\rho^6)F_{Ss}^\pi(0)-2(1+2\rho^2+2\rho^4+4\rho^6)F_S^\pi(0)\,.
\end{aligned}
\tag{52}
$$

12 A final relation: $K_{\ell 4}$, πK scattering and scalar formfactors

The final relation we found is between $K_{\ell 4}$, πK scattering and the scalar formfactors. The version below is the simplest we found.

$$
\begin{aligned}
&(1-\rho^2)^2(1+\rho^2)B_0\left[12\sqrt{2}\frac{F_0}{\rho}\left(g_p - g_p' + \left((1-\frac{1}{4}\rho^2\right)g_p'' + \frac{1}{2}\rho^2 f_t\right)\right.\\
&\qquad\qquad\qquad\left. -16\pi\frac{1+\rho}{\rho}a_0^- + 70\pi(1+\rho)(20\rho^2+\rho^4)a_3^-\right]\\
&= 12(1+\rho^2)\rho^4 F_S^{K\pi}(0) + 12\rho^6 F_{Su}^K(0) + 24\rho^8 F_{Sd}^K(0)\\
&\quad +2\rho^4(1-\rho^4)(1-4\rho^4)F_{Ss}^\pi(0) - 2\rho^4(1+2\rho^2+2\rho^4+4\rho^6)F_S^\pi(0)\,. \qquad (53)
\end{aligned}
$$

13 Conclusions

We have performed a systematic search for combinations that allow a test of ChPT at NNLO order. We have therefore looked at the three masses and three decay constants, 11 $\pi\pi$ threshold parameters, 14 πK threshold parameters, 6 $\eta \to 3\pi$ parameters, 10 observables in $K_{\ell 4}$, 18 in the scalar formfactors and 11 in the vectorformfactors. This means a total of 76 quantities. We found a total of 35 relations between these. Most of these had been noticed earlier but we did find several new ones. We have presented the relations in a form as simple as we found but given the total number they can be rewritten in many equivalent forms.

These are relations that should allow independent determinations of combinations of the NNLO LECs in ChPT. For the vector formfactors this was already done in [29, 28] and partly for the $\pi\pi$, πK system [12, 15, 16] and scalar formfactors [24]. Here we studied in detail the relations for the $\pi\pi$, πK scattering and $K_{\ell 4}$ since for these cases enough experimental and/or dispersion theory results exist. Fig. 1 is a summary of the numerical relations.

The resulting picture is that ChPT at NNLO works in most cases but there are some problems. The $\pi\pi$ system alone is working well, the πK system alone works satisfactorily but with some problems. The same can be said for the combinations of both systems. A common part in these two cases is the presence of a_3^-. Comparing πK scattering and $K_{\ell 4}$ there is a clear contradiction. In fact, both sides of the relation seem to be difficult to explain within ChPT. It was already noticed in [21] that getting such a large negative curvature in $K_{\ell 4}$ was difficult. It should be noted that none of the quantities involved in the tested relations was used as input for the fit of the NLO LECs in [11].

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A Relation between threshold and subtreshold parameters

For completeness we quote here the relations between the threshold and subthreshold parameters for the tree level part, i.e. that analytic dependence on s, t and u. The $\pi\pi$ ones can also be found in [13] and [21]. For the πK case we have already used the relation (17)

$$
\begin{aligned}
\pi a_0^0 &= 6b_5 + b_4 + (3/2)b_3 + (3/8)b_2 + (5/32)b_1\,,\\
\pi b_0^0 &= -2b_6 + 18b_5 + 3b_4 + 3b_3 + (1/4)b_2\,,\\
\pi a_0^2 &= b_4 + (1/16)b_1\,,\\
\pi b_0^2 &= -2b_6 + 3b_4 - (1/8)b_2\,,\\
\pi a_1^1 &= (2/3)b_6 + (1/3)b_4 + (1/24)b_2\,,\\
\pi b_1^1 &= (4/3)b_6 + (1/2)b_4 - (1/6)b_3\,,\\
\pi a_2^0 &= (16/15)b_6 + (7/30)b_4 + (1/30)b_3\,,\\
\pi b_2^0 &= (17/15)b_6 - (1/5)b_5\,,\\
\pi a_2^2 &= (4/15)b_6 + (1/30)b_4 + (1/30)b_3\,,\\
\pi b_2^2 &= (1/3)b_6 - (1/5)b_5\,,\\
\pi a_3^1 &= (1/35)b_6 + (1/35)b_5\,.
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
\pi\,(\rho+1)\,a_0^- &= 24\rho^3 c_{01}^- + (3/2)\rho c_{00}^-\,,\\
\pi\,(\rho+1)\,b_0^- &= (36\rho + 24\rho^2 + 36\rho^3)c_{01}^- - 3\rho c_{10}^- + ((3/4)\rho^{-1} + (3/4)\rho)c_{00}^-\,,\\
\pi\,(\rho+1)\,a_1^- &= 12\rho^2 c_{01}^- + \rho c_{10}^- + (1/4)c_{00}^-\,,\\
\pi\,(\rho+1)\,b_1^- &= (12 - 6\rho + 12\rho^2)c_{01}^- + (-1/2 + (1/2)\rho^{-1} + (1/2)\rho)c_{10}^- + -(1/8)\rho^{-1}c_{00}^-\,,\\
\pi\,(\rho+1)\,a_2^- &= (24/5)\rho c_{01}^- + (1/5)c_{10}^-\,,\\
\pi\,(\rho+1)\,b_2^- &= (-(12/5) + (12/5)\rho^{-1} + (12/5)\rho)c_{01}^- - (1/10)\rho^{-1}c_{10}^-\,,\\
\pi\,(\rho+1)\,a_3^- &= (24/35)c_{01}^-\,,\\
\pi\,(\rho+1)\,a_0^+ &= 6\rho^2 c_{01}^+ + (3/8)c_{00}^+\,,\\
\pi\,(\rho+1)\,b_0^+ &= -12\rho^2 c_{11}^+ + (6 + 3\rho + 6\rho^2)c_{01}^+ - (3/4)c_{10}^+ - (3/16)\rho^{-1}c_{00}^+\,.\,,
\end{aligned}
$$

$$
\begin{aligned}
\pi\,(\rho+1)\,a_1^+ &= 4\rho^2 c_{11}^+ + 2\rho c_{01}^+ + (1/4)c_{10}^+\,,\\
\pi\,(\rho+1)\,b_1^+ &= (4-2\rho+4\rho^2)c_{11}^+ + (\rho^{-1}+\rho)c_{01}^+ - c_{20}^+ - (1/8)\rho^{-1}c_{10}^+\,,\\
\pi\,(\rho+1)\,a_2^+ &= (8/5)\rho c_{11}^+ + (1/5)c_{01}^+ + (1/5)c_{20}^+\,,\\
\pi\,(\rho+1)\,b_2^+ &= (-(2/5)+(4/5)\rho^{-1}+(4/5)\rho)c_{11}^+ - (1/10)\rho^{-1}c_{01}^+ - (6/5)c_{30}^+ - (1/10)\rho^{-1}c_{20}^+\,,\\
\pi\,(\rho+1)\,a_3^+ &= (6/35)c_{11}^+ + (6/35)c_{30}^+ \qquad (55)
\end{aligned}
$$

It can be checked that these satisfy the relations given in Sects. 3, 4 and 5.

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