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Lunds Universitet
Institutionen för Astronomi och Teoretisk Fysik
Avdelningen för Teoretisk Högenergifysik
Sölvegatan 14 A
22362 Lund, Sweden

Technische Universität Darmstadt
Fachbereich Physik
Institut für Kernphysik
Schlossgartenstraße 9
64289 Darmstadt, Germany

Bachelor Thesis

Exploring the AdS/QCD correspondence

Sven Möller

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Supervised by Prof. Johan Bijnens

We study the AdS/QCD correspondence, which provides a new way to produce hadronic models, in a flavour-broken hard-wall approach to obtain $K_{\ell 3}$ transition form factors. We also calculate meson masses, decay constants, and pion form factors both in a hard-wall and a soft-wall approach. The latter is necessary to obtain correct Regge trajectories. We obtain results for light unflavoured and strange vector, pseudovector, pseudoscalar, and scalar mesons. All results are compared to and agree well with current experimental data.

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1 Introduction

1.1 Overview

The AdS/CFT correspondence [1] has provided a new approach to calculate quantities in strongly coupled theories like quantum chromodynamics (QCD) at low energies. AdS/QCD, the extension of the original conjecture to more realistic gauge theories, has been an increasingly fruitful field of research for the past decade. This text aims at exploring the intriguing results which can be obtained from the correspondence.

Chapter 1 gives a short introduction to the holographic principle, of which the AdS/CFT correspondence is a special case, mainly using plausability arguments. After this, the idea of the AdS/CFT correspondence will be stated together with an instruction of how to use and extend it to the QCD case. This involves some mathematical basics about the anti de Sitter space (AdS), the space in which the fields we deal with live.

Having all the necessary techniques at hand, we can begin to build up a model describing mesons and many of their properties. In Chapter 2 this will be done in a hard-wall setting with $N_\mathrm{f} = 3$. For this we mainly follow [2], whose results we confirm. This includes the calculation of meson masses, decay constants, and form factors, especially the $K_{\ell 3}$ form factor.

In Chapter 3, we present new calculations in several soft-wall models, adopting ideas from [3, 4, 5]. Masses, decay constants, and form factors will be calculated and compared to experimental data.

Finally, the peculiarities of dealing with scalar mesons will be discussed in Chapter 4 and masses as well as two-point correlators will be calculated. This will also serve as an example to explain how one has to deal with the boundary conditions occurring in AdS/QCD models (treated extensively in [6]).

1.2 The Holographic Principle

In theories of quantum gravity, the holographic principle is the assumption that to the description of the dynamics of a region of spacetime exists an equivalent description, localized on the boundary of that region. Quantum gravity (QG) is the attempt to self-consistently unify the theories of quantum mechanics and general relativity, for which the well-known string theory and the loop quantum gravity (LQG) are candidates. The choice of the word *holographic* stems from the fact that on a hologram a three-dimensional picture is essentially stored on a two-dimensional surface.

Holographic features were first discovered in connection with black holes. Let us try to understand how this comes about [7]. Black holes are not really black. They radiate photons which have a spectrum characteristic of a black body with (for the simplest types of black holes, the Schwarzschild black holes) a temperature

$$T_{\mathrm{H}} = \frac{\hbar}{8\pi k_{\mathrm{B}} M} \tag{1.1}$$

(in geometrized units, i.e. $c = G_{\mathrm{N}} = 1$), where M is the mass of the black hole. This effect was predicted by Stephen Hawking, who also provided the theoretical arguments for its existence (so far however, Hawking radiation has not been observed). The Hawking radiation is caused by quantum fluctuations near the horizon of the black hole. Important for the following steps is that there is something like a temperature of a black hole.

Another observation is that the area A of black holes, i.e. the area of its horizon, always increases with time,[1] so we can write

$$\frac{\mathrm{d}A}{\mathrm{d}t} \geq 0. \tag{1.2}$$

(This is referred to as Hawking's area theorem.) Moreover, when two black holes merge, the area of the resulting black hole is always larger than the sum of the areas of the original black holes. The surface area of a Schwarzschild black hole is given by $A = 16\pi M^2$, which we can then write in differential form as

$$\mathrm{d}M = \frac{1}{32\pi M}\mathrm{d}A = \frac{\hbar}{\pi k_{\mathrm{B}} M}\mathrm{d}\left(\frac{k_{\mathrm{B}} A}{4\hbar}\right). \tag{1.3}$$

Here, $\mathrm{d}M$ is the change in the total energy of the black holes and, identifying the Hawking temperature T_{H}, we can write

$$\mathrm{d}E = T_{\mathrm{H}}\mathrm{d}S, \tag{1.4}$$

where we defined

$$S = \frac{k_{\mathrm{B}} A}{4\hbar}. \tag{1.5}$$

The area theorem then suggests that S, in fact, behaves in a way we expect it from an entropy, namely as in the second law of thermodynamics. The above and further considerations then lead to the theory of thermodynamics of black holes. Of relevance for this introduction is the observation that the entropy of a black hole is essentially given by its surface. Another argument why a black hole should indeed have an entropy is that if one throws a gas with a certain energy into a black hole, the entropy of the gas disappears into the black hole. In order not to violate the second law, one must assume that this loss of entropy is compensated by an increase in the entropy of the black hole. Also note that the Schwarzschild black hole can be completely characterized

[1] One should maybe remark that if Hawking radiation exists, then this principle is violated since, by loosing mass, a black hole also looses area. The following steps are in principle still correct but when speaking of the entropy which should obey the second law of thermodynamics, one has to consider the sum of the entropy of black hole and the one of the remaining universe. So if the black hole looses area and thus entropy by radiating, then the entropy in the rest of the universe must rise by at least the same amount.

by its mass M. (This is referred to as no-hair theorem. For a general black hole, a similar statement is true but one needs the mass, the electric charge, and the angular momentum.) Therefore the increase in the entropy of the black hole only depends on the mass of the gas and hence the only way this can be true in all situations is if the entropy of the black hole is in some sense maximal. This was argued by Jacob Beckenstein. He used this and the fact that matter collapses into a black hole when it is too dense, to put an upper bound on the entropy of a region of space and, assuming weak gravity, spherical symmetry and a few more conditions, the entropy bound of that region of space is related to its boundary area. This is referred to as *spherical entropy bound*. Entropy in turn is in some sense a measure for the information content of a system.[2]

A major step was taken by Gerardus 't Hooft [8] (1993) in elevating this entropy bound to a general principle, the *holographic principle*[3], stating that the description of a volume of space can be thought of as encoded on the boundary of that region. Leonard Susskind [10] (1995) gave this principle an exact string theoretic interpretation. To understand how this was motivated, let us go back to the black holes emitting Hawking radiation. This radiation, as calculated by Hawking, is purely thermal and does not depend on the material falling into the black hole. However, if the matter entering the black hole found itself in a pure quantum mechanical state, this state would be transformed into the mixed state of Hawking radiation and the information about the original quantum state would be lost, thus contradicting Liouville's theorem (in its quantum mechanical formulation, the von Neumann equation) and consequently quantum mechanics. This poses a paradox, known as the *black hole information paradox*.

There are several ways, how this paradox can be solved.[4] Information can be preserved if one assumes that Hawking radiation is not completely thermal but has small quantum corrections. A possible mechanism which could explain this was found by 't Hooft. He explained how incoming particles could affect the outgoing particles by deforming the horizon with their gravitational field. The deformed horizon would then produce different outgoing particles than the original one. Most remarkable is the fact that this deformation is quite similar to the deformation of a world sheet in string theory emitting and absorbing particles. This led 't Hooft to the conclusion that the correct description of the black hole would have to be given in a string theoretic framework.

At that time, Susskind had also been working on the holographic principle, mostly independent of 't Hooft. He claimed that the oscillating horizon gives a complete description of the ingoing and outgoing particles' states. The theory of strings and world sheets was exactly equipped with such a holographic behaviour. He argued that black holes themselves could be viewed as long, highly excited string states, which is a remarkable

[2] How exactly one measures information and how this is related to the physical notion of entropy, is a far from trivial problem and has led to the formation of information theory, which since its foundation by Claude Shannon in the 40's and 50's of the last century has developed into a major branch of applied mathematics.

[3] For a recent overview article on the holographic principle, see for example [9].

[4] An interesting anecdote related to the outcome of the black hole information paradox is the Thorne-Hawking-Preskill bet made 1997 by Stephen Hawking, Kip Thorne, and John Preskill. See for example [11] or [12] for recent developments.

feature since it relates strings to classical objects, i.e. black holes. Most importantly, the black hole information paradox can be solved when using string theory as a way of describing quantum gravity.

1.3 The AdS/CFT Correspondence

There are not many, but still some, noteworthy applications of the holographic principle. The best-known is perhaps the AdS/CFT correspondence, first conjectured by Juan Maldacena in 1997 [1]. It has provided the best, i.e. most accurate, explicit formulation of the holographic principle so far.[5] The original AdS/CFT correspondence states that there is a correspondence between a weakly coupled gravity theory (type IIB string theory) on the product of a 5-sphere S^5 and a five-dimensional anti de Sitter space AdS_5 on the one side and the strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) on the four-dimensional boundary of the AdS_5, which is a special kind conformal field theory (CFT), hence the moniker *AdS/CFT*. Conformal field theories are quantum field theories exhibiting a high level of symmetry. To be precise, they are invariant under conformal, i.e. angle preserving transformations, including scale invariance.

Neither is it the aim of this text to explain the mathematical derivation of the correspondence nor is it intended to explore all the implications of such a correspondence. First of all, this would go far beyond the scope of this work and secondly, conformal field theories are in general much too special to describe any real physical system. Realistic field theories like QCD are not conformal. The development since the original conjecture has been twofold. While some researchers have been working on a more precise mathematical formulation of the correspondence, for which no proof exists so far, others have started to extend the conjecture to more realistic, non-conformal field theories such as QCD. (QCD can however be viewed as an approximate conformal field theory in the limit of high energies.) This generalization is dubbed non-AdS/non-CFT or in specific AdS/QCD. The latter has been developed and treated extensively during the last five years. It should be remarked that it is a consequence of the AdS/CFT correspondence that this extension to QCD implicitely contains the large N_c limit.

The main idea of AdS/QCD is that calculations impossible or hard to do in QCD directly are in some sense dual to (hopefully simpler) string calculations in a higher dimensional space. The main problem of calculations in QCD is that strongly interacting particles, such as quarks, are subject to confinement, which means that their coupling strengths become large at small energies. (The complementary effect is called asymptotic freedom and describes the decrease in coupling for high energies or small distances respectively.) Results from perturbation theory, the usual approach in quantum field theories, are therefore only valid for QCD at large energies, i.e. asymptotically free quarks.

[5] A general overview of the topic can be found in [13].

1.4 The Anti de Sitter Space

By the above stated correspondence, we should relate our gauge theory to a dual description in terms of a string theory living on $\mathrm{AdS}_5 \times S^5$. The metric in this space is given by

$$\mathrm{d}s^2 = \frac{u^2}{L^2}\left(\mathrm{d}t^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2\right) - \frac{L^2}{u^2}\mathrm{d}u^2 - L^2\mathrm{d}\Omega_5^2. \tag{1.6}$$

Here $t = x^0$,x^1,x^2,x^3 represent the usual spacetime coordinates, u is the additional fifth coordinate (L is some curvature radius of the space), and Ω_5 is the five-dimensional solid angle on the hypersphere. Let us explore the anti de Sitter space a little bit further. In general we speak of the n-dimensional AdS_n space. It is a vacuum solution of Einstein's field equations (generalized to n dimensions)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.7}$$

(again in geometrized units), where vacuum means that $T_{\mu\nu} = 0$, with a negative cosmological constant Λ. Moreover, we demand that the AdS space is a homogeneous and isotropic solution.

Without the cosmological constant, the solution can be a perfectly flat space but with Λ being negative, the AdS space has a constant negative scalar curvature (the Ricci scalar). A negative curvature corresponds to a hyperbolically curved space, in contrast to the surface of a sphere, which has a constant positive curvature. Of course, these pictures stem from the isometric embedding of these spaces as hypersurfaces in an $n+1$-dimensional space. The Ricci scalar however is an intrinsic property. Roughly, a negative curvature can be described by the fact that geodesics which are parallel to begin with start moving away from each other while in a spherical or elliptic space, they come closer to each other. Mathematical generalizations of the AdS space include more than one time-like dimension but this is of no physical relevance. So, when speaking of an n-dimensional AdS_n space, one refers to 1 time-like coordinate and $n-1$ space-like coordinates. If we embed the AdS space in a flat space of one additional dimension, it is possible to visualize it (see Figure 1.1). Its points are then given by the hypersurface obeying

$$t^2 + \tau^2 - (x^1)^2 - \ldots - (x^{n-1})^2 = L^2 \tag{1.8}$$

with the infinitesimal distance given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{d}\tau^2 - (\mathrm{d}x^1)^2 - \ldots (\mathrm{d}x^{n-1})^2. \tag{1.9}$$

More interesting in the context of this work is the fact that the AdS_n space can be equipped with a certain coordinate patch (only coverng half of the space). With this coordinate system, the metric is (now with $n = 5$)

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left(\mathrm{d}t^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 - \mathrm{d}z^2\right). \tag{1.10}$$

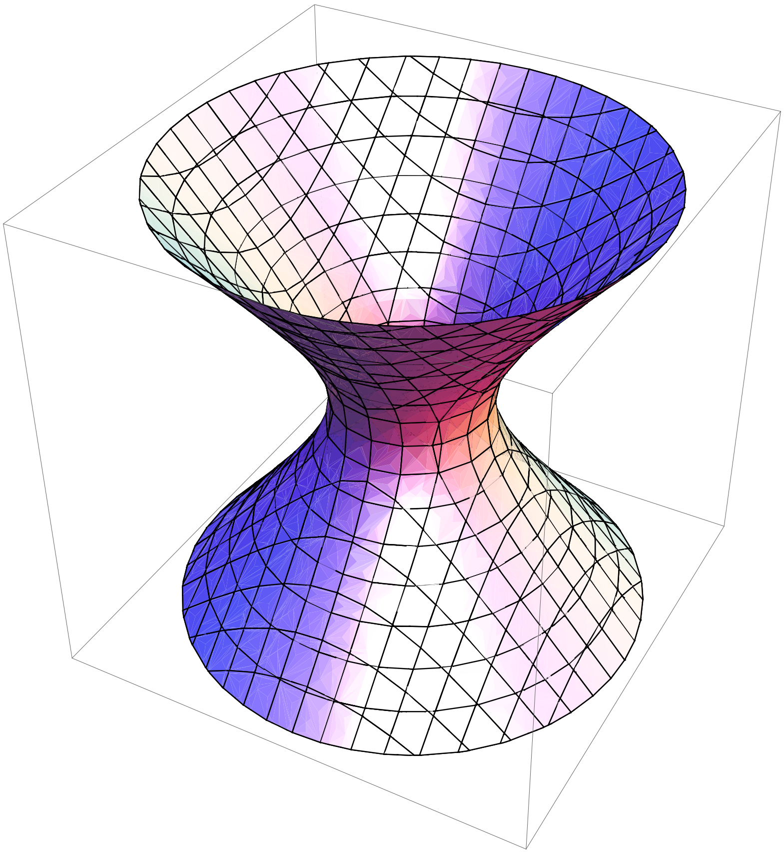

Figure 1.1: Image of the $1+1$-dimensional AdS space in a flat $2+1$-dimensional space, where the spacelike x-axis is the rotational axis. The additional dimension is (chosen to be) timelike and expressed by the coordinate τ. τ and t lie normal to the x-axis and to each other.

It is related to equation (1.6) by the transformation $z = \frac{1}{u}$. From this form it is easy to see that the AdS metric is conformally equivalent to the metric of a flat (half-space) Minkowski spacetime with one timelike and four spacelike dimensions.

The factor

$$a(z) = \frac{L}{z}, \tag{1.11}$$

which appears in front of the brackets of equation (1.10), is often referred to as warp factor. In more general spaces, it can take a different form.

1.5 The AdS$_5$ Metric

The AdS$_5$ space will be the space in which all further calculations will be conducted. Hence, let us explore the metric given by equation (1.10). There is no general consensus on the nomenclature but the conventions presented in this text seem to be the most common ones. The five coordinates $(t, x^1, x^2, x^3, z) = (x^0, x^1, x^2, x^3, x^5)$ are labelled with indices from 0 to 5, leaving out the 4. We can read off the metric tensor g_{MN} form the infinitesimal distance. By definition[6]

$$ds^2 = \frac{L^2}{z^2}\left(\mathrm{d}t^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 - \mathrm{d}z^2\right) =: g_{MN}\mathrm{d}x^M\mathrm{d}x^N. \tag{1.12}$$

In matrix form, the metric tensor is given by

$$g_{MN} = \frac{L^2}{z^2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} =: \frac{z^2}{L^2}\eta_{MN}. \tag{1.13}$$

The covariant metric tensor is determined from $g_{ML}g^{LN} = \delta_M^N$ (the Kronecker delta) and we get

$$g^{MN} = \frac{z^2}{L^2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} =: \frac{L^2}{z^2}\eta^{MN}. \tag{1.14}$$

For later convenience, we have defined $\eta^{MN} = \eta_{MN} = \mathrm{diag}(1, -1, -1, -1, -1)$. (These are not good tensors in the sense that we could raise or lower their indices, especially $\eta^{MN} \neq g^{MM'}g^{NN'}\eta_{M'N'}$.)

It is often convenient to deal with x^0, x^1, x^2, x^3 and z separately. For this we will introduce the convention that a capital roman letter, like M or N, stands for an index running over $0, 1, 2, 3, 5$ whereas lower case Greek letters, like μ or ν, only run over $0, 1, 2, 3$. So, for example x^μ stands for (t, x^1, x^2, x^3) and $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric tensor in four dimensions.

By g we will denote the determinant of g_{MN}, i.e.

$$g = \det(g_{MN}) = \frac{L^{10}}{z^{10}}. \tag{1.15}$$

The coordinate z corresponds to an energy scale, where low z stands for high and large z stands for low energies (called UV and IR). This means that a higher energy (or momentum transfer) QCD physics is dual to the behaviour of fields closer to the AdS boundary at $z = 0$. Since $z = \frac{1}{u}$ is a reciprocal energy scale, $z = 0$ corresponds to infinity. This is reflected by the outcome of many calculations where singularities occur in certain expressions for $z = 0$. It is therefore often necessary to introduce a UV cutoff L_0 for z. Whenever an expression containing L_0 appears, it should always be viewed in the limit of $L_0 \to 0$. When conducting numerical calculations, one has to choose an L_0, which is sufficiently small. Also, in one particular approach, one introduces a IR cutoff L_1, which plays the role of Λ_{QCD}. In these so called hard-wall models, the fifth dimension is hence compactified, but this will be treated in more detail shortly.

[6] In this text, the convention will be used that $A := B$ means that A is defined as B. Similarly in $A =: B$, B is defined as A.

1.6 Working with the Correspondence

Now that the necessary framework has been established, we must learn how exactly the AdS/QCD correspondence is applied. The general idea is that one has a quantity in QCD, which one wishes to calculate, such as masses, decay constants, or form factors. The operators in QCD, whose expectation values give these quantities, are then related via the correspondence to a field in the AdS_5 space. Here, one can make the necessary calculations and in the end translate the results back into the language of QCD. Because this process of switching between the two dual descriptions is like translating between two languages, one speaks of the *holographic dictionary*.

Let us state a little bit more precisely how this correspondence is applied. The original AdS/CFT correspondence states that for every operator $\mathcal{O}(x^\mu)$ of the conformal field theory, there exists a unique $\phi(x^\mu, z)$ field living in the five-dimensional AdS space. These fields are called bulk fields. They are related to the boundary field (playing the role of the source) $\phi_0(x^\mu)$ on the boundary of AdS_5 by the relation

$$\phi(x^\mu, 0) = z^{4-\Delta}\phi_0(x^\mu), \tag{1.16}$$

where Δ is the conformal dimension of the operator $\mathcal{O}(x^\mu)$. Now, let $\mathcal{S}[\phi(x^\mu, z)]$ be the gravity or string action of $\phi(x, z)$, then the correspondence takes the form [13]:

$$\exp\left(\mathcal{S}[\phi(x^\mu, 0)]\right) = \left\langle \exp\left(\int \mathrm{d}^4x\phi_o(x^\mu)\mathcal{O}(x^\mu)\right)\right\rangle_{\mathrm{CFT}}. \tag{1.17}$$

As we will see, this offers a relatively simple way of calculating correlation functions on the gauge theory side. If we define

$$Z := \exp\left(\mathcal{S}[\phi(x^\mu, 0)]\right), \tag{1.18}$$

then it follows that the n-point correlator can by calculated by taking repeated functional derivatives of Z with respect to the source, i.e.

$$\left\langle \mathcal{O}(x^\mu_{(1)})\ldots\mathcal{O}(x^\mu_{(m)})\right\rangle = \frac{\delta^m Z}{\delta\phi_0(x^\mu_{(1)})\ldots\delta\phi_0(x^\mu_{(m)})}. \tag{1.19}$$

Since QCD is not a conformal field theory, the direct application of the AdS/CFT correspondence would be meaningless. This problem is fixed by effectively breaking the conformal symmetry in the 5D bulk, which can be achieved in several ways. The in many ways simplest approach is to prevent the ability of the fields $\phi(x^\mu, z)$ to penetrate deeply into the bulk by introducing a hard-wall cutoff at $z = L_1$ (see Figure 1.2). This gives a *hard-wall* or IR brane, hence the name hard-wall model for these types of calculations. L_1 can be thought of as being related to the QCD cutoff via $L_1 \sim \frac{1}{\Lambda_{\mathrm{QCD}}}$. And in the same way, it sets the scale for masses. Hard-wall models are very economic and computationally simple but produce unphysical Regge trajectories, meaning that hadronic states, corresponding to eigenmodes of the bulk fields $\phi(x^\mu, z)$, have mass eigenvalues that behave like

$$m_n \sim n. \tag{1.20}$$

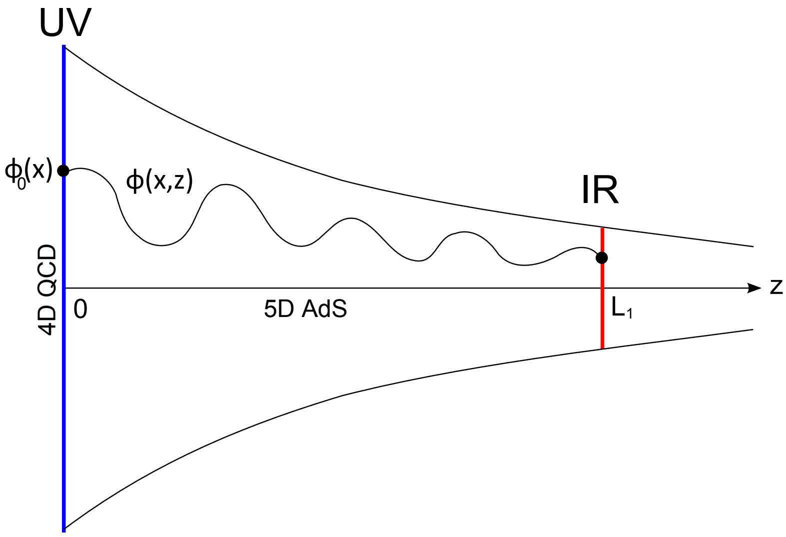

Figure 1.2: Visualization of the hard-wall approach.

This occurs in all models in which one only allows the bulk field to exist up to a limited depth in the bulk. The behaviour consistent with phenomenology are however Regge trajectories of the form

$$m_n^2 \sim n. \tag{1.21}$$

Hard-wall models may be able to make good predictions for ground states but higher radial excitations cannot be studied meaningfully in the hard-wall approach. This has led to the development of the soft-wall models. Here, one breaks the conformal symmetry by introducing a *background field* $\Phi(z)$ (also known as dilaton field), the form of which is chosen to produce the correct Regge trajectories. The range of z is unbounded. Since the conformal symmetry is gradually broken by the gravity background, one refers to these models as *soft-wall* models. Whilst one indeed produces physical mass spectra, soft-wall models are considerably more difficult from a computational point of view and one encounters some finiteness problems. We shall see later what is meant by that and how to resolve the arising issues.

For the sake of completeness, one should also mention that two different methods of making use of the correspondence exist. The idea behind the models presented so far is a *bottom-up* approach. Here, one starts from four-dimensional QCD and attempts to construct a higher dimensional dual theory by incorporating QCD phenomenology. So, for example, one chooses the dilaton field exactly in a way that one obtains correct Regge trajectories. In contrast to this is the *top-down* approach, where one starts from a string theory on the $\mathrm{AdS}_n \times K$, where K is a compact manifold, e.g. a hypersphere. Then, one attempts to derive a low-energy, i.e. strongly coupled, theory resembling QCD on the boundary of the AdS_n-space. While the latter approach might be favourable from a theoretical point of view, it is usually the bottom-up approach which is suitable for phenomenological analysis wherefore we will only deal with this approach in this text.

One goal of this text is to derive the equations of motion governing the fields in the five-dimensional space, find solutions with appropriate boundary conditions, and interpret the results in terms of QCD quantities. We will first do this in the context of a hard-wall model, and then afterwards extend our considerations to soft-wall models, for which we have to overcome some difficulties.

A considerable amount of work will be spent on deriving the equations of motion for the bulk fields starting from the 5D action. One can then find hadrons, two-point correlators, decay constants, and form factors from the solutions of these differential equations. Whenever it is possible, we will try to find analytical solutions but often it is not. So, a large part of the results are found numerically, especially in the soft-wall approach. The main idea is to find values for the free parameters of the model leading to the best agreement with experimental results.

The majority of articles in the field of AdS/QCD work in a flavour symmetric case, considering only the up and down quark, which are assumed to have equal masses. Following [2], we will also incorporate the strange quark, for which a flavour symmetric approach certainly would not give good results. Hence, one must allow flavour symmetry to be broken. Up/down (or isospin) symmetry will be kept and this is in general considered to be a very good approximation.

The models we will derive allow to calculate the masses of vector, pseudovector (axial vector), and pseudoscalar mesons, and even strange scalar mesons. Light scalar mesons are more difficult to obtain, and there is no general consensus on how to treat them. We will thus also consider techniques how scalar mesons can be explicitly built into the model. We will do so in the context of a hard-wall and a soft-wall approach.

2 Hard-Wall Model

In this chapter, we will build up a hard-wall model describing QCD with equal mass up and down quarks and also including a strange quark. So we have that the number of flavours is $N_\mathrm{f} = 3$. We have to begin by incorporating duals in AdS_5 to the QCD operators of interest. These are the current operators $J^a_{L\mu} = \bar{q}_L \gamma_\mu t^a q_L$ and $J^a_{R\mu} = \bar{q}_L \gamma_\mu t^a q_R$ and the quark bilinear $\bar{q}_L q_R$, where

$$q_{L,R} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}_{L,R} \tag{2.1}$$

and the t^a $(a = 1, 2, \ldots, 8)$ are related to the Gell-Mann matrices λ^a via $t^a = \frac{\lambda^a}{2}$ with $\mathrm{Tr}[t^a, t^b] = \frac{1}{2}\delta^{ab}$. (If we had only been considering the case $N_\mathrm{f} = 2$, we would use the Pauli matrices instead.)

The above QCD operators are related to their dual fields via [14, 15]

$$J^a_{L\mu} \to L^a_M(x^\mu, z) \tag{2.2}$$

and

$$J^a_{R\mu} \to R^a_M(x^\mu, z) \tag{2.3}$$

as well as

$$\bar{q}_L q_R \to \frac{2}{z} X(x^\mu, z). \tag{2.4}$$

(The factor 2 in front of the X is purely conventional, the $\frac{1}{z}$ however not.) The two gauge fields $L^a_M(x^\mu, z)$ and $R^a_M(x^\mu, z)$ and the scalar field $X(x^\mu, z)$ will be our main objects of investigation in this text. There is of course an infinite amount of operators in QCD and hence also an infinite amount of bulk fields in the AdS_5 space. The most relevant ones in the meson sector are however those three inroduced above. All other fields will be neglected. The masses of these operators are related via

$$m^2 L^2 = (\Delta - p)(\Delta + p - 4), \tag{2.5}$$

where Δ is the dimension of the QCD (p-form) operator. This results in the masses shown in Table 2.1.

With this preparation, we can write down the 5D action we will be working with. It is [14, 15]:

$$\mathcal{S} = \int \mathrm{d}^5 x \sqrt{g}\, \mathrm{Tr} \left(|D_M X|^2 - m_X^2 |X|^2 - \frac{1}{4g_5^2} \left(F_L^2 + F_R^2 \right) \right) \tag{2.6}$$

4D: $\mathcal{O}(x^\mu)$	5D: $\phi(x^\mu, z)$	p	Δ	m^2L^2
$\bar{q}_L\gamma_\mu t^a q_L$	$L^a_M(x^\mu, z)$	1	3	0
$\bar{q}_L\gamma_\mu t^a q_R$	$R^a_M(x^\mu, z)$	1	3	0
$\bar{q}_L q_R$	$\frac{2}{z}X(x^\mu, z)$	0	3	-3

Table 2.1: Masses of bulk fields.

or written out

$$\begin{aligned}\mathcal{S} = \int \mathrm{d}^5x\sqrt{g}\,\mathrm{Tr}\Bigg((D_MX)^\dagger(D^MX) + \frac{3}{L^2}X^\dagger X \\ - \frac{1}{4g_5^2}\left(F^L_{MN}F^{MN}_L + F^R_{MN}F^{MN}_R\right)\Bigg).\end{aligned} \tag{2.7}$$

Here the field strength F^L_{MN} is defined by

$$F^L_{MN} := \partial_M L_N - \partial_N L_M - i[L_M, L_N] \tag{2.8}$$

and F^R_{MN} analogously. We will expand the gauge fields in terms of the t^a:

$$L_N = L^a_N t^a, \tag{2.9}$$

where the coefficients L^a_N are real numbers. D_M is the covariant derivative through which the scalar field and the gauge fields interact. It is given by

$$D_MX = \partial_MX - iL_MX + iXR_M. \tag{2.10}$$

In a chirally symmetric world, i.e. if all quark masses are zero, the action has $SU(3)_L\times SU(3)_R$ symmetry, i.e. it is invariant under

$$X \to X' = U_LXU_R^\dagger \tag{2.11}$$

as well as

$$L_N \to L'_N = U_LL_N \tag{2.12}$$

and

$$R_N \to R'_N = U_RR_N, \tag{2.13}$$

where $U_L, U_R \in SU(3)$. The quark condensate $\langle\bar{q}q\rangle$ spontaneously breaks this symmetry into the vector subgroup $SU(3)_V$. The corresponding Goldstone Bosons are the pseudoscalar mesons (an octet of them). They would in fact be massless if we had chiral symmetry. However, since quarks have masses to begin with, the $SU(3)_L\times SU(3)_R$ symmetry is only an approximate one, thus giving the mesons a mass, which are then referred to as pseudo-Goldstone bosons. If we restrict ourselves to $N_\mathrm{f} = 2$, the $SU(2)_L\times SU(2)_R$ is almost exact since the up and down quarks are so light. This results in a very low pion mass compared to the other mesons. One also says that the quark masses explicitly break the chiral symmetry, as opposed to the spontaneous symmetry breaking of the quark condensate.

We will also consider the vector and axial vector (pseudovector) fields A and V defined via $L = V + A$ and $R = V - A$.

The Vacuum Solution

Let us first determine the vacuum solution $X_0 := \langle X \rangle$. It is precisely this vacuum expectation value X_0 which spontaneously breaks the (approximate) chiral symmetry by forming the quark condensate. The vacuum field X_0 is determined by turning off all fields except X_0 and solving the equation of motion resulting from the action for a x^μ-independent field. With only the $X_0(z)$ field left, the action simplifies to

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^5x \sqrt{g}\,\mathrm{Tr}\left(\partial_z X_0^\dagger \partial^5 X_0 + \frac{3}{L^2} X_0^\dagger X_0\right) \\ &= \int \mathrm{d}^5x \frac{L^5}{z^5}\,\mathrm{Tr}\left(-\frac{z^2}{L^2}(\partial_z X_0)^2 + \frac{3}{L^2} X_0^\dagger X_0\right).\end{aligned} \tag{2.14}$$

Using

$$\mathrm{Tr}(A^\dagger A) = \sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}|^2 , \tag{2.15}$$

we get

$$\mathcal{S} = \sum_{i,j=1}^{3} \int \mathrm{d}^5x \left(-\frac{L^3}{z^3}(\partial_z X_{0ij})^2 + 3\frac{L^3}{z^5} X_{0ij}^2\right). \tag{2.16}$$

The equation of motion for $X_0(z)$ can be derived by making a variational argument. Since the individual components X_{0ij} of X_0 are independent, this can be done for each of them separately. So, we look at the action for one component

$$\mathcal{S}_{ij} = \int \mathrm{d}^5x \left(-\frac{L^3}{z^3}(\partial_z X_{0ij})^2 + 3\frac{L^3}{z^5} X_{0ij}^2\right) \tag{2.17}$$

and look at an infinitesimal displacement δX_{0ij}, which gives

$$\mathcal{S}_{ij} + \delta\mathcal{S}_{ij} = \int \mathrm{d}^5x \left(-\frac{L^3}{z^3}(\partial_z (X_{0ij} + \delta X_{0ij}))^2 + 3\frac{L^3}{z^5}(X_{0ij} + \delta X_{0ij})^2\right). \tag{2.18}$$

After dropping the quadratic terms in δX_{0ij}, one gets

$$\delta\mathcal{S}_{ij} = \int \mathrm{d}^5x \left(-2\frac{L^3}{z^3}\partial_z X_{0ij}\partial_z \delta X_{0ij} + 6\frac{L^3}{z^5} X_{0ij}\delta X_{0ij}\right). \tag{2.19}$$

Partially integrating with respect to z in the first term gives

$$\begin{aligned}\delta\mathcal{S}_{ij} &= \int \mathrm{d}^5x \left(+2\partial_z\left(\frac{L^3}{z^3}\partial_z X_{0ij}\right)\delta X_{0ij} + 6\frac{L^3}{z^5} X_{0ij}\delta X_{0ij}\right) \\ &= 2\int \mathrm{d}^5x \left(+\partial_z\left(\frac{L^3}{z^3}\partial_z X_{0ij}\right) + 3\frac{L^3}{z^5} X_{0ij}\right)\delta X_{0ij}.\end{aligned} \tag{2.20}$$

Since the action $\mathcal{S}_{ij}$ should be extremal, $\delta\mathcal{S}_{ij}$ should vanish for all infinitesimal displacements δX_{0ij}. So we can read off the resulting equation of motion. It is

$$\partial_z \left(\frac{L^3}{z^3} \partial_z X_{0ij} \right) + 3\frac{L^3}{z^5} X_{0ij} = 0 \tag{2.21}$$

for each component of X_0.

The general solution to this homogeneous linear second order ordinary differential equation is given by the polynomial

$$v_{ij}(z) := 2X_{0ij} = A_{ij}z + B_{ij}z^3. \tag{2.22}$$

(We will always require any normalizable field to vanish in the UV since the integral over z is due to the metric given by $\int \frac{\mathrm{d}z}{z}$, which can only be finite if the integrand tends to zero for $z \to 0$.) By looking at boundary conditions determined by the AdS/QCD correspondence, one can relate the coefficients A_{ij} to the mass matrix elements M_{ij} and the B_{ij} to the quark condensates $\Sigma_{ij} = \langle \bar{q}_i q_j \rangle$ [15]. (How to deal with the boundary conditions explicitly to determine these coefficients, is partly shown in Section 4.2.) One gets

$$v_{ij}(z) = \zeta \frac{M_{ij}}{L} z + \frac{1}{\zeta} \frac{\Sigma_{ij}}{L} z^3, \tag{2.23}$$

where a rescaling parameter[7] ζ as advocated in [16] was also introduced. It is given by

$$\zeta = \frac{\sqrt{N_\mathrm{c}}}{2\pi} = \frac{\sqrt{3}}{2\pi}. \tag{2.24}$$

(This will also be treated in more detail in Section 4.2.)

We assume up/down (isospin) symmetry and set $m_q := m_u = m_d$. So we get

$$M = \begin{pmatrix} m_q & 0 & 0 \\ 0 & m_q & 0 \\ 0 & 0 & m_s \end{pmatrix} \tag{2.25}$$

and

$$\Sigma = \begin{pmatrix} \sigma_q & 0 & 0 \\ 0 & \sigma_q & 0 \\ 0 & 0 & \sigma_s \end{pmatrix}. \tag{2.26}$$

Let us then write

$$v_q(z) := \zeta \frac{m_q}{L} z + \frac{\sigma_q}{\zeta L} z^3 \tag{2.27}$$

and

$$v_s(z) := \zeta \frac{m_s}{L} z + \frac{\sigma_s}{\zeta L} z^3. \tag{2.28}$$

[7] In the end the mass matrix elements and the quark condensates will be treated as parameters of the model and fitted to experimental data, which means that the results are the same with or without the rescaling parameter. The parameters however have a physical meaning themselves and the rescaling parameter ensures that their values correspond to quark masses and condensates in QCD.

With this

$$X_0 = \frac{1}{2}\begin{pmatrix} v_q & 0 & 0 \\ 0 & v_q & 0 \\ 0 & 0 & v_s \end{pmatrix}. \tag{2.29}$$

The Expanded 5D Action

The field X can be expanded as [2]

$$X(x,z) = e^{i\pi^a(x^\mu,z)t^a} X_0(z) e^{i\pi^a(x^\mu,z)t^a}, \tag{2.30}$$

where $\pi = \pi^a t^a$ is the pion field expressed in terms of Gell-Mann-matrices and $X_0(z)$ is the vacuum field, which we have just derived. With flavour symmetry, X_0 would be a multiple of the unit matrix and the above expression would simplify to $X_0 e^{2i\pi^a t^a}$. Here, in three-flavour AdS/QCD this is not the case. At this point, we are neglecting the scalar part of X. We will treat it in Chapter 4.

With the expression for the vacuum field $v(z)$, we can now derive the equations of motion for the fields L and R, which we trade in for V and A, and X, which we trade in for π. Hence, we insert the pion field expansion for X into the action (2.7). The expression involves two matrix exponentials. A matrix exponential is, however, no object one can easily deal with. Especially, the derivative of such an expression, if the exponent is not simply linear in the variable differentiated for, becomes a complicated integral expression. The only chance to derive equations of motions from the action is to expand it in terms of the fields V, A and π. We will consider terms up to quadratic order in the fields, which is sufficient to obtain the masses and other observables.

To simplify the resulting expressions, we will explicitly evaluate the sum over the index a of t^a. Doing this, one multiply encounters the expressions $\mathrm{Tr}\left([t^a, X_0][t^b, X_0]\right)$ and $\mathrm{Tr}\left(\{t^a, X_0\}\{t^b, X_0\}\right)$. Thus, a closer investigation of these terms seems appropriate. One finds that they are zero whenever $a \neq b$. Hence we define the "masses" $M_V^a(z)$ and $M_A^a(z)$ via

$$\frac{1}{2} {M_V^a}^2 = -\,\mathrm{Tr}\left([t^a, X_0][t^a, X_0]\right) \tag{2.31}$$

and

$$\frac{1}{2} {M_A^a}^2 = \mathrm{Tr}\left(\{t^a, X_0\}\{t^a, X_0\}\right). \tag{2.32}$$

Expressing X_0 explicitly in terms of v_q and v_s, we get

$${M_V^a}^2 = \begin{cases} 0 & a = 1,2,3 \\ \frac{1}{4}(v_q - v_s)^2 & a = 4,5,6,7 \\ 0 & a = 8 \end{cases} \tag{2.33}$$

and

$${M_A^a}^2 = \begin{cases} v_q^2 & a = 1,2,3 \\ \frac{1}{4}(v_q + v_s)^2 & a = 4,5,6,7 \\ \frac{1}{3}(v_q^2 + 2v_s^2) & a = 8 \end{cases}. \tag{2.34}$$

These preparations made, one finds that up to quadratic order in the fields π, V, and A the bulk action is given by

$$\begin{aligned}\mathcal{S} = \int \mathrm{d}^5x \sum_a \Bigg(&-\frac{L}{4g_5^2 z}\left(\partial_M V_N^a - \partial_N V_M^a\right)^2 + \frac{{M_V^a}^2 L^3}{2z^3}{V_M^a}^2 \\ &-\frac{L}{4g_5^2 z}\left(\partial_M A_N^a - \partial_N A_M^a\right)^2 + \frac{{M_A^a}^2 L^3}{2z^3}\left(\partial_M \pi^a - A_M^a\right)^2 \Bigg).\end{aligned} \tag{2.35}$$

(The complete calculation can be found in Appendix A.1.) We are using a shorthand notation, where the contraction over η^{MN} is implicit. This means for example that

$$\begin{aligned}(\partial_M V_N^a - \partial_N V_M^a)^2 &= \eta^{MM'}\eta^{NN'}(\partial_M V_N^a - \partial_N V_M^a)(\partial_{M'} V_{N'}^a - \partial_{N'} V_{M'}^a) \\ &= \frac{L^4}{z^4} g^{MM'} g^{NN'}(\partial_M V_N^a - \partial_N V_M^a)(\partial_{M'} V_{N'}^a - \partial_{N'} V_{M'}^a) \\ &= \frac{L^4}{z^4}(\partial_M V_N^a - \partial_N V_M^a)(\partial^M V^{Na} - \partial^N V^{Ma})\end{aligned} \tag{2.36}$$

or similarly

$$(\partial_M \pi^a - A_M^a)^2 = \frac{L^2}{z^2}(\partial_M \pi^a - A_M^a)(\partial^M \pi^a - A^{Ma}). \tag{2.37}$$

Let us also define

$$\alpha^a(z) = \frac{g_5^2 {M_V^a}^2 L^2}{z^2} \tag{2.38}$$

and

$$\beta^a(z) = \frac{g_5^2 {M_A^a}^2 L^2}{z^2}, \tag{2.39}$$

which will simplify many equations. Note that since L appeared to the power -2 in ${M_V^a}^2$ and ${M_A^a}^2$, $\alpha^a(z)$ and $\beta^a(z)$ do not depend on L. More generally, the parameter L has no influence on the results of the calculations since it is only a choice of scale. By setting $L = 1$ one could in principle remove MeV as the last unit. To avoid ambiguities, the sum over a is marked explicitely with a sigma sign in the above action. This convention will be kept throughout the text, which means that there is no implicit summation over a (Einstein notation) unless it is obvious from the context.

Gauge Fixing

We can now take a closer look at the axial sector of the action (2.35), i.e. all the terms involving A. As a consequence of (2.11-2.13), it is invariant under the gauge transformation

$$\begin{aligned}A_M^a \to {A'}_M^a &= A_M^a - \partial_M \lambda^a \\ \pi^a \to {\pi'}^a &= \pi^a - \lambda^a\end{aligned} \tag{2.40}$$

This can be seen directly from

$$\begin{aligned}\partial_M {A'}_N^a - \partial_N {A'}_M^a &= \partial_M (A_N^a - \partial_N \lambda^a) - \partial_N (A_M^a - \partial_M \lambda^a) \\ &= \partial_M A_N^a - \partial_M \partial_N \lambda^a - \partial_N A_M^a + \partial_N \partial_M \lambda^a \\ &= \partial_M A_N^a - \partial_N A_M^a\end{aligned} \tag{2.41}$$

and

$$\begin{aligned}\partial_M \pi'^a - A'^a_M &= \partial_M(\pi^a - \lambda^a) - (A^a_M - \partial_M \lambda^a) \\ &= \partial_M \pi^a - A^a_M.\end{aligned} \tag{2.42}$$

We can then for example choose λ^a in a way that $\partial_z \lambda^a = A^a_z$, which means that we are free to set $A^a_z = 0$. We will see, that this has as a consequence that we do not get light unflavoured scalar mesons, as they vanish due to this gauge transformation.

For the vector sector (i.e. those terms in the action involving V), the mass term looks different and hence destroys the gauge freedom. However for $a = 1, 2, 3, 8$, ${M^a_V}^2 = 0$, which follows from the assumed isospin symmetry. So, at least for those a, we can choose $V^a_z = 0$ following the same arguments we made for A.

2.1 Vector Sector

Having derived the action (2.35), we can now look at the equation of motion it implies for the fields. Let us first look at the vector sector. We have

$$\begin{aligned}\mathcal{S}_V &= \int \mathrm{d}^5x \sum_a \left(-\frac{L}{4g_5^2 z} (\partial_M V^a_N - \partial_N V^a_M)^2 + \frac{\alpha^a(z) L}{2g_5^2 z} {V^a_M}^2 \right) \\ &= \int \mathrm{d}^5x \frac{L}{4g_5^2} \sum_a \left(-\frac{1}{z} \left(\eta^{MM'} \eta^{NN'} (\partial_M V^a_N - \partial_N V^a_M)(\partial_{M'} V^a_{N'} - \partial_{N'} V^a_{M'}) \right) \right. \\ &\quad \left. + \frac{2\alpha^a(z)}{z} \eta^{MM'} V^a_M V^a_{M'} \right).\end{aligned} \tag{2.43}$$

Then, one finds using similar techniques as before (see Section A.2 of appendix for full derivation) that the equation of motion is given by

$$\boxed{\eta^{ML} \partial_M \left(\frac{1}{z} (\partial_L V^a_N - \partial_N V^a_L) \right) + \frac{\alpha^a(z)}{z} V^a_N = 0.} \tag{2.44}$$

The above equation can be written as

$$\begin{aligned}\eta^{\mu\lambda} \partial_\mu \left(\frac{1}{z} (\partial_\lambda V^a_\nu - \partial_\nu V^a_\lambda) \right) - \partial_z \left(\frac{1}{z} (\partial_z V^a_\nu - \partial_\nu V^a_z) \right) + \frac{\alpha^a(z)}{z} V^a_\nu = 0 \\ \eta^{\mu\lambda} \partial_\mu \left(\frac{1}{z} (\partial_\lambda V^a_z - \partial_z V^a_\lambda) \right) + \frac{\alpha^a(z)}{z} V^a_z = 0.\end{aligned} \tag{2.45}$$

The vector field V^a_ν can be decomposed into a transversal and a longitudinal part (Helmholtz decomposition or fundamental theorem of vector calculus)

$$V^a_\mu(x, z) = V^a_{\mu\perp}(x, z) + V^a_{\mu\parallel}(x, z), \tag{2.46}$$

where the transversal part obeys $\eta^{\mu\nu} \partial_\mu V_{\nu\perp} = 0$ or equivalently $\partial^\mu V^a_{\mu\perp} = 0$. On the other hand, the longitudinal part can be written as $V^a_{\mu\parallel} = \partial_\mu \xi^a$ for some ξ^a. The equation of motion then reads

$$
\begin{aligned}
\frac{1}{z}\eta^{\mu\lambda}\partial_\mu\partial_\lambda V^a_{\nu\perp} - \partial_z\left(\frac{1}{z}(\partial_z V^a_{\nu\perp})\right) + \frac{\alpha^a(z)}{z}V^a_{\nu\perp} - \partial_z\left(\frac{1}{z}(\partial_z\partial_\nu\xi^a)\right)& \\
+\frac{\alpha^a(z)}{z}\partial_\nu\xi^a + \partial_z\left(\frac{1}{z}(\partial_\nu V^a_z)\right) &= 0 \\
\frac{1}{z}\eta^{\mu\lambda}\partial_\mu\partial_\lambda V^a_z + \frac{\alpha^a(z)}{z}V^a_z - \frac{1}{z}\eta^{\mu\lambda}\partial_\mu\partial_\lambda\partial_z\xi^a &= 0.
\end{aligned}
\tag{2.47}
$$

To simplify this, we introduce ϖ^a and φ^a so that $V^a_z = -\partial_z\varpi^a$ and $\xi^a = \varphi^a - \varpi^a$. Then the equation of motion simplifies to

$$
\begin{aligned}
\frac{1}{z}\eta^{\mu\lambda}\partial_\mu\partial_\lambda V^a_{\nu\perp} - \partial_z\left(\frac{1}{z}(\partial_z V^a_{\nu\perp})\right) + \frac{\alpha^a(z)}{z}V^a_{\nu\perp} - \partial_z\left(\frac{1}{z}(\partial_\nu\partial_z\varphi^a)\right)& \\
+\frac{\alpha^a(z)}{z}\partial_\nu(\varphi^a - \varpi^a) &= 0 \\
\alpha^a(z)\partial_z\varpi^a + \eta^{\mu\lambda}\partial_\mu\partial_\lambda\partial_z\varphi^a &= 0.
\end{aligned}
\tag{2.48}
$$

It is easier to solve these equations in momentum space. Thus, we apply the Fourier Transform. Since many different conventions regarding its definition exist, one should mention that throughout the text the following convention is used: The 4D Fourier Transform of $f(x^\mu, z)$ is given by $\widehat{f}(k^\nu, z) = \int \mathrm{d}^4x e^{i\eta_{\nu\mu}k^\nu x^\mu} f(x^\mu, z)$. Then, by using the differentiation rules, the above partial differential equations are transformed into ordinary ones. One obtains

$$
\begin{aligned}
-\frac{1}{z}k^2\widehat{V}^a_{\nu\perp} - \partial_z\left(\frac{1}{z}(\partial_z\widehat{V}^a_{\nu\perp})\right) + \frac{\alpha^a(z)}{z}\widehat{V}^a_{\nu\perp} + ik_\nu\partial_z\left(\frac{1}{z}(\partial_z\widehat{\varphi}^a)\right)& \\
-i\frac{\alpha^a(z)}{z}k_\nu(\widehat{\varphi}^a - \widehat{\varpi}^a) &= 0 \\
\alpha^a(z)\partial_z\widehat{\varpi}^a - k^2\partial_z\widehat{\varphi}^a &= 0,
\end{aligned}
\tag{2.49}
$$

where we defined $k^2 := \eta_{\mu\nu}k^\mu k^\nu$. If we now multiply the upper equation by k^ν, all terms involving $\widehat{V}^a_{\nu\perp}$ will vanish since $\partial^\nu V^a_{\nu\perp} = 0$. We divide the resulting equation by k^2 and together with the second equation get a pair of equations for the longitudinal part and the z-component:

$$
\begin{aligned}
\partial_z\left(\frac{1}{z}(\partial_z\widehat{\varphi}^a)\right) - \frac{\alpha^a(z)}{z}(\widehat{\varphi}^a - \widehat{\varpi}^a) &= 0 \\
\alpha^a(z)\partial_z\widehat{\varpi}^a - k^2\partial_z\widehat{\varphi}^a &= 0.
\end{aligned}
\tag{2.50}
$$

It then follows that the transverse terms vanish independently, which gives the equation for the transverse part:

$$
\partial_z\left(\frac{1}{z}(\partial_z\widehat{V}^a_{\mu\perp})\right) + \frac{k^2 - \alpha^a(z)}{z}\widehat{V}^a_{\mu\perp} = 0. \tag{2.51}
$$

These equations can now be solved independently, as done in the following.

2.1.1 The Transverse Part

We will begin by solving the differential equation for the transverse part (2.51). Depending on the boundary conditions one obtains the bulk-to-boundary propagator or normalizable hadron modes.

Bulk-to-Boundary Propagator
Let us write the vector field $\widehat{V}^a_{\mu\perp}$ as a product of its UV-boundary value $\widehat{V}^{0a}_{\mu\perp}(k^\nu)$ and the bulk-to-boundary propagator (or profile function) $\mathcal{V}^a(k^\nu, z)$

$$\widehat{V}^a_{\mu\perp}(k^\nu, z) = \widehat{V}^{0a}_{\mu\perp}(k^\nu)\mathcal{V}^a(k^\nu, z), \tag{2.52}$$

where $\mathcal{V}^a(k^2, L_0) = 1$ by definition. $\widehat{V}^{0a}_{\mu\perp}(k^\nu)$ acts as the (Fourier transform of the) source of the 4D vector current operator dual to the bulk field $V^a_{\mu\perp}$. While in this case the coupling to the source determines the boundary condition in the UV, for hadrons we will have to choose a different UV boundary condition so that the solution is normalizable. $\mathcal{V}^a$ obviously fulfills the same differential equation (2.51) as $\widehat{V}^a_{\mu\perp}$, namely

$$\partial_z \left(\frac{1}{z}(\partial_z \mathcal{V}^a) \right) + \frac{k^2 - \alpha^a(z)}{z}\mathcal{V}^a = 0. \tag{2.53}$$

To obtain a unique solution we must also add an additional boundary condition. It is a common assumption in hard-wall models that the derivative with respect to z of any field should vanish at the IR brane, i.e. $\partial_z \mathcal{V}^a(k^2, L_1) = 0$. This guarantees the boundary terms in the IR to vanish.

Since equation (2.53) is of order two, given two boundary conditions, we can in principle solve the given boundary value problem (BVP). In general, if α^a depends on z, a solution to the above differential equation can only be found numerically.

Analytical Solution
If however $a = 1, 2, 3, 8$, then $\alpha^a(z) = 0$ and thus the differential equation simplifies and one obtains an analytic solution. The solution can be expressed in terms of Bessel functions

$$\mathcal{V}^a(k^2, z) = \frac{z}{L_0}\frac{J_1(kz)Y_0(kL_1) - Y_1(kz)J_0(kL_1)}{J_1(kL_0)Y_0(kL_1) - Y_1(kL_0)J_0(kL_1)}. \tag{2.54}$$

In the limit $L_0 \to 0$ this becomes

$$\mathcal{V}^a(k^2, z) = \frac{\pi}{2}kz\left(\frac{J_1(kz)Y_0(kL_1)}{J_0(kL_1)} - Y_1(kz)\right). \tag{2.55}$$

The situation for $a = 4, 5, 6, 7$ is more complicated. Here $\alpha^a(z)$ does not vanish. In general $\alpha^a(z)$ is an even polynomial in z of order 4 ($\alpha^a(z) = A + Bz^2 + Cz^4$). In this case, there exists no analytical solution. If we however assume a certain symmetry, namely that $\sigma_q = \sigma_s$, then α^a becomes a constant and the solution can again be written in terms of Bessel functions. It is

$$\mathcal{V}^a(k^2, z) = \frac{\pi}{2}\sqrt{\alpha^a - k^2}z\left(-\frac{I_1(\sqrt{\alpha^a - k^2}z)Y_0(-i\sqrt{\alpha^a - k^2}L_1)}{I_0(\sqrt{\alpha^a - k^2}L_1)} + iY_1(-i\sqrt{\alpha^a - k^2}z)\right). \tag{2.56}$$

To work with real arguments only, let us make a case distinction. For $k^2 > \alpha^a$ we define $\widetilde{k} = \sqrt{k^2 - \alpha^a}$. Then

$$\mathcal{V}^a(k^2, z) = \frac{\pi}{2}\widetilde{k}z \left(J_1(\widetilde{k}z)\frac{Y_0(\widetilde{k}L_1)}{J_0(\widetilde{k}L_1)} - Y_1(\widetilde{k}z) \right). \tag{2.57}$$

If on the other hand $k^2 < \alpha^a$, then we define $\widetilde{K} = \sqrt{\alpha^a - k^2}$ and get the solution

$$\mathcal{V}^a(k^2, z) = \widetilde{K}z \left(I_1(\widetilde{K}z)\frac{K_0(\widetilde{K}L_1)}{I_0(\widetilde{K}L_1)} + K_1(\widetilde{K}z) \right). \tag{2.58}$$

One can easily check that the solution for $\alpha^a = 0$ (2.55) is a special case of the solution for constant α^a above, which it has to be if the calculations are correct.

The profile function will be needed later, since it appears in expressions for form factors.

2.1.2 Longitudinal part

The equation (2.50) for the longitudinal part and the z-component of V^a (in momentum space) is

$$\begin{aligned} \partial_z\left(\frac{1}{z}(\partial_z\widehat{\varphi}^a)\right) - \frac{\alpha^a(z)}{z}(\widehat{\varphi}^a - \widehat{\varpi}^a) = 0 \\ \alpha^a(z)\partial_z\widehat{\varpi}^a - k^2\partial_z\widehat{\varphi}^a = 0, \end{aligned} \tag{2.59}$$

where we defined $V^a_{\mu\|} = \partial_\mu\xi^a$, $V^a_z = -\partial_z\varpi^a$, and $\xi^a = \varphi^a - \varpi^a$.

For $a = 1, 2, 3, 8$, $\alpha^a = 0$ and we obtain no nonzero solutions at all since $\widehat{\varphi}^a = 0$ vanishes because of the differential equation and $\widehat{\varpi}^a = 0$ since we could gauge away V^a_z and $\widehat{V}^a_z = -\partial_z\widehat{\varpi}^a$. Hence also $\widehat{\xi}^a = \widehat{\varphi}^a - \widehat{\varpi}^a$ vanishes.

When α^a is constant and nonzero (i.e. for $\sigma_q = \sigma_s$ and $a = 4, 5, 6, 7$), $\widehat{\varpi}^a$ and $\widehat{\varphi}^a$ are just multiples of each other. It is then possible to decouple the above equations and obtain a single differential equation for $\widehat{\varphi}^a - \widehat{\varpi}^a = \widehat{\xi}^a$:

$$\partial_z\left(\frac{1}{z}(\partial_z\widehat{\xi}^a)\right) + \frac{k^2 - \alpha^a}{z}\widehat{\xi}^a = 0. \tag{2.60}$$

This is exactly the differential equation we had for the Fourier transform of the transverse part. We can then look at the profile function for the longitudinal part $\mathcal{W}^a(k^2, z)$ given via the relation

$$\xi^a(k^\mu, z) = \xi^a_0(k^\mu)\mathcal{W}^a(k^2, z), \tag{2.61}$$

which results in exactly the same solution we had for the transverse part (2.57 and 2.58), so $\mathcal{V}^a = \mathcal{W}^a$. In general, if $\sigma_s \neq \sigma_q$, this does not occur. Then, we must solve the pair of differential equations (2.59) for $\widehat{\varphi}^a$ and $\widehat{\varpi}^a$. A choice of UV boundary conditions could be $\widehat{\varphi}^a(k^2, L_0) = 0$, $\widehat{\varpi}^a(k^2, L_0) = -1$, which corresponds to $\mathcal{W}^a(k^2, 0) = 1$. In the IR we have $\partial_z\widehat{\varphi}^a(k^2, L_1) = \partial_z\widehat{\varpi}^a(k^2, L_1) = 0$. In the general case, the solution must be found numerically.

2.1.3 Two-Point functions

As explained in the introduction, using the AdS/QCD correspondence, we can easily calculate two point correlators (also called two-point functions or Green functions). Let us consider the transverse two point-function function $i\int \mathrm{d}^4x\, e^{i\eta_{\lambda\rho}k^\lambda x^\rho}\left\langle J_\perp^{\mu a}(x)J_\perp^{\nu b}(0)\right\rangle$ and its longitudinal counterpart $i\int \mathrm{d}^4x\, e^{i\eta_{\lambda\rho}k^\lambda x^\rho}\left\langle J_\parallel^{\mu a}(x)J_\parallel^{\nu b}(0)\right\rangle$. We calculate them by differentiating the action (2.35) twice with respect to the source V_μ^0 after inserting (2.52) and (2.61). This gives

$$i\int \mathrm{d}^4x\, e^{i\eta_{\lambda\rho}k^\lambda x^\rho}\left\langle J_\perp^{\mu a}(x)J_\perp^{\nu b}(0)\right\rangle = -P_\perp^{\mu\nu}\delta^{ab}\frac{\partial_z\mathcal{V}^a(k^2,L_0)}{g_5^2L_0} \tag{2.62}$$

and

$$i\int \mathrm{d}^4x\, e^{i\eta_{\lambda\rho}k^\lambda x^\rho}\left\langle J_\parallel^{\mu a}(x)J_\parallel^{\nu b}(0)\right\rangle = -P_\parallel^{\mu\nu}\delta^{ab}\frac{\partial_z\widehat{\varphi}^a(k^2,L_0)}{g_5^2L_0}, \tag{2.63}$$

where we have defined the transverse projector $P_\perp^{\mu\nu}=\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)$ and the longitudinal projector $P_\parallel^{\mu\nu}=\frac{k^\mu k^\nu}{k^2}$. (How such a calculation is done in principle, can be seen in Chapter 4.) Comparing the above results with the QCD result to leading order, one can determine the parameter g_5 of the model [14, 15]. One gets

$$g_5^2=\frac{12\pi^2}{N_\mathrm{c}}=4\pi^2. \tag{2.64}$$

2.1.4 Normalizable Solutions

Let us now look for normalizable modes of the 5D fields. The solutions of the above derived equations of motion correspond to mesons. Compared to the calculations above we have to change the boundary conditions to account for that.

Our splitting of the vector field into a transversal and a longitudinal part now has a concrete physical meaning. Vector mesons we obtain as solutions of the differential equation for the transverse part (2.51) and scalar mesons as solutions for the longitudinal part (2.59).

2.1.4.1 Vector Mesons

Let us study the vector mesons in detail. We will denote their wave functions by $\psi^a(z)$. In the case where α^a is constant, i.e. for $\sigma_q=\sigma_s$ or $a=1,2,3,8$, there is an analytical solution in terms of Bessel functions. In the other cases one can find the solution numerically.

We want $\psi^a(z)$ to be a normalizable solution of the differential equation for the transverse part (2.51). This means we want the integral

$$\int_{L_0}^{L_1}\frac{\mathrm{d}z}{z}\psi^{a2}(z) \tag{2.65}$$

to be finite. For this we must impose the boundary condition $\psi^a(L_0) = 0$. In addition, we maintain the Neumann boundary condition $\partial_z \psi^a(L_1) = 0$ so that the boundary terms vanish. Since the differential equation is linear and the boundary conditions are homogeneous, every multiple of a solution is again a solution. This arbitrariness can be removed by specifying another boundary condition like $\partial_z \psi^a(L_0) = c$, where $c \neq 0$. (The actual value of c is unimportant at this point since we will normalize the solution anyway.) But this means that the boundary value problem is overdetermined since we have an order two differential equation, but three boundary conditions. A solution only exists for certain values of the parameter k^2. Since solutions of this kind of differential equation are in general oscillatory, we get an infinite discrete spectrum of $k^2 =: M_n^{a2}$ ($n = 1, 2, \ldots$) for which a solution exists. M_n^a is of course the mass of the corresponding eigenmode, which we call $\psi_n^a(z)$. Higher n correspond to radial excitations.

On can then identify the different vector mesons in the octet for $a = 1, \ldots, 8$. Since we were assuming up/down symmetry, certain cases of a are identical. For $a = 1, 2, 3$ we get an infinite tower of ρ mesons (ρ^+,ρ^-, and ρ^0) and for $a = 4, 5, 6, 7$ we get K^* mesons (K^{*+}, K^{*-}, K^{*0}, and $\overline{K}^{*0}$)[2].

We normalize the solutions such that the integral (2.65) equals to one. In addition, the eigenmodes $\psi_n^a(z)$ obey an orthogonality relation. Hence

$$\int_{L_0}^{L_1} \frac{\mathrm{d}z}{z} \psi_n^a(z) \psi_m^a(z) = \delta_{nm}. \tag{2.66}$$

To show this orthogonality, we look at

$$\int_{L_0}^{L_1} \mathrm{d}z \frac{\alpha^a(z) - M_n^{a2}}{z} \psi_n^a(z) \psi_m^a(z) = \underbrace{\left. \frac{\partial_z \psi_n^a(z) \psi_m^a(z)}{z} \right|_{L_0}^{L_1}}_{=0} - \int_{L_0}^{L_1} \frac{\mathrm{d}z}{z} \partial_z \psi_n^a(z) \partial_z \psi_m^a(z), \tag{2.67}$$

where we used partial integration and the differential equation (2.51) to find that $\frac{\partial_z \psi_n^a(z)}{z}$ is the primitive of $\frac{\alpha^a(z) - M_n^{a2}}{z} \psi_n^a(z)$. Then, we can interchange m and n and by symmetry get

$$\int_{L_0}^{L_1} \mathrm{d}z \frac{\alpha^a(z) - M_n^{a2}}{z} \psi_n^a(z) \psi_m^a(z) = \int_{L_0}^{L_1} \mathrm{d}z \frac{\alpha^a(z) - M_m^{a\,2}}{z} \psi_n^a(z) \psi_m^a(z), \tag{2.68}$$

which implies (using the linearity of the integral)

$$M_n^{a2} \int_{L_0}^{L_1} \frac{\mathrm{d}z}{z} \psi_n^a(z) \psi_m^a(z) = M_m^{a\,2} \int_{L_0}^{L_1} \frac{\mathrm{d}z}{z} \psi_n^a(z) \psi_m^a(z). \tag{2.69}$$

Since $M_n^a \neq M_m^a$ for $n \neq m$, this can only mean that

$$\int_{L_0}^{L_1} \frac{\mathrm{d}z}{z} \psi_n^a(z) \psi_m^a(z)) = 0 \tag{2.70}$$

for $n \neq m$, which was the assertion.

Analytical solution

We have seen that in many cases, there exists an analytical solution of the differential equation in terms of Bessel functions, namely if α^a is constant, which also includes the case when it is zero.

If we had not imposed boundary conditions, the general solution to the differential equation (2.51) with constant α^a would have been

$$\psi^a(z) = -ic_1 z I_1(\sqrt{\alpha^a - k^2} z) + c_2 z Y_1(-i\sqrt{\alpha^a - k^2} z). \tag{2.71}$$

Imposing the boundary condition that $\psi^a(z)$ vanishes at the UV boundary, we see that term containing I_1 fulfills this, the term containing Y_1 however not, so we have to drop it. We then rewrite $\psi^a(z)$ as

$$\psi^a(z) = czJ_1(\sqrt{k^2 - \alpha^a} z). \tag{2.72}$$

The Neumann boundary condition in the IR gives the condition

$$J_0(\sqrt{k^2 - \alpha^a} L_1) = 0 \tag{2.73}$$

(using $\partial_z(zJ_1(z)) = zJ_0(z)$), which generates the discrete mass spectrum $k^2 = M_n^{a2}$. If r_n denotes the n-th zero of J_0, the masses are given by

$$M_n^a = \sqrt{\frac{r_n^2}{L_1^2} + \alpha^a}. \tag{2.74}$$

Using the approximation of J_0 for large z

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right), \tag{2.75}$$

we find that they are approximately given by

$$\sqrt{M_n^{a2} - \alpha^a} \approx \frac{\pi}{L_1}\left(n - \frac{1}{4}\right) \tag{2.76}$$

($n = 1, 2, \ldots$). Here, it is important to note that this corresponds to a mass spectrum of the form $M_n^2 \sim n^2$, which is unphysical.

Using the mass condition (2.74) and normalizing the solution, we get

$$\psi_n^a(z) = \frac{\sqrt{2} z J_1\left(\sqrt{M_n^{a2} - \alpha^a} z\right)}{L_1 J_1\left(\sqrt{M_n^{a2} - \alpha^a} L_1\right)} \tag{2.77}$$

or, using r_n,

$$\psi_n^a(\tilde{z}) = \sqrt{2}\tilde{z}\frac{J_1(r_n\tilde{z})}{J_1(r_n)}, \tag{2.78}$$

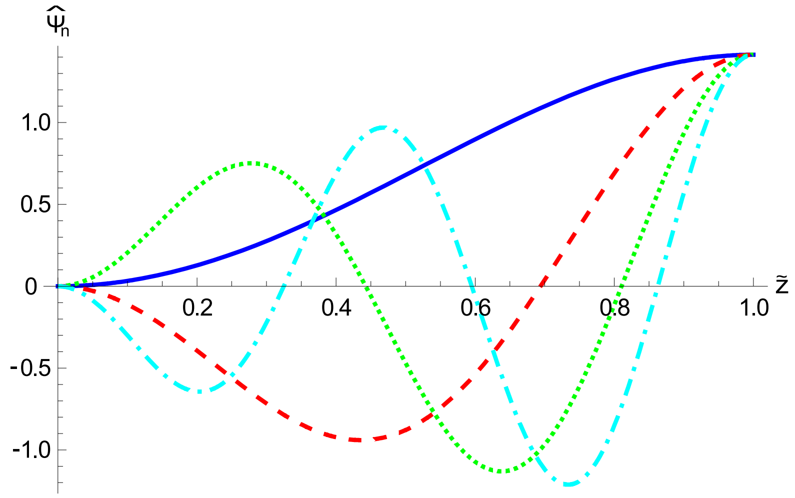

Figure 2.1: The first four modes of $\psi_n^a(\tilde{z})$.

where we introduced $\tilde{z} = \frac{z}{L_1}$. The four lightest vector meson modes are plotted in Figure 2.1.

We can explicitly check the orthonormality of the $\psi_n^a(z)$:

$$\int_0^{L_1} \frac{\mathrm{d}z}{z} \psi_n^a(z)\psi_m^a(z) = \left\{ \begin{array}{ll} \frac{r_m J_0(r_m)J_1(r_n) - r_n J_0(r_n)J_1(r_m)}{r_n^2 - r_m^2} = 0 & \text{if } m \neq n \\ 1 - \frac{J_0(r_n)J_2(r_n)}{J_1^2(r_n)} = 1 & \text{if } m = n \end{array} \right\} = \delta_{mn}. \quad (2.79)$$

Two-Point Function as a Sum over Meson Modes

Let us turn back to the profile function $\mathcal{V}^a(k^2, z)$. In the analytical case there is a $J_0(\widetilde{k}L_1)$ in the denominator. This creates poles when $\widetilde{k}L_1$ is a zero of J_0, which is exactly the mass condition. This indicates that we can in general write the profile function as a sum over meson poles.

To derive this, we first make a general ansatz and write

$$\mathcal{V}^a(k^2, z) = \sum_n c_n^a(k^2)\psi_n^a(z). \quad (2.80)$$

Now we want to calculate the coefficients $c_n^a(k^2)$. For this we multiply the above equation by $\frac{\psi_m^a(z)}{z}$ and integrate. The orthonormality condition (2.79) yields

$$c_m^a(k^2) = \int_{L_0}^{L_1} \mathrm{d}z \frac{\psi_m^a(z)\mathcal{V}^a(k^2, z)}{z}. \quad (2.81)$$

To evaluate this integral, we make use of the following trick:

$$\begin{aligned} c_m^a(k^2) &= \int_{L_0}^{L_1} \frac{\mathrm{d}z}{z} \psi_m^a(z)\mathcal{V}^a(k^2, z) \\ &= \frac{1}{{M_m^a}^2 - k^2} \int_{L_0}^{L_1} \mathrm{d}z \frac{(\alpha^a(z) - k^2) - (\alpha^a(z) - {M_m^a}^2)}{z} \psi_m^a(z)\mathcal{V}^a(k^2, z). \end{aligned} \quad (2.82)$$

So we have

$$
\begin{aligned}
(M_m^{a\,2} - k^2)c_m^a(k^2) &= \int_{L_0}^{L_1} \mathrm{d}z \frac{\alpha^a(z) - k^2}{z} \psi_m^a(z)\mathcal{V}^a(k^2, z) \\
&\quad - \int_{L_0}^{L_1} \mathrm{d}z \frac{\alpha^a(z) - M_m^{a\,2}}{z} \psi_m^a(z)\mathcal{V}^a(k^2, z) \\
&= \underbrace{\frac{\psi_m^a(z)\partial_z \mathcal{V}^a(k^2, z)}{z}\bigg|_{L_0}^{L_1}}_{=0} - \int_{L_0}^{L_1} \frac{\mathrm{d}z}{z} \partial_z \psi_m^a(z) \partial_z \mathcal{V}^a(k^2, z) \qquad (2.83) \\
&\quad - \frac{\partial_z \psi_m^a(z)\mathcal{V}^a(k^2, z)}{z}\bigg|_{L_0}^{L_1} + \int_{L_0}^{L_1} \frac{\mathrm{d}z}{z} \partial_z \psi_m^a(z) \partial_z \mathcal{V}^a(k^2, z) \\
&= \frac{\partial_z \psi_m^a(L_0)}{L_0}.
\end{aligned}
$$

Solving for $c_m^a(k^2)$ finally gives

$$
c_m^a(k^2) = -\frac{1}{k^2 - M_m^{a\,2}} \left. \frac{\partial_z \psi_m^a(z)}{z} \right|_{L_0}. \tag{2.84}
$$

If we define

$$
F_n^a := \frac{\partial_z \psi_n^a(L_0)}{g_5 L_o}, \tag{2.85}
$$

then we can write the profile function as

$$
\mathcal{V}^a(k^2, z) = \sum_n \frac{-g_5 F_n^a \psi_n^a(z)}{k^2 - M_n^{a2}}. \tag{2.86}
$$

In the analytical case, in the limit of $L_0 \to 0$ we get

$$
F_n^a = \frac{\sqrt{2} r_n}{g_5 L_1^2 J_1(r_n)} \tag{2.87}
$$

One can also identify F_n^a as the decay constant of the n-th KK vector meson [2]. This will help us later to compare our model to experimental data. By our choice of normalization, F_n^a has the dimension of a mass squared.

2.1.4.2 Scalar Mesons

If we are looking for normalizable solutions of the longitudinal part corresponding to scalar mesons, we have to solve the pair of equations (2.59) for $\widehat{\varphi}_n^a$ and $\widehat{\varpi}_n^a$. When α^a is just a constant, then we have seen that we can instead solve the single equation (2.60) for $\widehat{\xi}_n^a$. Again, in both cases, the subscript n indicates a discrete mass spectrum. Reasonable boundary conditions are $\widehat{\varphi}_n^a(L_0) = \widehat{\varpi}_n^a(L_0) = 0$ (or $\widehat{\xi}_n^a(L_0) = 0$) and $\partial_z \widehat{\varphi}_n^a(L_1) = \partial_z \widehat{\varpi}_n^a(L_1) = 0$ (or $\partial_z \widehat{\xi}_n^a(L_1) = 0$). This would produce exactly the same solutions for ξ_n^a as the ones we got for ψ_n^a in the transverse case, which means that vector and scalar mesons will have the same mass in the $\sigma_s = \sigma_q$ limit.

In general we must consider the equation (2.59). This can only be done numerically. It turns out that this differential equation is identical to the one for the longitudinal part of the axial sector, which we will discuss in detail in the following Section 2.2. We will also adopt the normalization discussed in that section.

We argued that for $a = 1, 2, 3, 8$ the longitudinal modes are unphysical. So, we can only consider $a = 4, 5, 6, 7$, which corresponds to a K_0^* (κ) meson.

2.2 Axial Sector

We will now treat the axial sector of the action (2.35) in very much the same way as we treated the vector part. Using $\beta^a(z)$ the relevant action reads as

$$\mathcal{S}_A = \int \mathrm{d}^5x \sum_a \left(-\frac{L}{4g_5^2 z} \left(\partial_M A_N^a - \partial_N A_M^a\right)^2 + \frac{\beta^a(z)L}{2zg_5^2} \left(\partial_M \pi^a - A_M^a\right)^2 \right). \tag{2.88}$$

The equation of motion for the axial field A can be derived similarly to the one for the vector field. One finds

$$\eta^{ML}\partial_M \left(\frac{1}{z}(\partial_L A_N^a - \partial_N A_L^a) \right) + \frac{\beta^a(z)}{z}(A_N^a - \partial_N \pi^a) \tag{2.89}$$

or equivalently

$$\begin{aligned} \eta^{\mu\lambda}\partial_\mu \left(\frac{1}{z}(\partial_\lambda A_\nu^a - \partial_\nu A_\lambda^a) \right) - \partial_z \left(\frac{1}{z}(\partial_z A_\nu^a - \partial_\nu A_z^a) \right) + \frac{\beta^a(z)}{z}(A_\nu^a - \partial_\nu \pi^a) = 0 \\ \eta^{\mu\lambda}\partial_\mu \left(\frac{1}{z}(\partial_\lambda A_z^a - \partial_z A_\lambda^a) \right) + \frac{\beta^a(z)}{z}(A_z^a - \partial_z \pi^a) = 0. \end{aligned} \tag{2.90}$$

We again introduce the longitudinal and transverse part of A_μ^a and write $A_{\mu\|}^a = \partial_\mu \phi^a$. Then we get

$$\begin{aligned} \frac{1}{z}\eta^{\mu\lambda}\partial_\mu\partial_\lambda A_{\nu\perp}^a - \partial_z \left(\frac{1}{z}(\partial_z A_{\nu\perp}^a) \right) + \frac{\beta^a(z)}{z} A_{\nu\perp}^a - \partial_z \left(\frac{1}{z}(\partial_z \partial_\nu \phi^a) \right) \\ + \frac{\beta^a(z)}{z}\partial_\nu \phi^a - \frac{\beta^a(z)}{z}\partial_\nu \pi^a = 0 \\ -\frac{1}{z}\eta^{\mu\lambda}\partial_\mu\partial_\lambda\partial_z \phi^a - \frac{\beta^a(z)}{z}\partial_z \pi^a = 0, \end{aligned} \tag{2.91}$$

were we dropped the A_z^a terms since we showed that they can be set to zero. Switching to momentum space gives

$$\begin{aligned} -\frac{1}{z}k^2 A_{\nu\perp}^a - \partial_z \left(\frac{1}{z}(\partial_z A_{\nu\perp}^a) \right) + \frac{\beta^a(z)}{z} A_{\nu\perp}^a + ik_\nu \partial_z \left(\frac{1}{z}(\partial_z \phi^a) \right) \\ -ik_\nu \frac{\beta^a(z)}{z}\phi^a + ik_\nu \frac{\beta^a(z)}{z}\pi^a = 0 \\ k^2 \frac{1}{z}\partial_z \phi^a - \frac{\beta^a(z)}{z}\partial_z \pi^a = 0. \end{aligned} \tag{2.92}$$

Following the same arguments as made for the vector sector, we split this into an equation for the transverse part and one for the longitudinal part. For the transverse part, we get exactly the same solutons as in the vector sector (replacing α^a by β^a), namely

$$\partial_z \left(\frac{1}{z} (\partial_z \widehat{A}^a_{\mu\perp}) \right) + \frac{k^2 - \beta^a(z)}{z} \widehat{A}^a_{\mu\perp} = 0, \tag{2.93}$$

and the solutions are analogous. The equations for the longitudinal part of the axial field and the π field read:

$$\begin{aligned} \partial_z \left(\frac{1}{z} (\partial_z \widehat{\phi}^a) \right) - \frac{\beta^a(z)}{z} (\widehat{\phi}^a - \widehat{\pi}^a) = 0 \\ k^2 \partial_z \widehat{\phi}^a - \beta^a(z) \partial_z \widehat{\pi}^a = 0. \end{aligned} \tag{2.94}$$

Note that β^a will not simplify to a constant in the case of $\sigma_q = \sigma_s$.

One can combine the above pair of coupled equations into a single differential equation of second order by defining $y^a(k^2, z) = \frac{1}{z} \partial_z \widehat{\phi}^a(k^2, z)$. The above equations then read

$$\begin{aligned} -\frac{z}{\beta^a(z)} \partial_z \left(y^a \right) + \widehat{\phi}^a - \widehat{\pi}^a = 0 \\ \frac{k^2 z}{\beta^a(z)} y^a - \partial_z \widehat{\pi}^a = 0. \end{aligned} \tag{2.95}$$

Differentiating the upper equation and inserting the expression for $\partial_z \widehat{\pi}^a$ obtained from the lower equation, one gets

$$\partial_z \left(\frac{z}{\beta^a(z)} \partial_z y^a(k^2, z) \right) + z \left(\frac{k^2}{\beta^a(z)} - 1 \right) y^a(k^2, z) = 0. \tag{2.96}$$

Analogously to the longitudinal part of the vector field we use the boundary conditions $\widehat{\phi}^a(k^2, L_0) = 0$, $\widehat{\pi}^a(k^2, L_0) = -1$ and $\partial_z \widehat{\phi}^a(k^2, L_1) = \partial_z \widehat{\pi}^a(k^2, L_1) = 0$ to obtain the profile function, coupled to the 4D pseudoscalar source. This corresponds to the boundary conditions $y^a(k^2, L_1) = 0$ and, using equation (2.94), $\partial_z y^a(k^2, L_0) = \frac{\beta^a(L_0)}{L_0}$.

An analytic solution cannot be found, but close to the UV cutoff the profile function can be written as [2]

$$y^a(k^2, L_0) = \beta^a(L_0) \log(k L_0) + c \log(k L_0)(k L_0)^2, \tag{2.97}$$

where c is some constant.

2.2.1 Normalizable Solutions

Again, normalizable solutions for the transverse and longitudinal part correspond to hadrons. The transverse modes are pseudovector (axial) mesons. The solutions are exactly those of the vector mesons with α^a replaced by β^a.

Longitudinal modes describe pseudoscalar hadrons, i.e. pions (π^+, π^-, and π^0) for $a = 1, 2, 3$, and kaons (K^+, K^-, K^0, $\overline{K}^0$) for $a = 4, 5, 6, 7$. Let us study these again in detail. To do so, we turn back to our original pair of differential equations (2.94) for the longitudinal part. We will denote the normalizable solutions $\widehat{\pi}_n^a(z)$ and $\widehat{\phi}_n^a(z)$ respectively. For them to be normalizable ($\int \frac{\mathrm{d}z}{z}\widehat{\pi}_n^{a2}$ and $\int \frac{\mathrm{d}z}{z}\widehat{\phi}_n^{a2}$ are finite), we have to impose the UV boundary conditions $\widehat{\phi}_n^a(L_0) = 0$ and $\widehat{\pi}_n^a(L_0) = 0$. In addition we want that $\partial_z\widehat{\phi}_n^a(L_1) = 0$ and $\partial_z\widehat{\pi}_n^a(L_1) = 0$ (the two last conditions imply each other as can be easily seen from the differential equation). It is apparent that for any solution, a multiple of that solution is also a solution of the ODE (including the boundary conditions). So we have to fix one additional boundary condition, e.g. the derivative of $\widehat{\phi}_n^a$ in $z = L_0$. This arbitrariness will vanish when the solutions are normalized.

If all the parameters are known (m_q,m_s,σ_q,σ_s,L_1), the above boundary value problem can be solved numerically and it turns out that (exactly as in the vector case) there is an infinite but discrete set of values for $k^2 =: m_n^{a\,2}$ for which the solutions to the BVP exist, as was already indicated by the index n. The two first modes are shown in Figure 2.2.

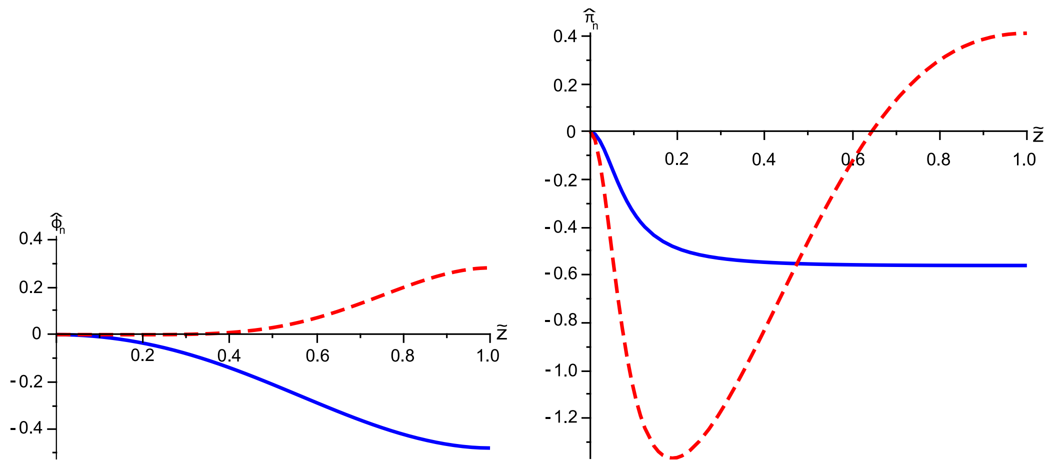

Figure 2.2: Plot of the first two modes of $\widehat{\phi}_n^a(\tilde{z})$ (left) and $\widehat{\pi}_n^a(\tilde{z})$ (right) in units of L_1. The blue (solid) graphs corresponds to $n = 1$ and the red (dashed) ones to $n = 2$.

We could just as well have studied the second order differential equation (2.96) instead. Normalizable modes ($\widehat{\pi}_n^a(z)$ and $\widehat{\phi}_n^a(z)$) correspond to normalizable solutions of equation (2.96), which we shall call $\eta_n^a(z)$. The boundary conditions above can be written in terms of η_n^a. One gets the IR boundary condition $\eta_n^a(L_1) = 0$ and, by using equation (2.94), $\partial_z\eta_n^a(L_0) = 0$ in the UV.

One also finds a generalized orthogonality relation for the η_n^a's. After normalization, which for convenience is chosen to depend on the mass, one gets

$$\int_{L_0}^{L_1} \mathrm{d}z \frac{z}{\beta^a(z)}\eta_n^a(z)\eta_m^a(z) = \frac{\delta_{mn}}{m_n^{a\,2}}. \tag{2.98}$$

The calculation is analogous to yet sligtly more complicated than the one we made for the ψ_n^a's and can be found in Appendix A.3.

Two-Point Function as a Sum over Meson Modes

We can also try to write $y^a(k^2, z)$ as a sum over meson poles. Using the orthonormality relation and partial integration one gets

$$c_n^a(K^2) = \frac{m_n^{a\,2}}{k^2 - m_n^{a\,2}} \eta_m^a(L_0) \tag{2.99}$$

and thus

$$y^a(k^2, z) = \sum_n \frac{m_n^{a\,2} \eta_n^a(L_0) \eta_n^a(z)}{k^2 - m_n^{a\,2}}. \tag{2.100}$$

Again, the complete calculation can be found in Appendix A.4. We define again

$$f_n^a := -\frac{\partial_z \widehat{\phi}_n^a(L_0)}{g_5 L_0} = \frac{-\eta_n^a(L_0)}{g_5}, \tag{2.101}$$

which turns out to be the decay constant of the n-th pseudoscalar hadron [2]. In this case f_n^a has the dimension of a mass.

We can then integrate (2.100) to obtain

$$\widehat{\phi}^a = \sum_n \frac{-g_5 m_n^{a\,2} f_n^a \widehat{\phi}_n^a(z)}{k^2 - m_n^{a\,2}} \tag{2.102}$$

and

$$\widehat{\pi}^a = \sum_n \frac{-g_5 m_n^{a\,2} f_n^a \widehat{\pi}_n^a(z)}{k^2 - m_n^{a\,2}}. \tag{2.103}$$

2.3 Form Factors

Kaon-to-Pion Transition Form Factor

The above considerations give sufficient material to construct further observables based on them. Since we went through all the trouble of including the strange quark, it suggests itself that we should consider expressions involving kaons. Thus, let us look at the form factors of the $K_{\ell 3}$ transition. It describes a decay $K \to \pi \ell \nu$. The transition form factors are defined via

$$\left\langle \pi^-(k) \right| J^\mu_{\Delta S = \pm 1} \left| K^0(q) \right\rangle = f_+(K^2)(k^\mu + q^\mu) + f_-(K^2)(q^\mu - k^\mu), \tag{2.104}$$

where $K^2 = (k-q)^2$ and k and q are the pion and kaon four-momenta. One could have also defined the form factors based on $K^+ \to \pi^0$. $J^\mu_{\Delta S = \pm 1}$ is the strangeness changing vector current. One can also define

$$f_0(K^2) := f_+(K^2) + \frac{K^2}{m_K^2 - m_\pi^2} f_-(K^2). \tag{2.105}$$

Expressions for f_+ and f_0 in terms of the fields in our model can be derived [2][8]. They are

$$f_0(K^2) = \int_{L_0}^{L_1} \mathrm{d}z \Bigg(\left(\widehat{\varphi}^4(K^2,z) - \widehat{\varpi}^4(K^2,z)\right) \Bigg\{ \frac{1}{z}\partial_z\widehat{\phi}_1^1\partial_z\widehat{\phi}_1^7 + \frac{g_5^2 v_q(v_q+v_s)}{2z^3}(\widehat{\phi}_1^1 - \widehat{\pi}_1^1)(\widehat{\phi}_1^7 - \widehat{\pi}_1^7)$$
$$+ \frac{K^2}{2z}\widehat{\phi}_1^1\widehat{\phi}_1^7 + \frac{g_5^2 K^2}{8z^3(m_K^2 - m_\pi^2)}\Big[(v_s - v_q)(3v_q + v_s)(\widehat{\phi}_1^1 - \widehat{\pi}_1^1)(\widehat{\phi}_1^7 - \widehat{\pi}_1^7)$$
$$- 4v_q v_s \widehat{\phi}_1^1(\widehat{\phi}_1^7 - \widehat{\pi}_1^7) + (v_q + v_s)(3v_q - v_s)(\widehat{\phi}_1^1 - \widehat{\pi}_1^1)\widehat{\phi}_1^7\Big]\Bigg\}$$
$$+ \frac{\partial_z\widehat{\varpi}^4(K^2,z)}{(m_K^2 - m_\pi^2)}\Bigg\{\frac{m_K^2 + m_\pi^2 - K^2}{2z}\left(\partial_z\widehat{\phi}_1^1\,\widehat{\phi}_1^7 - \widehat{\phi}_1^1\partial_z\widehat{\phi}_1^7\right) + \frac{g_5^2(v_s - v_q)(3v_q+v_s)}{8z^3}\partial_z(\widehat{\pi}_1^1\widehat{\pi}_1^7)$$
$$+ \frac{g_5^2 v_q(v_q+v_s)}{2z^3}\left(\widehat{\pi}_1^1\partial_z\widehat{\pi}_1^7 - \partial_z\widehat{\pi}_1^1\,\widehat{\pi}_1^7\right) - \frac{m_K^2 - m_\pi^2}{2z}\frac{\partial_z\alpha^4(z)}{\alpha^4(z)}\,\widehat{\phi}_1^1\widehat{\phi}_1^7\Bigg\}\Bigg) \tag{2.106}$$

and

$$f_+(K^2) = \int_{L_0}^{L_1} \mathrm{d}z\mathcal{V}^4(K^2,z)\left[\frac{1}{z}\partial_z\widehat{\phi}_1^1(z)\partial_z\widehat{\phi}_1^7(z) + \frac{g_5^2}{2z^3}v_q(v_q+v_s)\left(\widehat{\phi}_1^1(z) - \widehat{\pi}_1^1(z)\right)\left(\widehat{\phi}_1^7(z) - \widehat{\pi}_1^7(z)\right)\right]. \tag{2.107}$$

They might look quite complicated, but they only contain already derived quantities, so they can be easily computed - numerically of course. The subscript indicates a meson mode, while without the subscript, the profile function is meant.

The form factor $f_+(K^2)$ is usually fitted to a quadratic polynomial ([17], p. 717)

$$f_+(K^2) = f_+(0)\left(1 + \lambda'_+\frac{K^2}{m_\pi^2} + \frac{1}{2}\lambda''_+\left(\frac{K^2}{m_\pi^2}\right)^2\right) \tag{2.108}$$

and $f_0(K^2)$ to a linear one

$$f_0(K^2) = f_0(0)\left(1 + \lambda_0\frac{K^2}{m_\pi^2}\right). \tag{2.109}$$

The observables $f_+(0)$, λ'_+, λ''_+, and λ_0 can then be compared to experimental values. (It follows directly from equation (2.105) that $f_+(0) = f_0(0)$.)

Pion Form Factor

Another form factor we can calculate is the pion form factor. Similar to the kaon-to-pion form factor, it is defined via

$$\langle\pi(k)|\, J_V^\mu\,|\pi(q)\rangle = (k^\mu + q^\mu)F_\pi(K^2), \tag{2.110}$$

[8] This is the first time, this has ever been done. The derivation involves three-point functions and quite some experience in using the translation dictionary.

where J_V^μ is the vector current and again $K^2 = (k-q)^2$. In contrast to the kaon-to-pion form factor, there is only an F_+ and no F_- part, which is a consequence of the fact that the pion form factor corresponds to a conserved current.

It can be expressed in our model by [5][9]

$$F_\pi(K^2) = \int_{L_0}^{L_1} \frac{\mathrm{d}z}{z} \mathcal{V}^a(K^2, z) \left((\partial_z \widehat{\phi}_1^a(z))^2 + \beta^a(z)(\widehat{\pi}_1^a(z) - \widehat{\phi}_1^a(z))^2 \right), \tag{2.111}$$

where $a = 1, 2, 3$. For spacelike momentum transfer ($K^2 < 0$), this is an interesting observable and we can compare our results to experimental data.

2.4 Results

Having done all the necessary preparations, we are now able to actually compute the masses, decay constants, and form factors, which we discussed above. The five input parameters are L_1, m_q, m_s, σ_q and σ_s. As mentioned before, the higher radial excitations ($n = 2, 3, \ldots$) are not well represented in hard-wall models. Calculating them and comparing them to their experimental values would be a fruitless effort. Thus, we are restricted to the case $n = 1$. This still gives us plenty of quantities we can compute. We will consider two models. Model AI is calculated under the assumption $\sigma_s = \sigma_q$ and we will determine the model parameters successively, trying to reproduce certain exact known observables. This way, we can see very nicely, how the different observables depend on the input parameters. In Model AII, we work in the general case, where $\sigma_s \neq \sigma_q$, and make a global fit to all observables.

2.4.1 Model AI

Working in the $\sigma_q = \sigma_s$ limit, we have four parameters left to fix using experimental data. We will use observables which are known to a very high precision. The ρ mass $M_\rho = (775.49 \pm 0.34)$ MeV[10] corresponds to M_1^a for $a = 1, 2, 3$ in vector sector of our model. For the π, we use the mass $m_\pi = 139.57$ MeV and the decay constant $f_\pi = (92.4 \pm 0.35)$ MeV corresponding to m_1^a and f_1^a for $a = 1, 2, 3$ in the pseudoscalar sector. One could have taken the π^0 mass instead, but since the mass difference of about 5 MeV is mainly due to electromagnetic corrections, which are not included in our model, it does not really play a role. Finally, we use the kaon mass $m_K = 495.7$ MeV corresponding to m_1^a for $a = 4, 5, 6, 7$ also in the pseudoscalar sector. Here, since the mass difference between the $K^\pm$ and the K^0 is due to the differing mass of the up and down quark and we assumed up/down symmetry, we take an average mass ($m_{K^0} = 497.6$ MeV, $m_{K^\pm} = 493.7$ MeV).

Since the ρ mesons do not couple to the vacuum expectation value $v(z)$, their mass only depends on L_1, namely via the mass relation (2.74). From this, one gets $L_1 =$

[9] The different prefactor compared to their formula comes from different normalization conventions.

[10] All meson masses and decay constants throughout the whole text are taken from [17] (PDG) if not noted otherwise.

$\frac{r_1}{M_\rho} = (322.47 \text{ MeV})^{-1}$. Then, since the pion's mass and decay constant do not depend on m_s, one can determine the parameters m_q and σ_q by matching $m_1^{1,2,3}$ and $f_1^{1,2,3}$ with m_π and f_π. One gets $m_q = 8.291$ MeV and $\sigma_q = (213.66 \text{ MeV})^3$. Finally, to find m_s, we try to satisfy the mass condition for m_K, which now depends on m_q, $\sigma_q = \sigma_s$ and m_s. One finds $m_s = 188.48$ MeV.

Having determined all parameters, one can then put the model to the test by calculating masses and decay constants for further particles. We will consider the (unconfirmed) scalar meson $K_0^*(800)$ or κ with $m_{K_0^*} = (672 \pm 40)$ MeV. In our model, this corresponds to the mass of the first mode of scalar mesons for $a = 4, 5, 6, 7$. Furthermore, we consider the $K^{*\pm}$ and K^{*0} again viewed together as K^* with $M_{K^*} = 893.8$ MeV. In the model, these are the lightest modes of the vector mesons for $a = 4, 5, 6, 7$. As pseudovector mesons, we will consider the $a_1(1260)$ with $m_{a_1} = (1230 \pm 40)$ MeV ($a = 1, 2, 3$) and the $K_1(1270)$ with $m_{K_1} = (1272 \pm 7)$ MeV ($a = 4, 5, 6, 7$).

The results of the computations are shown in Table 2.2. Capital letters M and F are used to indicate solutions of transverse parts corresponding to vector and pseudovector mesons with the normalization of the wave function discussed in Section 2.1.4.1. Lower case letters m and f indicate solutions of the longitudinal parts corresponding to scalar and pseudoscalar mesons with the normalization discussed in Section 2.2.1. The masses

Observable	Sector	a	n	Model AI [MeV]	Measured [MeV]
m_π	pseudoscalar	1,2,3	1	(fit)	139.57
f_π	pseudoscalar	1,2,3	1	(fit)	92.4 ± 0.35
m_K	pseudoscalar	4,5,6,7	1	(fit)	495.7
f_K	pseudoscalar	4,5,6,7	1	103.8	113 ± 1.4
$m_{K_0^*}$	scalar	4,5,6,7	1	791.0	672
$f_{K_0^*}$	scalar	4,5,6,7	1	27.6	
M_ρ	vector	1,2,3	1	(fit)	775.49 ± 0.34
$\sqrt{F_\rho}$	vector	1,2,3	1	329.3	345 ± 8[2]
M_{K^*}	vector	4,5,6,7	1	791.0	893.8
$\sqrt{F_{K^*}}$	vector	4,5,6,7	1	329.7	
M_{a_1}	pseudovector	1,2,3	1	1366.2	1230 ± 40
$\sqrt{F_{a_1}}$	pseudovector	1,2,3	1	488.8	433 ± 13[2]
M_{K_1}	pseudovector	4,5,6,7	1	1458.1	1272 ± 7
$\sqrt{F_{K_1}}$	pseudovector	4,5,6,7	1	511.1	

Table 2.2: Comparison of masses and decay constants as calculated in Model AI with experimental data.

seem to be in a good agreement with experimental data.

We can then also compare our values for $f_+(0)$, λ'_+, λ''_+ and λ_0 to experimental data and data from lattice gauge theory and chiral perturbation theory. The results are presented in Table 2.3 and shown in Figure 2.3. These results are also in a good agreement with experimental data and results from other approaches. One should add

Observable	Model AI	Lattice	χPT	Data [18]
$f_+(0)$	0.966	0.968(11) [19] 0.9742(41) [21] 0.9560(84) [23]	0.961(8) [20] 0.978(10) [22] 0.984(12) [24] 0.974(11) [25]	
λ'_+	0.0249	0.0237(23)(21) [23]		0.0249(11)
λ''_+	0.00206			0.0016(5)
λ_0	0.0122	0.0128(22)(45) [23]		0.0134(12)

Table 2.3: Results for form factors compared to lattice gauge theory, chiral perturbation theory, and experimental data.

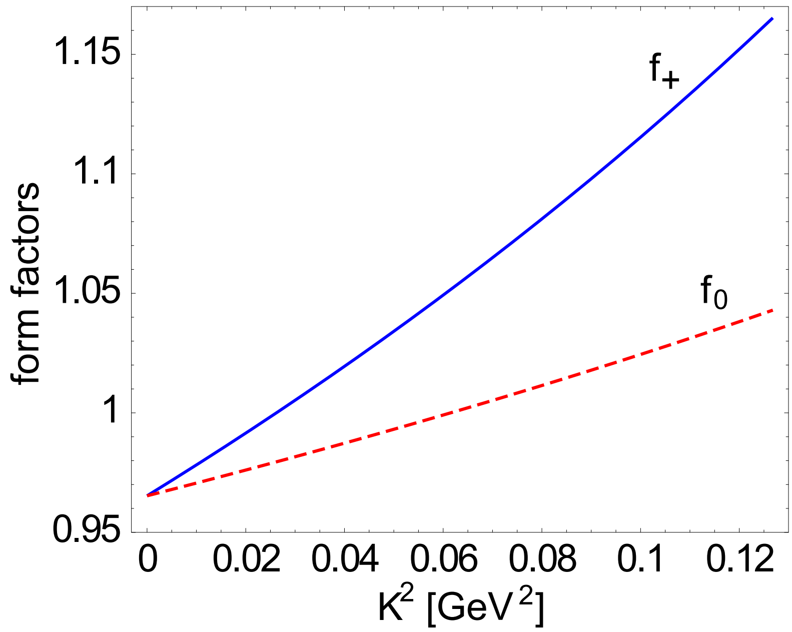

Figure 2.3: Kaon-to-pion transition form factors as calculated in Model AI.

that $f_+(k^2)$ is almost linear in k^2, which results in a rather large numerical uncertainty for the value of λ''_+.

A general comment about the numerical calculations should be made. Most of this work involves solving initial or boundary value problems numerically. For this the parameter L_0 is necessary. A value of 10^{-8} turned out to be a reasonably good choice for L_0 since the computational results did not change significantly when changing L_0 from 10^{-7} to 10^{-8}. Stability problems could occur for even smaller values of L_0 depending on the type of solver used.

Our results confirm the numbers given in [2]. Small differences stem from the exact numerical method applied, but (except for λ''_+) the deviation is well below 1%, indicating that the numerics giving the results in this text and theirs were carried out properly.

In addition, changing the solver or switching to a completely different program never resulted in a significant change of the results. (Most of our calculations were done in Matlab using built-in solvers for initial and boundary value problems.)

2.4.2 Model AII

Let us now make a global fit to the observables, to which we have experimental data. These 14 quantities are m_π, f_π, m_K, f_K, $m_{K_0^*}$, M_ρ, F_ρ, M_{K^*}, M_{a_1}, F_{a_1}, M_{K_1}, λ'_+, λ''_+, and λ_0. Allowing σ_s to differ from σ_q, we now have five parameters to determine. These we obtain by making a least square fit, where we try to minimize the sum of the squares of the relative errors, using a multidimensional minimization algorithm. This results in the following values for the model parameters: $L_1 = (342.8\text{ MeV})^{-1}$, $m_q = 8.077$ MeV, $m_s = 213.7$ MeV, $\sigma_q = (208.3\text{ MeV})^3$, and $\sigma_s = (217.9\text{ MeV})^3$. The corresponding values for the observables are shown in Table 2.4. We see that $\sigma_q \approx \sigma_s$,

Observable	Model AII [MeV]	Measured [MeV]
m_π	138.9	139.57
f_π	88.3	92.4 ± 0.35[17]
m_K	532.5	495.7
f_K	104.5	113 ± 1.4[2]
$m_{K_0^*}$	703.1	672
$f_{K_0^*}$	39.2	
M_ρ	824.4	775.49 ± 0.34
$\sqrt{F_\rho}$	350.1	345 ± 8[2]
M_{K^*}	862.1	893.8
$\sqrt{F_{K^*}}$	352.6	
M_{a_1}	1288.9	1230 ± 40
$\sqrt{F_{a_1}}$	462.0	433 ± 13[2]
M_{K_1}	1440.9	1272 ± 7
$\sqrt{F_{K_1}}$	501.2	
$f_+(0)$	0.954	
λ'_+	0.0236	0.0249(11)[18]
λ''_+	0.0017	0.0016(5)[18]
λ_0	0.0144	0.0123(12)[18]

Table 2.4: Comparison of masses, decay constants, and form factors as calculated in Model AII.

which a posteriori justifies why we chose them to be equal in Model AI. The results are also in good accordance with those obtained from [2], considering that they fitted to 15 observables ($f_+(0)$ additionally) and might have used a different weighting of the errors.

2.4.3 Pion Form Factor

Finally, let us calculate the pion form factor for spacelike momentum transfer with the input parameters from Models AI and AII. Especially results in the high $|K^2|$ region are of interest since they are difficult to calculate with other theoretic approaches. The result and a comparison to experimental data can be seen in Figure 2.4. The form factor

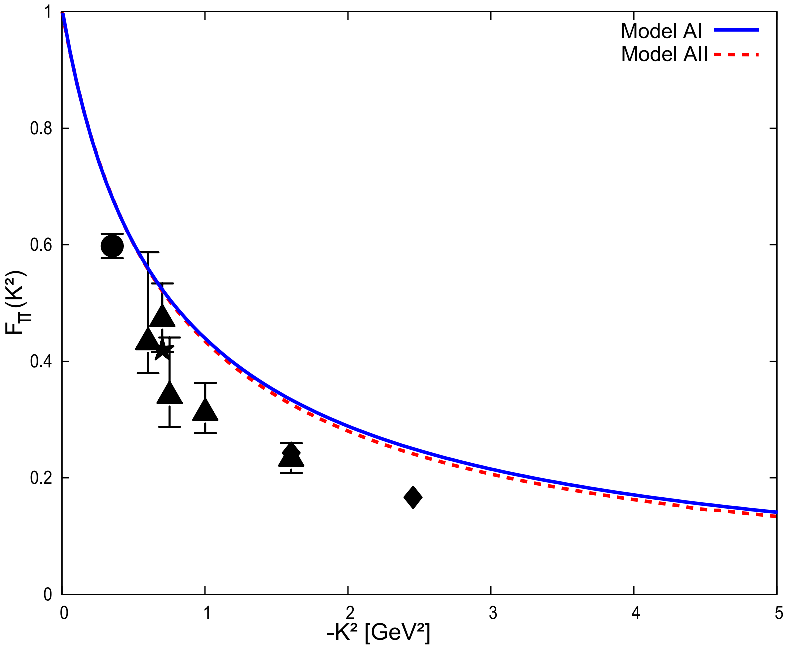

Figure 2.4: Spacelike pion form factor as calculated in Model AI and AII with comparison to experimental data, which was gathered by [5]. The triangles are data from DESY, reanalyzed by [26], the diamonds are from Jefferson Lab [27], and the circles and the star are also from DESY [28, 29].

in our model has the correct shape and is consistently a little bit above the experimental values. Small changes in the input parameters do not have a significant impact on the outcome.

3 Soft-Wall Model

We have now seen how to apply the AdS/QCD correspondence to gain concrete results for masses and other observables, which agree reasonably well with experiment. We did so in the context of a flavour-asymmetric hard-wall model. The drawback of hard-wall models in general is, that their mass spectra have unphysical trajectories, making it impossible to predict higher radial excitations of the particles we studied above. This problem can be solved in a soft-wall setting (first proposed by [3]), which we will explore in the following. The main steps are the same, the only difference is a background field and an unbounded fifth dimension. The AdS_5 metric is the same as before (1.12).

The action is now given by

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^5x\sqrt{g}e^{-\Phi(z)}\,\mathrm{Tr}\left(|D_MX|^2 + \frac{3}{L^2}\,|X|^2 - \frac{1}{g_5^2}\left(F_L^2 + F_R^2\right)\right)\\ &= \int \mathrm{d}^5x\sqrt{g}e^{-\Phi(z)}\,\mathrm{Tr}\left((D_MX)^\dagger(D^MX) + \frac{3}{L^2}X^\dagger X\right.\\ &\quad\left. - \frac{1}{g_5^2}\left(F^L_{MN}F_L^{MN} + F^R_{MN}F_R^{MN}\right)\right).\end{aligned} \tag{3.1}$$

The field $\Phi(z)$ is the background dilaton field. It appears in all integrals over z in addition to the metric factor $\frac{1}{z}$, which we had before. Those integrals are now always taken from zero (or L_0) to infinity.

It is important to know that the way the masses scale with n depends on the exact choice of the background field. In QCD with linear confinement, one expects a behaviour $m_n^2 \sim n$ [30]. In soft-wall models, correct Regge trajectories can be obtained, if the background field is of the asymptotic form [3]

$$\Phi(z) \overset{z\to\infty}{=} c^2z^2, \tag{3.2}$$

where c now plays the role of Λ_{QCD} and hence we expect it to be about of the same magnitude as $\frac{1}{L_1}$. The easiest field fulfilling that condition, is of course exactly given by $\Phi(z) = c^2z^2$. This choice has however a certain well-known drawback, which we will discuss when looking at the vacuum solution.

We will follow the same steps as before, i.e. we will expand X in terms of its vacuum solution and the pion field. The z-dependent vacuum solution $X_0 = \langle X\rangle$ is obtained by looking at the action with all fields except X turned off. One gets

$$\begin{aligned} \mathcal{S} &= \int \mathrm{d}^5 x \sqrt{g} e^{-\Phi(z)} \operatorname{Tr}\left(\partial_z X_0^\dagger \partial^5 X_0 + \frac{3}{L^2} X_0^\dagger X_0\right) \\ &= \sum_{i,j=1}^{3} \int \mathrm{d}^5 x e^{-\Phi(z)} \left(-\partial_z X_{0ij} \frac{L^3}{z^3} \partial_z X_{0ij} + 3\frac{L^3}{z^5} X_{0ij}^2\right) \end{aligned} \tag{3.3}$$

(compare to (2.17)).

One again makes a variational argument for the action $\mathcal{S}_{ij}$, which contains all terms with X_{0ij}. This is done exactly as in the hard-wall case, the only difference being the factor of $e^{-\Phi(z)}$. It does, however, not greatly alter the calculations. One gets

$$\delta \mathcal{S}_{ij} = \int \mathrm{d}^5 x \left(+\partial_z \left(e^{-\Phi(z)} \frac{L^3}{z^3} \partial_z X_{0ij}\right) + 3e^{-\Phi(z)} \frac{L^3}{z^5} X_{0ij}\right) \delta X_{0ij}. \tag{3.4}$$

Since the action $\mathcal{S}_{ij}$ should be extremal, $\delta\mathcal{S}_{ij}$ should vanish independent of the infinitesimal displacement δX_{0ij}. So we can read off the resulting equation of motion. It is

$$\partial_z \left(e^{-\Phi(z)} \frac{L^3}{z^3} \partial_z X_{0ij}\right) + 3e^{-\Phi(z)} \frac{L^3}{z^5} X_{0ij} = 0 \tag{3.5}$$

for each component of X_0.

Assuming that $\Phi(z) = c^2 z^2$, the above equation becomes

$$z^2 X_{0ij}''(z) - \left(2c^2 z^3 + 3z\right) X_{0ij}'(z) + 3X_{0ij}(z) = 0. \tag{3.6}$$

This ODE is closely related to Kummer's equation and its solution can be expressed as a sum of two linearly independent hypergeometric functions, namely ${}_1F_1$, Kummer's function (of the first kind), and U, Kummer's function of second kind (or Tricomi confluent hypergeometric function). One gets

$$X_{0ij} = AczU\left(\frac{1}{2}; 0; (cz)^2\right) + B(cz)^3 \, {}_1F_1\left(\frac{3}{2}; 2; (cz)^2\right). \tag{3.7}$$

The constants A and B are determined by expanding the above expression up to third order in z. Then one gets an expression with a first and a third order term, which resembles the vacuum solution in the hard-wall case. The constants will be chosen, such that we obtain

$$2X_{0ij} =: v_{ij}(z) = \frac{m_{ij} z}{L} + \frac{\sigma_{ij} z^3}{L} + \mathcal{O}(z^5). \tag{3.8}$$

One easily sees that the constant A must be equal to

$$\frac{m_{ij}}{Lc} \Gamma\left(\frac{3}{2}\right). \tag{3.9}$$

The problem with the second term in (3.7) is that it diverges too strongly as z goes to infinity. Hence the integral $\int \mathrm{d}z \frac{e^{-\Phi(z)}}{z} X_0^2(z)$ does not converge, which means that the solution is not normalizable and cannot correspond to a physical quantity. So, we have to drop the second term. The vacuum solution then reads

$$v_{ij}(z) = \frac{m_{ij}}{L}\Gamma\left(\frac{3}{2}\right) zU\left(\frac{1}{2}; 0; (cz)^2\right). \tag{3.10}$$

If we look at the expansion of our solution in low z, it now reads as

$$\frac{m_{ij}z}{L} + \frac{c^2 m_{ij}}{2L}\left(1 + \gamma + 2\log\left(\frac{cz}{2}\right)\right) z^3. \tag{3.11}$$

One can then again identify the coefficient of the second term with the quark condensate and establish a proportionality between the quark masses and condensates, which is unphysical. In reality, spontaneous symmetry breaking also occurs in the chiral limit, where all quark masses are zero. One cannot hope to achieve reasonable results with a model that has such a severe shortcoming (and indeed one does not). The parameters m_{ij} and σ_{ij} responsible for the explicit and spontaneous breaking of chiral symmetry should be independent in any model that is dual to QCD.

Several approaches of how to include both parameters independently into the soft-wall model have been studied. In [5], a model is proposed, where the vacuum expectation value is chosen in a way that it behaves asymptotically (both in the UV and the IR) as the vacuum solution above, but has M and Σ as two independent parameters, ignoring the fact that the vacuum solution is then not an exact solution of the vacuum equation of motion anymore.

Another approach is studied in [4]. They consider an additional quartic term in the potential term in the action. The dilaton field is modified to be consistent with the choice of the vacuum expectation value. The quartic term is needed to obtain the required asymptotic behaviour for $v(z)$ and $\Phi(z)$. (Very recently [31] also have extended this model with a modified metric in the UV region.)

In the following, we will study both of the approaches and also try to extend them to the $N_\mathrm{f} = 3$ case.

3.1 Modified Action Model

The action will be complemented by a quartic potential term and now reads

$$\mathcal{S} = \int \mathrm{d}^5x\sqrt{g}e^{-\Phi(z)}\,\mathrm{Tr}\left(|D_M X|^2 + \frac{3}{L^2}|X|^2 + \kappa|X|^4 - \frac{1}{g_5^2}\left(F_L^2 + F_R^2\right)\right). \tag{3.12}$$

The differential equation for $v_{ij}(z) = 2X_{0ij}(z)$ then becomes

$$\partial_z\left(\frac{e^{-\Phi(z)}}{z^3}\partial_z v_{ij}(z)\right) + \frac{e^{-\Phi(z)}}{z^5}\left(3v_{ij}(z) + \frac{\kappa L^2}{2}v_{ij}^3(z)\right) = 0, \tag{3.13}$$

which can be easily seen by analogy to the case without a quartic term. This differential equation is not linear anymore. In the case $\kappa = 0$, we are back to the original case studied above.

We will now turn the tables and start with a vacuum expectation value (for the up and down quark) fulfilling certain asymptotic behaviour requirements. The dilaton field is then determined via

$$\partial_z \Phi(z) = \frac{z^3}{L^3 \partial_z v_q(z)} \left(\partial_z \left(\frac{L^3}{z^3} \partial_z v_q(z) \right) + 3 \frac{L^3}{z^5} v_q(z) + \frac{L^5}{z^5} \frac{\kappa}{2} v_q^3(z) \right), \tag{3.14}$$

which follows from the differential equation (3.13) above. This determines the dilaton field $\Phi(z)$ uniquely up to a constant, which means $e^{-\Phi(z)}$ will be determined up to a constant factor. For consistency, we however demand that $e^{-\Phi(0)} = 1$, i.e. $\Phi(0) = 0$.

Unfortunately it turns out that it is not possible - at least not without further ado - to extend this model to the flavour non-symmetric case, since the dilaton field would also have to obey equation (3.14), with v_q replaced by v_s.

We already know what the small-z behaviour of v_q should be. We want

$$v_q(z) \overset{z \to 0}{=} \frac{m_q \zeta}{L} z + \frac{\sigma_q}{\zeta L} z^3. \tag{3.15}$$

In [4] another constraint is added, which is derived from phenomenology. When we look at the solution we had in the unmodified case, this was given by $v_{ij}(z) = \frac{m_{ij}}{L} \Gamma\left(\frac{3}{2}\right) z U\left(\frac{1}{2}; 0; (cz)^2\right)$. This solution approaches a constant as $z \to \infty$. While this of course has the necessary finiteness property we demand from the vacuum solution, it leads to a restoration of chiral symmetry in the mass spectrum for large n, which is not a feature of QCD. One rather observes that the highly excited mesons exhibit parallel trajectories, which means that chiral symmetry is not restored, since the mass difference between two different mesons stays the same rather than tending to zero. In [4], it is argued that this can only be the case if $v_q(z)$ is a linear function for large z, i.e.

$$v_q(z) \overset{z \to \infty}{=} \frac{\gamma}{L} z. \tag{3.16}$$

One simple way of writing a $v_q(z)$, which fulfills the above constraints is by making the ansatz

$$v_q(z) = \frac{z}{L} \left(A + B \tanh(Cz^2) \right). \tag{3.17}$$

This gives

$$L v_q(z) \overset{z \to 0}{=} Az + BCz^3 \tag{3.18}$$

and

$$L v_q(z) \overset{z \to \infty}{=} (A+B) z. \tag{3.19}$$

Inserting this into the expression for the dilaton field gives in the limit of large z (with $\Phi(0) = 0$):

$$\Phi(z) \overset{z \to \infty}{=} \frac{\kappa}{4} (A+B)^2 z^2, \tag{3.20}$$

which was the required form to produce correct Regge trajectories.

By matching the coefficients A, B, C, and γ with the constraints, one gets

$$\gamma = \sqrt{\frac{4\lambda}{\kappa}}, \tag{3.21}$$

$$A = \zeta m_q, \tag{3.22}$$

$$B = \gamma - A, \tag{3.23}$$

and

$$C = \frac{\sigma_q}{\zeta B}. \tag{3.24}$$

Here, λ is - as we will see soon - the average slope of the radial trajectories of the mesons, which is approximately the same for all mesons. The parameter κ has unlike the other ones no direct physical analogue but can be given a concrete meaning, as shown below.

Once, one has all the parameters determined, one has the vacuum solution $v_q(z)$ and the dilaton field $\Phi(z)$ fixed. However, so far we have only included the up and the down quark, which we assumed to have equal mass. One drawback of this model is that the strange quark cannot easily be included. Of course, now that $\Phi(z)$ is known one could try to find v_s as solution to the vacuum equation of motion. One has however the constraints dictated in the IR by m_s and in the UV by γ. In principle, one could hope to find a solution to a second order boundary differential equation with two constraints. Here, this is not the case since by choosing the condition in the IR, the value at the UV boundary is uniquely determined if one in general imposes a finiteness condition in the UV. So, we will have to accept that we can only model strangeless mesons in this approach. This will still give us a lot of observables to work with. Especially, in comparison to the hard-wall model, we can consider higher radial excitations.

So now, let us assume, we have our v_q. Then

$$X_0 = \frac{1}{2}\begin{pmatrix} v_q & 0 \\ 0 & v_q \end{pmatrix}. \tag{3.25}$$

We do not need to make all the calculations again in the two-dimensional case (using Pauli matrices instead of Gell-Mann matrices), we can just define ${M_V^a}^2$, ${M_A^a}^2$, $\alpha^a(z)$, and $\beta^a(z)$ exactly as we did before and restrict ourselves to $a = 1, 2, 3$. (Since we will study a soft-wall model where we can include the third quark in the next section, we will also consider $a = 4, 5, 6, 7, 8$ in the following calculations.)

The action can be expanded in the same way as done in the hard-wall approach and one gets up to second order in π, A, and V:

$$\begin{aligned} \mathcal{S} = \int \mathrm{d}^5 x e^{-\Phi(z)} \sum_a \Bigg(&-\frac{L}{4g_5^2 z}\left(\partial_M V_N^a - \partial_N V_M^a\right)^2 + \frac{{M_V^a}^2 L^3}{2z^3} {V_M^a}^2 \\ &-\frac{L}{4g_5^2 z}\left(\partial_M A_N^a - \partial_N A_M^a\right)^2 + \frac{{M_A^a}^2 L^3}{2z^3}\left(\partial_M \pi^a - A_M^a\right)^2 \Bigg). \end{aligned} \tag{3.26}$$

The only difference to the hard-wall action is a factor of $e^{-\Phi(z)}$. The quartic term does not appear here explicitly, but is of course hidden in the dependence of $\alpha^a(z)$ and $\beta^a(z)$ on the vacuum solution, which in turn depend on the quartic term.

Again, A_z^a can be chosen to be zero and so can V_z^a for $a = 1, 2, 3, 8$.

3.1.1 Vector Sector

The vector sector of the action is

$$\begin{aligned}\mathcal{S}_V &= \int \mathrm{d}^5 x e^{-\Phi(z)} \sum_a \left(-\frac{L}{4g_5^2 z} (\partial_M V_N^a - \partial_N V_M^a)^2 + \frac{\alpha^a(z) L}{2g_5^2 z} {V_M^a}^2 \right) \\ &= \int \mathrm{d}^5 x e^{-\Phi(z)} \frac{L}{4g_5^2} \sum_a \left(-\frac{1}{z} \left(\eta^{MM'} \eta^{NN'} (\partial_M V_N^a - \partial_N V_M^a)(\partial_{M'} V_{N'}^a - \partial_{N'} V_{M'}^a) \right) \right. \\ &\quad \left. + \frac{2\alpha^a(z)}{z} \eta^{MM'} V_M^a V_{M'}^a \right) .\end{aligned} \tag{3.27}$$

The equation of motion for V can then be derived and is

$$\eta^{ML} \partial_M \left(\frac{e^{-\Phi(z)}}{z} (\partial_L V_N^a - \partial_N V_L^a) \right) + e^{-\Phi(z)} \frac{\alpha^a(z)}{z} V_N^a = 0. \tag{3.28}$$

We can split this up and get

$$\begin{aligned}\eta^{\mu\lambda} \partial_\mu \left(\frac{e^{-\Phi(z)}}{z} (\partial_\lambda V_\nu^a - \partial_\nu V_\lambda^a) \right) - \partial_z \left(\frac{e^{-\Phi(z)}}{z} (\partial_z V_\nu^a - \partial_\nu V_z^a) \right) + e^{-\Phi(z)} \frac{\alpha^a(z)}{z} V_\nu^a &= 0 \\ \eta^{\mu\lambda} \partial_\mu \left(\frac{e^{-\Phi(z)}}{z} (\partial_\lambda V_z^a - \partial_z V_\lambda^a) \right) + e^{-\Phi(z)} \frac{\alpha^a(z)}{z} V_z^a &= 0.\end{aligned} \tag{3.29}$$

Making a decomposition of V_μ^a into a transversal and a longitudinal part, writing the longitudinal part as $V_{\mu\|}^a = \partial_\mu \xi^a$, writing the z-component as $V_z^a = -\partial_z \varpi^a$, and introducing $\varphi^a = \xi^a + \varpi^a$, we get

$$\begin{aligned}\frac{e^{-\Phi(z)}}{z} \eta^{\mu\lambda} \partial_\mu \partial_\lambda V_{\nu\perp}^a - \partial_z \left(\frac{e^{-\Phi(z)}}{z} (\partial_z V_{\nu\perp}^a) \right) + e^{-\Phi(z)} \frac{\alpha^a(z)}{z} V_{\nu\perp}^a & \\ -\partial_z \left(\frac{e^{-\Phi(z)}}{z} (\partial_\nu \partial_z \varphi^a) \right) + e^{-\Phi(z)} \frac{\alpha^a(z)}{z} \partial_\nu (\varphi^a - \varpi^a) &= 0 \\ \alpha^a(z) \partial_z \varpi^a + \eta^{\mu\lambda} \partial_\mu \partial_\lambda \partial_z \varphi^a &= 0.\end{aligned} \tag{3.30}$$

Taking the Fourier transform, this gives

$$\partial_z \left(\frac{e^{-\Phi(z)}}{z} (\partial_z \widehat{V}_{\mu\perp}^a) \right) + e^{-\Phi(z)} \frac{k^2 - \alpha^a(z)}{z} \widehat{V}_{\mu\perp}^a = 0 \tag{3.31}$$

for the transverse part and

$$\partial_z\left(\frac{e^{-\Phi(z)}}{z}(\partial_z\widehat{\varphi}^a)\right)-e^{-\Phi(z)}\frac{\alpha^a(z)}{z}(\widehat{\varphi}^a-\widehat{\varpi}^a)=0$$
$$\alpha^a(z)\partial_z\widehat{\varpi}^a-k^2\partial_z\widehat{\varphi}^a=0 \tag{3.32}$$

for the longitudinal part.

As the reader may have noticed, the general form of the differential equations has not changed compared to the hard-wall model. The only difference is that $e^{-\Phi(z)}$ appears at various places in the equations. Furthermore, α^a and β^a will certainly never simplify to a constant, except for those cases, when $\alpha^a=0$. As a consequence, unless $\alpha^a=0$, there exist only numerical solutions.

3.1.1.1 Vector Mesons

We are again looking for normalizable solutions of the differential equation for the transverse part.

Approximate Solution

To gain a little bit more understanding about the solutions, we will first study the case, where $a=1,2,3$ (ρ mesons), so that $\alpha^a=0$ and we will use the original simple dilaton field $\Phi(z)=c^2z^2$. (This is in fact a very good approximation, especially for higher radial excitations.) In this case, we can find analytical solutions.

The general solution to the somewhat simplified equation (3.31) is given by (writing again ψ^a and later ψ^a_n for the normalizable modes)

$$\psi^a(z)=Ac^2z^2\,{}_1F_1\left(1-\frac{k^2}{4c^2};2;c^2z^2\right)+Be^{c^2z^2}U\left(\frac{k^2}{4c^2};0;-c^2z^2\right). \tag{3.33}$$

In the above form the second term gives complex values, but its imaginary part is just a multiple of the first term, so we can restrict ourselves to the real part of the second term. We want the modes to be normalizable. That means the integral

$$\int_0^\infty \mathrm{d}z\frac{e^{-\Phi(z)}}{z}(\psi^a(z))^2 \tag{3.34}$$

has to converge. One condition for that is, that the value of ψ^a has to vanish at $z=0$. If one evaluates the above expressions, one gets

$$c^2z^2\,{}_1F_1\left(1-\frac{k^2}{4c^2};2;c^2z^2\right)\bigg|_{z=0} \tag{3.35}$$

and

$$e^{c^2z^2}U\left(\frac{k^2}{4c^2};0;-c^2z^2\right)\bigg|_{z=0}=\left(\Gamma\left(1+\frac{k^2}{4c^2}\right)\right)^{-1}\neq 0. \tag{3.36}$$

So we have to drop the second term. What is left, is

$$\psi^a(z) = Ac^2z^2\ {}_1F_1\left(1-\frac{k^2}{4c^2};2;c^2z^2\right). \tag{3.37}$$

The integral of this expression does not converge in general, where the problem now lies in the large-z region. There is however a discrete set of values for k^2, such that the integral converges. One finds

$$M_n^{a2} := k^2 = 4nc^2, \tag{3.38}$$

for $n = 1, 2, 3, \ldots$, which is a Regge trajectory of the desired form. After normalization to unity, one gets

$$\psi_n^a(z) = \sqrt{2n}c^2z^2\ {}_1F_1\left(1-n;2;c^2z^2\right), \tag{3.39}$$

which can be explicitly written as a polynomial of degree $2n$ in z:

$$\psi_n^a(z) = \sqrt{2n}\sum_{k=0}^{n-1}\frac{(1-n)_k(c^2z^2)^{k+1}}{(2)_k k!}, \tag{3.40}$$

where $(n)_k$ is the Pochhammer symbol and defined as

$$(n)_k = \frac{\Gamma(n+k)}{\Gamma(n)}. \tag{3.41}$$

It is interesting to note that while in the hard-wall model we had to impose a somewhat artificial boundary condition at the IR boundary to get a mass spectrum, in soft-wall models this comes naturally as a condition for normalizability.

General Case

In general, the (unsimplified) dilaton field has a very complicated algebraic form. One has to find the solution numerically, but apart from that, the idea of how to find the meson modes is the same as presented above. One uses $\psi^a(L_0) = 0$ and $\partial_z\psi^a(L_0) = c \neq 0$ as initial values. The choice of c is arbitrary and will not affect the normalized solution, since the differential equation is linear. One then has to find those k^2, for which the solution is normalizable.

One can also show that the solutions in general obey an orthogonality relation and because of the normalization it holds:

$$\int_{L_0}^{\infty} \mathrm{d}z \frac{e^{-\Phi(z)}}{z}\psi_n^a(z)\psi_m^a(z) = \delta_{mn}. \tag{3.42}$$

(To prove this one can look at the proof of the same relation in the hard-wall case and replace every $\frac{1}{z}$ by $\frac{e^{-\Phi(z)}}{z}$ and similarly for the reciprocal.)

3.1.1.2 Scalar mesons

Due to the gauge freedom, we only had scalar mesons for $a = 4, 5, 6, 7$ in the hard-wall case. The same happens for the soft-wall model. Since we only included $a = 1, 2, 3$, our model will not give any scalar mesons. (There are however other ways of including the scalar mesons, discussed in Chapter 4.)

3.1.2 Axial Sector

The action for the axial sector is given by

$$\mathcal{S}_A = \int \mathrm{d}^5 x e^{-\Phi(z)} \sum_a \left(-\frac{L}{4g_5^2 z} \left(\partial_M A_N^a - \partial_N A_M^a\right)^2 + \frac{\beta^a(z) L}{2zg_5^2} \left(\partial_M \pi^a - A_M^a\right)^2 \right). \quad (3.43)$$

The equation of motion becomes

$$\eta^{ML} \partial_M \left(\frac{e^{-\Phi(z)}}{z} (\partial_L A_N^a - \partial_N A_L^a) \right) + e^{-\Phi(z)} \frac{\beta^a(z)}{z} (A_N^a - \partial_N \pi^a). \quad (3.44)$$

Decomposing A_μ^a into its transversal and longitudinal part, writing $A_{\mu\|}^a = \partial_\mu \phi^a$, and dropping A_z^a since it can be gauged away, one gets

$$\begin{aligned} &\frac{e^{-\Phi(z)}}{z} \eta^{\mu\lambda} \partial_\mu \partial_\lambda A_{\mu\perp}^a - \partial_z \left(\frac{e^{-\Phi(z)}}{z} (\partial_z A_{\nu\perp}^a) \right) + e^{-\Phi(z)} \frac{\beta^a(z)}{z} A_{\nu\perp}^a \\ &-\partial_z \left(\frac{e^{-\Phi(z)}}{z} (\partial_z \partial_\nu \phi^a) \right) + e^{-\Phi(z)} \frac{\beta^a(z)}{z} \partial_\nu \phi^a - e^{-\Phi(z)} \frac{\beta^a(z)}{z} \partial_\nu \pi^a = 0 \\ &\qquad -\frac{e^{-\Phi(z)}}{z} \eta^{\mu\lambda} \partial_\mu \partial_\lambda \partial_z \phi^a - e^{-\Phi(z)} \frac{\beta^a(z)}{z} \partial_z \pi^a = 0. \end{aligned} \quad (3.45)$$

We then perform the Fourier Transform and split this up in an equation for the transverse part and in a pair of equations for the longitudinal part. We get

$$\partial_z \left(\frac{e^{-\Phi(z)}}{z} (\partial_z \widehat{A}_{\mu\perp}^a) \right) + e^{-\Phi(z)} \frac{k^2 - \beta^a(z)}{z} \widehat{A}_{\mu\perp}^a = 0 \quad (3.46)$$

(transverse part) and

$$\begin{aligned} \partial_z \left(\frac{e^{-\Phi(z)}}{z} (\partial_z \widehat{\phi}^a) \right) - e^{-\Phi(z)} \frac{\beta^a(z)}{z} (\widehat{\phi}^a - \widehat{\pi}^a) = 0 \\ k^2 \partial_z \widehat{\phi}^a - \beta^a(z) \partial_z \widehat{\pi}^a = 0 \end{aligned} \quad (3.47)$$

(longitudinal part).

3.1.2.1 Pseudovector Mesons

Pseudovector mesons we get as solution of the equation for the transverse part. The equation is identical to the one for the transverse part of the vector sector (vector mesons) and can be solved numerically using the same boundary conditions. We will use the same normalization as for the vector mesons.

3.1.2.2 Pseudoscalar Mesons

Normalizable solutions to the pair of differential equations for the longitudinal part of the axial field correspond to pseudoscalar mesons. By defining $y^a(k^2, z) = \frac{e^{-\Phi(z)}}{z}\partial_z\widehat{\phi}^a(k^2, z)$ we can transform the two equations into a single equation for y^a:

$$\partial_z\left(\frac{z}{\beta^a(z)e^{-\Phi(z)}}\partial_z y^a\right) + \frac{z}{e^{-\Phi(z)}}\left(\frac{k^2}{\beta^a(z)} - 1\right)y^a = 0. \tag{3.48}$$

The normalizable solutions of this equation we will denote by $\eta^a_n(z)$ and they will be related to a discrete mass spectrum $k^2 = m^{a\,2}_n$. Again we get an orthogonality relation and due to our choice of normalization we get:

$$\int_{L_0}^{\infty} \mathrm{d}z \frac{z}{\beta^a(z)e^{-\Phi(z)}}\eta^a_n(z)\eta^a_m(z) = \frac{\delta_{mn}}{m^{a\,2}_n}. \tag{3.49}$$

3.1.3 Results

We have four parameters for our model, namely m_q, σ_q, λ, and κ. These we want to determine by matching the model, which we shall call Model B, to phenomenology.

Let us start with λ. The ρ mesons (vector mesons) do not couple to the vacuum solution ($\alpha^a = 0$). This suggests, that the ρ mesons are a good starting point to determine the parameter λ. We have seen that if we assume the background field to be simply $\Phi(z) = c^2z^2$, then we get an exact mass spectrum of the form $M^2_\rho = 4nc^2$. On the other hand, our modified dilaton field behaves for large z as $\Phi(z) \overset{z\to\infty}{=} \frac{\kappa}{4}(A + B)z^2 = \lambda z^2$. Since the modified field only differs from the original, simple one in the small z region and higher modes are more concentrated in the higher z region, it is a reasonable assumption that for large n, the results with the simple and modified dilaton field coincide. This is in fact the case. So, for n large, the ρ mesons will lie on a trajectory $M^2_\rho = 4n\lambda$. By using experimental data, we can hence fix our parameter λ. We use the radial excitations up to $n = 7$, but exclude $n = 1, 2$ since they obviously deviate from the linear behavior. We then determine $\lambda = 0.187$ GeV^2, which corresponds to $c = 432.5$ MeV (compare to $(L_1)^{-1}$ in the hard-wall case). The result is shown in Figure 3.1.

Our next goal is to fix the parameter κ, which is essentially responsible for the mass splitting between the vector (ρ) and pseudovector (a_1) mesons. To see this, we solve the differential equations for the transverse part in the vector and the axial sector for k^2, which becomes the mass. For the vector mesons, we get

$$M^2_{\rho,n} = -\frac{1}{\psi^a_n}\frac{z}{e^{-\Phi(z)}}\partial_z\left(\frac{e^{-\Phi(z)}}{z}\partial_z\psi^a_n(z)\right) + \underbrace{\alpha^a(z)}_{=0} \tag{3.50}$$

ans similarly

$$M^2_{a_1,n} = -\frac{1}{\psi^a_n}\frac{z}{e^{-\Phi(z)}}\partial_z\left(\frac{e^{-\Phi(z)}}{z}\partial_z\psi^a_n(z)\right) + \beta^a(z) \tag{3.51}$$

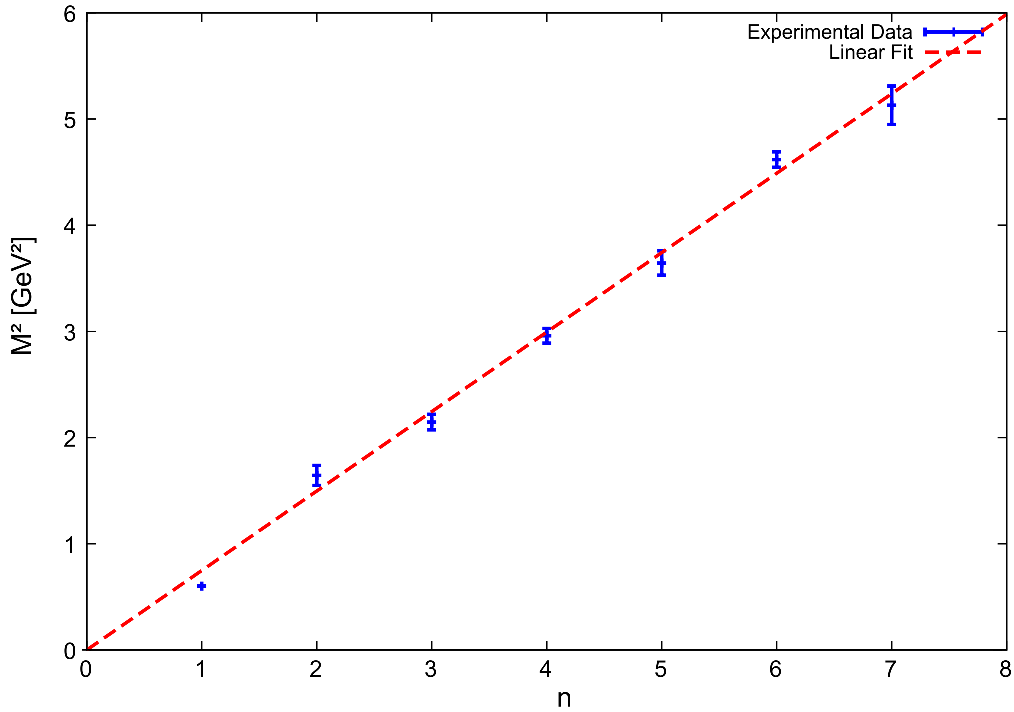

Figure 3.1: First seven ρ masses and linear fit to the last five.

for the pseudovector mesons. For large n this gives

$$\Delta(M^2) = \lim_{n\to\infty}(M^2_{a_1,n} - M^2_{\rho,n}) = \lim_{z\to\infty}(\beta^a(z) - \alpha^a(z)) = \frac{4g_5^2\lambda}{\kappa}. \tag{3.52}$$

Hence, we should determine κ from the squared mass difference in the limit of large n (see Figure 3.2). One gets that $\Delta(M^2) = 1.54$ GeV^2, which gives $\kappa \approx 19.2$. Only the last three masses ($n = 3, 4, 5$) were used to determine $\Delta(M^2)$ since we said that the mass difference becomes constant for *large* n.

The last step is to determine m_q and σ_q by matching the experimental values of m_π and f_π with our model. We find $m_q = 9.31$ MeV and $\sigma_q = (205.5\text{ MeV})^3$.

Finally, we can calculate the masses of the π, ρ, and a_1 mesons. The results are shown in Table 3.1. The results are in good agreement but at least for the a_1 and the ρ this should not surprise since we determined the constants such that we match their mass trajectories for large n.

We can also determine the decay constants of the π, ρ, and a_1. They are determined exactly as in the hard-wall case. The results are shown in Table 3.2.

Finally, let us calculate the pion form factor $F_\pi(k^2)$ in this model. In the soft-wall setting it is given by [5]

$$F_\pi(k^2) = \int_{L_0}^{\infty} \frac{\mathrm{d}z}{z} e^{-\Phi(z)} \mathcal{V}^a(k^2, z)\left((\partial_z\widehat{\phi}^a_1(z))^2 + \beta^a(z)(\widehat{\pi}^a_1(z) - \widehat{\phi}^a_1(z))^2\right), \tag{3.53}$$

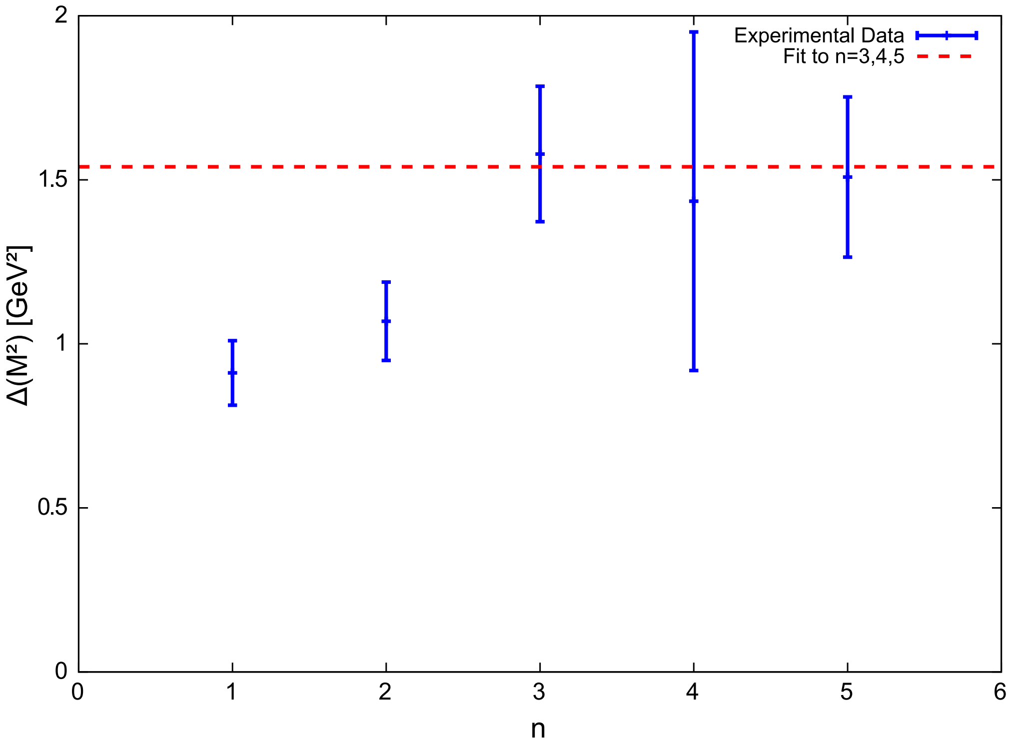

Figure 3.2: Mass differences between ρ and a_1.

where $a = 1, 2, 3$. $\mathcal{V}^a(k^2, z)$ is again the bulk-to-boundary propagator of the transverse part in the vector sector. It is obtained from the differential equation with the boundary condition $\mathcal{V}^a(k^2, L_0) = 1$ in the IR and a finiteness condition in the UV. The results can be seen in Figure 3.3. One sees that the soft-wall result is slightly more off than the hard-wall result.

One remark about the numerics should be made. While for the hard-wall computations a simple and quick BVP solver could be used, this is not possible in the soft-wall case. The problem is that instead of a real boundary value at a real IR boundary, we only have a vague finiteness condition at infinity. The only method in question is hence a shooting method, where one tries to minimize the function values at some right endpoint. This is a far from trivial procedure and automation often fails, especially when one tries to find higher radial states. However it is exactly done, it turns out to be more time-consuming than for the hard-wall model.

3.2 Approximate Vacuum Solution Model

Another idea to overcome the difficulties of the soft-wall model concerning the vacuum expectation value was proposed by [5]. We will now extend their model to the broken flavour symmetry case. Here, one also writes down the vacuum solution explicitly but one

Meson	n	Model B [MeV]	Measured [MeV]
π	1	(fit)	139.57
	2	1530.0	1300 ± 100
	3	1819.6	1816 ± 14
	4	2031.8	
	5	2217.5	
ρ	1	488.8	775.5 ± 1
	2	1157.2	1282 ± 37
	3	1448.4	1465 ± 25
	4	1694.9	1720 ± 20
	5	1906.9	1909 ± 30
	6	2096.2	2149 ± 17
	7	2269.3	2265 ± 40
a_1	1	1128.7	1230 ± 40
	2	1541.7	1647 ± 22
	3	1832.7	1930^{+30}_{-70}
	4	2043.1	2096 ± 122
	5	2226.8	2270^{+55}_{-40}

Table 3.1: Comparison of masses as calculated in Model B with experimental data.

Meson	Model B [MeV]	Measured [MeV]
π	(fit)	92.4 ± 0.35[17]
ρ	225.1	345 ± 8[2]
a_1	360.0	433 ± 13[2]

Table 3.2: Comparison of decay constants as calculated in Model B with experimental data.

keeps the simple dilaton field $\Phi(z) = c^2 z^2$. Of course $v(z)$ will not really be the vacuum solution anymore, but it is chosen in a way that it is similar to the exact vacuum solution (let us call it $v_0(z)$ from now on), which was of the form

$$v_0(z) = \frac{\zeta m}{L} \Gamma\left(\frac{3}{2}\right) z U\left(\frac{1}{2}; 0; (cz)^2\right). \tag{3.54}$$

We already stated that this is finite for $z \to \infty$. We will now construct a function, which is similar to v_0, both in the IR and in the UV region. In addition, we will explicitly include the parameter σ, such that the asymptotic IR behaviour is like $\frac{m\zeta}{L} z + \frac{\sigma}{L\zeta} z^3$. A good choice is [5]:

$$v(z)L = \left(m\zeta z + \frac{\sigma}{\zeta} z^3\right)\left(1 - e^{-\frac{A}{c^4 z^4}}\right) + B e^{-\frac{3}{4c^2 z^2}}. \tag{3.55}$$

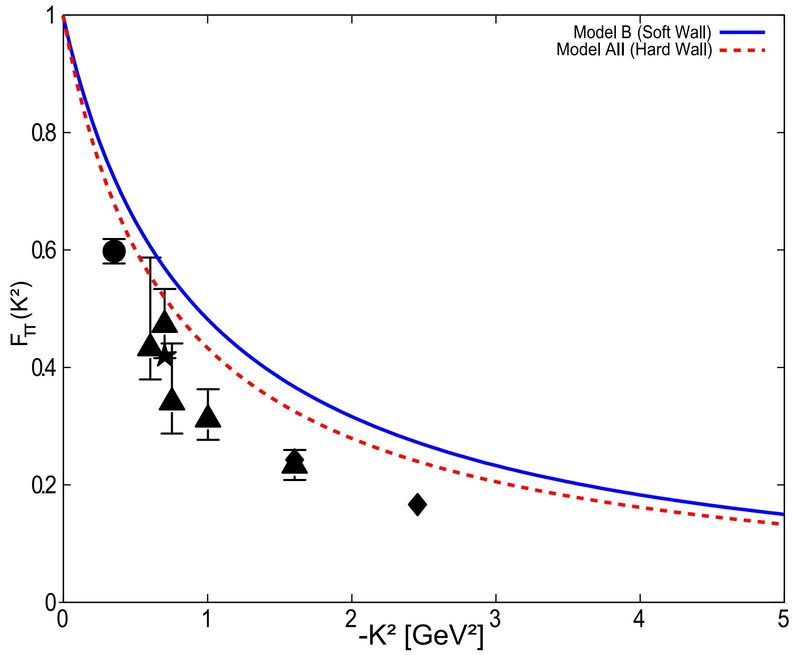

Figure 3.3: Spacelike pion form factor as calculated in Model B with comparison to experimental data [26, 27, 28, 29] and Model AII (hard-wall).

It has the correct behaviour in the IR and for large z has the asymptotic form $B\exp(-\frac{3}{4c^2z^2})$, which is the same as the one of v_0 if we set

$$B = \frac{m\sqrt{\pi}\zeta}{2c}. \tag{3.56}$$

The parameter A determines the intermediate behaviour of $v(z)$. Let us for simplicity reasons put $A = 1$.

Of course, the vacuum solution is not really a solution to the vacuum equation of motion any more but it has the correct behaviour in the IR and the UV. The hope is that the intermediate behaviour does not greatly affect the results. An a posteriori justification of the above choice would be reasonably good results of the model when trying to reproduce experimental data. It might very well be possible that the parameters we achieve in the end differ from the ones in the previous models, but this should not surprise too much. Note that all the derivations in the preceding chapter are still valid, $\Phi(z)$ and $v_q(z)$ have simply changed. Moreover it is now no problem at all to include the strange quark.

3.2.1 Results

The mass relation

$$M^2_{\rho,n} = 4c^2 n \tag{3.57}$$

we derived in the previous chapter for the ρ meson is now exact since we are working with the simple form of the dilaton field and the ρ does not couple to the vacuum solution v_q. This allows us to determine the parameter c directly by looking at the (higher) radial states of the ρ meson, as we did before in Model B to determine λ. The result is of course the same and we get $c = 432.5$ MeV. Then, we can determine m_q and σ_q by matching m_π and f_π. We get $m_q = 4.45$ MeV and $\sigma_q = (265.2 \text{ MeV})^3$. We can then calculate the masses of the a_1 and higher π masses as well as decay constants for what we call Model C. For the sake of simplicity, let us assume again that $\sigma_q = \sigma_s$. Then we can determine m_s by matching the Kaon mass m_K. Once all parameters are determined, we can calculate all the different meson masses and decay constants. The results are shown in Table 3.3.

Surprisingly, most of the calculated masses are quite close to the experimental values, with a few exceptions. The same can be said about the decay constants. Allowing $\sigma_q \neq \sigma_s$ and making a global fit, also leaving A and B free parameters should give even better results. This is however not easy to implement for the reasons mentioned earlier in this chapter.

Observable	n	Model C [MeV]	Measured [MeV]
m_π	1	(fit)	139.57
	2	1646.7	1300 ± 100
	3	1862.0	1816 ± 14
f_π	1	(fit)	92.4 ± 0.35[17]
m_K	1	(fit)	495.7
	2	1652.4	~ 1460
	3	1871.3	~ 1830
f_K	1	101.7	113 ± 1.4[2]
$m_{K_0^*}$	1	998.7	672 ± 40
	2	1363.5	1425 ± 50
	3	1634.1	1945 ± 30
$f_{K_0^*}$	1	15.6	
M_ρ	1	865.0	775.5 ± 1
	2	1223.3	1282 ± 37
	3	1498.3	1465 ± 25
	4	1730.1	1720 ± 20
	5	1934.3	1909 ± 30
	6	2118.9	2149 ± 17
	7	2288.7	2265 ± 40
$\sqrt{F_\rho}$	1	290.2	345 ± 8[2]
M_{K^*}	1	868.8	893.8
	2	1225.3	1414 ± 15
	3	1499.9	1717 ± 27
$\sqrt{F_{K^*}}$	1	289.0	
M_{a_1}	1	1222.6	1230 ± 40
	2	1480.4	1647 ± 22
	3	1638.6	1930^{+30}_{-70}
	4	1815.8	2096 ± 122
	5	2005.3	2270^{+55}_{-40}
$\sqrt{F_{a_1}}$	1	170.3	433 ± 13[2]
M_{K_1}	1	1246.7	1272 ± 7
	2	1513.7	1403 ± 7
	3	1675.7	1650 ± 50
$\sqrt{F_{K_1}}$	1	147.0	

Table 3.3: Comparison of masses and decay constants as calculated in Model C with experimental data.

4 Scalar Mesons

We have seen in the calculations of the preceding two chapters that scalar mesons were a little bit tricky to handle. In fact, for $N_\mathrm{f} = 2$, we did not obtain scalar mesons at all. The following chapter is an overview, how one can deal with scalar mesons, both in the context of a hard-wall and of a soft-wall model. Furthermore, we will explicitly calculate expressions like two-point correlators and will also pay closer attention to the boundary conditions.

4.1 Soft Wall

We begin by studying scalar mesons in a soft-wall approach. As it turns out the shortcoming of the simple dilaton field

$$\Phi(z) = (cz)^2, \tag{4.1}$$

which caused problems when dealing with the vacuum expectation value, does not affect our calculations concerning scalar mesons, so we use this form for simplicity reasons.

For convenience, we repeat the 5D action

$$\mathcal{S} = \mathrm{d}^5 x \sqrt{g} e^{-\Phi(z)} \operatorname{Tr} \left(|D_M X|^2 + \frac{3}{L^2} |X|^2 - \frac{1}{g_5^2} \left(F_L^2 + F_R^2 \right) \right) \tag{4.2}$$

and the vacuum solution

$$2X_0(z) = v(z) = \frac{M\zeta}{L} \Gamma\left(\frac{3}{2}\right) zU\left(\frac{1}{2}; 0; c^2 z^2\right). \tag{4.3}$$

4.1.1 Scalar Mesons

A scalar field S can be built in explicitly by writing [6]

$$X = e^{i\pi}(X_0 + S)e^{i\pi}. \tag{4.4}$$

We insert this into our action (4.2) and look at quadratic terms in S. A straightforward calculation gives

$$\mathcal{S}_S = \frac{1}{2} \int \mathrm{d}^5 x \sqrt{g} e^{-\Phi(z)} \left(g^{MN} \partial_M S^a \partial_N S^a + \frac{3}{L^2} S^a S^a \right), \tag{4.5}$$

where we expanded

$$S = S^a t^a. \tag{4.6}$$

(One could in principle also have included the scalar singlet.) The equation of motion is derived exactly as we derived the equation of motion for X_0 with the difference that the solution is now also x^μ-dependent. Since the steps are the same, we need not repeat it here. The result is analogous:

$$-\eta^{MN}\partial_M\left(e^{-\Phi(z)}\frac{L^3}{z^3}\partial_N S^a\right)+3e^{-\Phi(z)}\frac{L^3}{z^5}S^a=0 \tag{4.7}$$

Again, we switch over to momentum space by taking the Fourier transform. This gives

$$k^2\frac{L^3}{z^3}e^{-\Phi(z)}\widehat{S}^a+\partial_z\left(e^{-\Phi(z)}\frac{L^3}{z^3}\partial_z\widehat{S}^a\right)+\frac{3L^3}{z^5}e^{-\Phi(z)}\widehat{S}^a=0 \tag{4.8}$$

This differential equation can be somewhat simplified by introducing Y^a with

$$\widehat{S}^a=Y^a e^{\frac{c^2z^2+3\log(cz)}{2}}. \tag{4.9}$$

Then after a few simplification steps we have

$$\left(\frac{3}{4z^2}+2c^2-k^2+c^4z^2\right)Y^a-\partial_z^2Y^a=0. \tag{4.10}$$

After introducing $\widetilde{z}=cz$ and rearranging the terms, this gives a Schrödinger-like equation

$$-\partial_{\widetilde{z}}^2Y^a+V(\widetilde{z})Y^a=\frac{k^2}{c^2}Y^a, \tag{4.11}$$

where

$$V(\widetilde{z})=\widetilde{z}^2+2+\frac{3}{4\widetilde{z}^2}. \tag{4.12}$$

This differential equation can be solved analytically. The general solution is only normalizable if

$$m_n^2:=k^2=c^2(4n+2) \tag{4.13}$$

for $n=1,2,3\ldots$. Then the solution is given by

$$Y^a\propto e^{-\frac{\widetilde{z}^2}{2}}\widetilde{z}^{\frac{3}{2}}L^1_{n-1}(\widetilde{z}^2), \tag{4.14}$$

where L^1_{n-1} is a generalized (or associated) Laguerre polynomial. Substituting this back into the expression for $\widehat{S}^a$ and normalization to unity gives

$$\widehat{S}^a_n(\widetilde{z})=\sqrt{\frac{2}{n}}\widetilde{z}^3L^1_{n-1}(\widetilde{z}^2). \tag{4.15}$$

We have therefore shown that the scalar mesons lie on a trajectory of the form $m_n^2\sim n$, which they had to because of the choice of the dilaton field. We can compare the result to the vector mesons, for which we had

$$M_n^2=c^24n. \tag{4.16}$$

Hence scalar mesons are always heavier than vector mesons, which is consistent with phenomenology. To see, how good the trajectory for the scalar mesons is in agreement with experimental data, we fit the mass to the radial excitations of the (flavourless) scalar mesons. For definiteness let us choose the f_0. It turns out that the model fits the data very well if we exclude the lowest scalar $f_0(600)$. Since for light scalar mesons, mixing with scalar glueballs or other states occurs, it could be argued that there has been a misidentification of the lightest state. Moreover, if we assume that $a_0(980)$ and $f_0(980)$ are in the same octet and noting that $a_0(980)$ is the lightest of its kind, we could argue the same. With this, we get a value for c of $c = 408.3$ MeV, which is in good agreement with previous results.

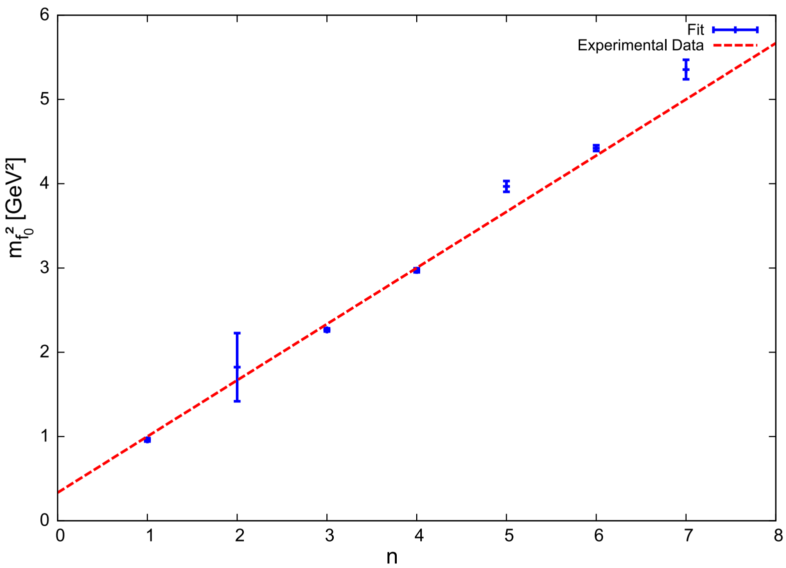

Figure 4.1: Calculated masses of the scalar f_0 mesons and measured values.

4.2 Hard Wall

Scalar mesons can of course also be treated in a hard-wall setting, as we will do in the following. On this occasion we will see, how to deal with the boundary conditions to fix the coefficients of the vacuum solution and we will calculate the scalar two-point correlator.

For convenience, let us consider a differently normalized action:

$$\mathcal{S} = \int \mathrm{d}^5 x \mathcal{L}_5 = \int \mathrm{d}^5 x \sqrt{g} \frac{M_5}{2} \operatorname{Tr} \left(|D_M X|^2 + \frac{3}{L^2} |X|^2 - \frac{1}{2} \left(F_L^2 + F_R^2 \right) \right). \tag{4.17}$$

This can be brought into the form of the action we have been using so far by a redefinition of fields and constants.

4.2.1 Vacuum Solution

We have already derived the vacuum equation of motion (now $v(z) := \langle X \rangle$)

$$\partial_z \left(\frac{L^3}{z^3} \partial_z X_0 \right) + 3 \frac{L^3}{z^3} X_0 = 0. \tag{4.18}$$

and the general solution

$$v(z) = c_1 z + c_2 z^3 \tag{4.19}$$

in Chapter 2. The coefficient matrices c_1 and c_2 depend on the boundary values $v(L_0)$ and $v(L_1)$. We will just regard them as constants and give them their physical meaning later, when we have expressions we can compare to QCD. We set

$$\widetilde{M}_q := \frac{L}{L_0} v(L_0), \tag{4.20}$$

which corresponds to $v'(0)$ in the limit $L_0 \to 0$, and

$$\xi = L v(L_1). \tag{4.21}$$

Expressing the coefficients c_1 and c_2 in terms of the boundary values gives

$$c_1 = \frac{\widetilde{M}_q L_1^3 - \xi L_0^2}{L L_1 (l_1^2 - L_0^2)} \tag{4.22}$$

and

$$c_2 = \frac{\xi - \widetilde{M}_q L_1}{L L_1 (L_1^2 - L_0^2)}. \tag{4.23}$$

We already know that a nonzero $\widetilde{M}_q$ (or c_1) corresponds to the explicit breaking of chiral symmetry, while c_2 corresponds to the spontaneous breaking of the chiral symmetry.

This tells us, that we obtain the value of ξ by dynamically minimizing the action. So we plug in our vacuum solution (4.19) into the 5D action (4.17) and integrate over z. This gives

$$\begin{aligned}
\mathcal{S} &= \int \mathrm{d}^5 x \frac{L^5}{z^5} M_5 \frac{1}{2} \operatorname{Tr} \left(-\frac{z^2}{L^2} (c_1 + 3 c_2 z^2)^2 + \frac{3}{L^2} (c_1 z + c_2 z^3)^2 \right) \\
&= \int \mathrm{d}^4 x L^3 M_5 \int_{L_0}^{L_1} \mathrm{d}z \operatorname{Tr} \left(\frac{c_1^2}{z^3} - 3 c_2^2 z \right) \\
&= - \int \mathrm{d}^4 x M_5 L \operatorname{Tr} \left(\frac{-\widetilde{M}_q^2 L_1^6 - 4 \widetilde{M}_q L_1^3 \xi L_0^2 - \xi^2 L_0^4 + 3 L_0^2 L_1^2 \xi^2 + 3 \widetilde{M}_q^2 L_1^4 L_0^2}{2 L_1^2 (L_1^2 - L_0^2) L_0^2 L_1^2} \right).
\end{aligned} \tag{4.24}$$

Taking the limit $L_0 \to 0$ we get

$$\mathcal{S} = - \int \mathrm{d}^4 x M_5 L \operatorname{Tr} \left(-\frac{\widetilde{M}_q^2}{2 L_0^2} - 2 \frac{\xi \widetilde{M}_q}{L_1^3} + \frac{3}{2} \frac{\xi^2}{L_1^2} + \frac{3}{2} \frac{\widetilde{M}_q^2}{L_1^4} \right). \tag{4.25}$$

In the chiral limit, this would lead to $c_2 = 0$ and hence give no spontaneous symmetry breaking. To avoid this, we add a potential term on the IR boundary to the 4D Lagrangian of the form [15]

$$\mathcal{L}_{\mathrm{IR}} = -\frac{L^4}{z^4}\, V(X)|_{L_1}\,, \tag{4.26}$$

where

$$V(X) = -\frac{1}{2}m_b^2 \,\mathrm{Tr}\,|X|^2 + \lambda\,\mathrm{Tr}\,|X|^4 \tag{4.27}$$

with parameters λ and m_b. This yields an effective 4D action

$$\begin{aligned}\mathcal{S}_{\mathrm{eff}} = -\int \mathrm{d}^4x\,\mathrm{Tr}\Bigg(& M_5 L\left(-\frac{\widetilde{M}_q^2}{2L_0^2} - 2\frac{\xi\widetilde{M}_q}{L_1^3} + \frac{3}{2}\frac{\xi^2}{L_1^2} + \frac{3}{2}\frac{\overline{M}_q^2}{L_1^4}\right) \\ & + \frac{L^4}{L_1^4}\left(-\frac{1}{2}m_b^2\left(\frac{\xi}{L}\right)^2 + \lambda\left(\frac{\xi}{L}\right)^4\right)\Bigg).\end{aligned} \tag{4.28}$$

Minimizing this with respect to ξ gives

$$\xi^2 = \frac{\mathbb{1}}{4\lambda}\left(m_b^2L^2 - 3M_5L\right) + \mathcal{O}(\widetilde{M}_q). \tag{4.29}$$

We now have a five-dimensional model depending on five parameters: $\widetilde{M}_q$, M_5, L_1, ξ and λ. Let us then redefine $\xi \to \xi\mathbb{1} + \mathcal{O}(\widetilde{M}_q)$, so we treat ξ as scalar parameter from now on.

4.2.2 Scalar Two-Point Correlator

We again include the scalar field by writing

$$X = e^{i\pi}(X_0 + S)e^{i\pi}, \tag{4.30}$$

where we are again working in the flavour symmetric case. Inserting the expression for X above in the action and selecting the quadratic terms in S, we get

$$\mathcal{S}_S = -\frac{M_5}{2}\int \mathrm{d}^5x\frac{L^3}{z^3}\,\mathrm{Tr}\left(\eta^M N\partial_M S\partial_N S + \frac{3}{z^2}S^2\right), \tag{4.31}$$

which using partial integration can be cast in the form

$$\mathcal{S}_S = -\frac{M_5L^3}{2}\int \mathrm{d}^5x\,\mathrm{Tr}\left(\frac{1}{z^3}S\eta^{\mu\nu}\partial_\mu\partial\nu S - S\partial_z(\frac{1}{z^3}\partial_z S) - \frac{3}{z^5}S^2\right). \tag{4.32}$$

We also have to consider boundary terms for the 4D Lagrangian, which occur when we plug in the expression for X in terms of X_0 and S and integrate by parts with respect to z. This gives the boundary term

$$-\int \mathrm{d}^4x\,\frac{M_5}{2}\frac{L^3}{z^3}\,\mathrm{Tr}\left(S\partial_z S + 2S\partial_z v\right)\Big|_{L_0}^{L_1}. \tag{4.33}$$

We must not forget the potential (4.27) on the IR-boundary, which, using the expression (4.29) to replace m_b^2 up to $\mathcal{O}(\widetilde{M}_q)$, reads

$$
\begin{aligned}
\mathcal{L}_{\mathrm{IR}} &= -\frac{L^4}{L_1^4}\,\mathrm{Tr}\left(-\frac{1}{2}m_b^2\,|X|^2+\lambda\,|X|^4\right) = -\frac{L^4}{L_1^4}\,\mathrm{Tr}\left(-\frac{1}{2}\left(\frac{4\lambda\xi^2}{L^2}+\frac{3M_5}{L}\right)|X|^2+\lambda\,|X|^4\right)\\
&= \left(\frac{2L^2\lambda\xi^2}{L_1^4}+\frac{3M_5L^3}{2L_1^4}\right)\mathrm{Tr}(v^2+2vS+S^2)-\frac{L^4\lambda}{L_1^4}\,\mathrm{Tr}(v^4+4v^3S+6v^2S^2+4vS^3+S^4),
\end{aligned}
\tag{4.34}
$$

where v is evaluated at the IR boundary. Keeping only the terms involving S, this gives

$$
\begin{aligned}
\mathcal{L}_{\mathrm{IR}} = &\left(\left(\frac{2L^2\lambda\xi^2}{L_1^4}+\frac{3M_5L^3}{2L_1^4}\right)2\frac{\xi}{L}-\frac{L^4\lambda}{L_1^4}4\frac{\xi^3}{L_1^3}\right)\mathrm{Tr}\,S\\
&+\left(\left(\frac{2L^2\lambda\xi^2}{L_1^4}+\frac{3M_5L^3}{2L_1^4}\right)-\frac{L^4\lambda}{L_1^4}6\frac{\xi^2}{L^2}\right)\mathrm{Tr}\,S^2+\mathcal{O}(S^3)\\
= &\,\frac{3M_5L^2\xi}{L_1^4}\,\mathrm{Tr}\,S-\left(\frac{4L^2\lambda\xi^2}{L_1^4}-\frac{3M_5L^3}{2L_1^4}\right)\mathrm{Tr}\,S^2+\mathcal{O}(S^3).
\end{aligned}
\tag{4.35}
$$

If we define

$$
V(S)|_{L_1} := m_S^2\,\mathrm{Tr}\,S^2\big|_{L_1}+\mathcal{O}(S^3)
\tag{4.36}
$$

with

$$
m_S^2 := \frac{4\lambda\xi^2}{L^2}-\frac{3M_5}{2L}+\mathcal{O}(\widetilde{M}_q),
\tag{4.37}
$$

the term reads as

$$
\mathcal{L}_{\mathrm{IR}} = -\left.\frac{L^4}{z^4}V(S)\right|_{L_1}+\frac{3M_5L^2\xi}{L_1^4}\,\mathrm{Tr}\,S|_{L_1}\,.
\tag{4.38}
$$

If we now look at the second term in (4.33) and evaluate it at the upper boundary we get

$$
\begin{aligned}
-\frac{M_5L^3}{L_1^3}\,\mathrm{Tr}\,S\partial_z v|_{L_1} &= -\frac{M_5L^3}{L_1^3}\,\mathrm{Tr}\,(c_1+3c_2z^2)S\big|_{L_1}\\
&= -\frac{M_5L^3}{L_1^3}\left(\frac{-\xi L_0^2}{LL_1(L_1^2-L_0^2)}+3\frac{\xi}{LL_1(L_1^2-L_0^2)}L_1^2\right)\mathrm{Tr}\,S|_{L_1}\\
&\overset{L_0\ll L_1}{=} -\frac{M_5L^3}{L_1^3}3\frac{\xi}{LL_1^3}L_1^2\,\mathrm{Tr}\,S|_{L_1} = -\frac{3M_5L^2\xi}{L_1^4}\,\mathrm{Tr}\,S|_{L_1}\,,
\end{aligned}
\tag{4.39}
$$

so it cancels with the second term in the potential term (4.38), resulting in a total boundary term in the 4D Lagrangian for S of

$$
\mathcal{L}_{\mathrm{bound}} = \frac{-M_5}{2}\frac{L^3}{z^3}\,\mathrm{Tr}\,S\partial_z S\Big|_{L_0}^{L_1}-\left.\frac{L^4}{z^4}V(S)\right|_{L_1}+M_5\frac{L^3}{z^3}\,\mathrm{Tr}\,S\partial_z v\Big|_{L_0}\,.
\tag{4.40}
$$

If we want the quadratic terms on the IR-boundary to cancel out, we must impose the boundary condition

$$
\left(M_5\partial_z+2\frac{L}{z}m_S^2\right)\mathrm{Tr}\,S\Big|_{L_1} = 0.
\tag{4.41}
$$

Let us now solve the equation of motion for S, which we derive from the action $\mathcal{S}_S$ (4.32). This gives

$$\frac{L^3}{z^3}\eta^{\mu\nu}\partial_\mu\partial_\nu S - \partial_z(\frac{L^3}{z^3}\partial_z S) - 3\frac{L^3}{z^5}S = 0. \tag{4.42}$$

In momentum space, this reads

$$-\left(\frac{3}{z^2} + k^2\right)\widehat{S}(k^\mu, z) + \frac{3}{z}\partial_z\widehat{S}(k^\mu, z) - \partial_z^2\widehat{S}(k^\mu, z) = 0, \tag{4.43}$$

where we again defined $k^2 = \eta_{\mu\nu}k^\mu k^\nu$.

The general solution can be expressed using Bessel functions of the first and second kind:

$$\widehat{S}(k^\mu, z) = a_1(k_\mu)z^2 J_1(kz) + a_2(k_\mu)z^2 Y_1(kz) \tag{4.44}$$

Next, we fix a_1 and a_2 by using the boundary conditions on the UV- and the IR-boundary. For the IR-boundary we have

$$\left(M_5\partial_z + 2\frac{L}{z}m_S^2\right)\mathrm{Tr}\,\widehat{S}\Big|_{L_1} = 0, \tag{4.45}$$

following from (4.41). The UV boundary value, we determine by relating $\widehat{S}\Big|_{L_0}$ to the source coupled to the QCD operator, which we call s:

$$\widehat{S}\Big|_{L_0} = \alpha\frac{L_0}{L}s, \tag{4.46}$$

where the proportionality factor α will be determined later by matching the expressions we receive to QCD. With these two boundary conditions, we can determine the coefficients a_1 and a_2, which are quite complicated expressions.

Then, we insert our solution for the scalar field S (or $\widehat{S}$) into the action. Since S obeys the equation of motion (4.42), the 5D Lagrangian (4.32) vanishes and does not contribute. So, we are only left with the boundary term (4.40). Since we chose a_1 and a_2 so that the quadratic terms on the IR-boundary cancel, we are left with

$$\mathcal{S} = \int \mathrm{d}^4x\,\frac{M_5L^3}{2z^3}\,\mathrm{Tr}\left(S\partial_z S + 2M_5 S\partial_z v\right)\Big|_{L_0} \tag{4.47}$$

(neglecting terms of order ≥ 2 in S).

Our goal is to calculate the two-point correlator

$$\Pi_S = -\int \mathrm{d}^4x e^{i\eta_{\mu\nu}k^\mu x^\nu}\left\langle J_S(x^\mu)J_S(0)\right\rangle, \tag{4.48}$$

where $J_S = \bar{q}q$, which is the operator dual to S. As explained in the beginning, the correlator can be obtained from our action $\mathcal{S}$ by taking the functional derivative twice with respect to the source s

$$\Pi_S = \frac{\delta^2\mathcal{S}}{\delta s^2}. \tag{4.49}$$

As we can see from equation (4.47), $\widehat{S}$ and hence s appear up to quadratic order (higher orders we neglected) in $\mathcal{S}$. Therefore

$$\mathcal{L} = \frac{1}{2}\Pi_S \operatorname{Tr}(s^2) + \Gamma_S \operatorname{Tr}(s). \tag{4.50}$$

So let us insert our solution for $\widehat{S}$ into the action and collect the terms with quadratic order in s. We begin by writing S in terms of $\widehat{S}$ in (4.47):

$$\mathcal{S} = \operatorname{Tr}\int \mathrm{d}^4x\, \frac{M_5 a^3}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{i\eta_{\mu\nu}k^\mu x^\nu}\widehat{S}(k^\mu, z)\left(\int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{i\eta_{\mu\nu}k^\mu x^\nu}\partial_z\widehat{S}(k^\mu, z) + 2M_5\partial_z v\right)\Bigg|_{L_0}. \tag{4.51}$$

We can ignore the v-term as is cannot produce quadratic terms in s. So we get

$$\begin{aligned}
\mathcal{S} &= \int \mathrm{d}^4x\, \frac{M_5L^3}{2z^3}\operatorname{Tr}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\int \frac{\mathrm{d}^4k'}{(2\pi)^4} e^{i\eta_{\mu\nu}k^\mu x^\nu}\widehat{S}(k^\mu, z) e^{i\eta_{\mu\nu}k'^\mu x^\nu}\partial_z\widehat{S}(k'^\mu, z)\Bigg|_{L_0} \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\int \frac{\mathrm{d}^4k'}{(2\pi)^4}\int \mathrm{d}^4x e^{i\eta_{\mu\nu}(k^\mu+k'^\mu)x^\nu}\ \frac{M_5L^3}{2z^3}\operatorname{Tr}\left(\widehat{S}(k^\mu, z)\partial_z\widehat{S}(k'^\mu, z)\right)\Bigg|_{L_0} \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\int \frac{\mathrm{d}^4k'}{(2\pi)^4}(2\pi)^4\delta(k^\mu + k'^\mu)\ \frac{M_5L^3}{2z^3}\operatorname{Tr}\left(\widehat{S}(k^\mu, z)\partial_z\widehat{S}(k'^\mu, z)\right)\Bigg|_{L_0} \\
&= \operatorname{Tr}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \frac{M_5L^3}{2z^3}\widehat{S}(k^\mu, z)\partial_z\widehat{S}(-k^\mu, z)\Bigg|_{L_0} \\
&= \operatorname{Tr}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \frac{M_5L^3}{2L_0^3}\alpha\frac{L_0}{L}s\partial_z\widehat{S}(k^\mu, z)\Bigg|_{L_0}.
\end{aligned} \tag{4.52}$$

Now we have to evaluate $\partial_z\widehat{S}$ using the values of a_1 and a_2 we determined earlier. Inserting this in the above formula and identifying the term in front of s^2, we get (using a computer algebra system)

$$\Pi_S = \alpha^2 M_5 L\left(\frac{1}{L_0^2} + \frac{k}{L_0}\frac{J_0(kL_0) + b(k)Y_0(kL_0)}{J_1(kL_0) + b(k)Y_1(kL_0)}\right) \tag{4.53}$$

with

$$b(k) := -\frac{kL_1J_2(kL_1) - \frac{8\lambda\xi^2}{M_5L}J_1(kL_1)}{kL_1Y_2(kL_1) - \frac{8\lambda\xi^2}{M_5L}Y_1(kL_1)}. \tag{4.54}$$

Let us study the correlator in the limit $L_0 \to 0$. The Bessel functions have the well-known approximations for small argument ($0 < x \ll \sqrt{\beta+1}$)

$$J_\beta(x) \approx \frac{1}{\Gamma(\beta+1)}\left(\frac{x}{2}\right)^\beta \tag{4.55}$$

and

$$Y_\beta(x) \approx \left\{\begin{array}{ll} \frac{2}{\pi}\left(\log(\frac{x}{2}) + \gamma\right) & \text{if } \beta = 0 \\ -\frac{\Gamma(\beta)}{\pi}\left(\frac{2}{x}\right)^\beta & \text{if } \beta > 0 \end{array}\right., \tag{4.56}$$

where γ is the Euler-Mascheroni constant. With this we get

$$\begin{aligned} \frac{k}{L_0}\frac{J_0(kL_0)+b(k)Y_0(kL_0}{J_1(kL_0)+b(k)Y_1(kL_0)} &\approx \frac{k}{L_0}\frac{1+b(k)\frac{2}{\pi}\left(\log(\frac{kL_0}{2})+\gamma\right)}{\frac{kL_0}{2}-\frac{b(k)}{\pi}\frac{2}{kL_0}} \\ \approx \frac{k}{L_0}\frac{1+b(k)\frac{2}{\pi}\log(\frac{kL_0}{2})}{\frac{b(k)}{\pi}\frac{2}{kL_0}} &\approx \frac{\pi k^2}{2b(k)}+k^2\log(\frac{kL_0}{2}) \\ \approx \frac{\pi k^2}{2b(k)}+\frac{1}{2}k^2\log(k^2L_0^2) & \end{aligned} \tag{4.57}$$

and so

$$\Pi_S(k^2) \approx \alpha^2 M_5 L\left(\frac{1}{L_0^2}+\frac{1}{2}k^2\log(p^2L_0^2)+\frac{\pi k^2}{2b(k)}\right). \tag{4.58}$$

The term $\frac{1}{L_0^2}$ is divergent for $L_0 \to 0$ but can be absorbed in a bare mass and a bare kinetic term for s [15]. This renormalization makes the correlator finite.

If we now look at the case of large momentum, i.e. $pL_1 \gg 1$, we have

$$\Pi_S(k^2) \approx \frac{\alpha^2 M_5 L}{2}k^2\log(k^2). \tag{4.59}$$

This we can match with the case of large momentum in QCD

$$\Pi_S^{\mathrm{QCD}}(k^2) \approx \frac{N_C}{\pi^2}k^2\log(k^2) \tag{4.60}$$

and get by comparison

$$\alpha^2 M_5 L = \frac{N_\mathrm{c}}{4\pi^2}. \tag{4.61}$$

From other calculations [32] it can be found that

$$M_5 L = \frac{N_\mathrm{c}}{12\pi^2} =: \widetilde{N}_C, \tag{4.62}$$

which gives $\alpha = \sqrt{3}$. Since the value of the quark masses are related to the vacuum expectation value of X on the UV-boundary, we can then obtain the relation

$$\widetilde{M_q} = \alpha M_q = \sqrt{3}M_q, \tag{4.63}$$

which gives us the correct normalization of the quark masses. The expression (4.61) is exactly the square of ζ, which we introduced in Chapter 2, and $\frac{1}{M_5L}$ corresponds to the value of g_5^2.

Π_S can also be approximated for small momentum k. For this we look at

$$\begin{aligned} b(k) :&= -\frac{kL_1J_2(kL_1)-\frac{8\lambda\xi^2}{M_5L}J_1(kL_1)}{kL_1Y_2(kL_1)-\frac{8\lambda\xi^2}{M_5L}Y_1(kL_1)} \approx -\frac{kL_1\frac{1}{2}\left(\frac{kL_1}{2}\right)^2-\frac{8\lambda\xi^2}{M_5L}\frac{kL_1}{2}}{-kL_1\frac{1}{\pi}\left(\frac{2}{kL_1}\right)^2+\frac{8\lambda\xi^2}{M_5L}\frac{1}{\pi}\frac{2}{kL_1}} \\ &\approx \frac{\frac{8\lambda\xi^2}{M_5L}\frac{kL_1}{2}}{-kL_1\frac{1}{\pi}\left(\frac{2}{kL_1}\right)^2+\frac{8\lambda\xi^2}{M_5L}\frac{1}{\pi}\frac{2}{kL_1}}. \end{aligned} \tag{4.64}$$

So

$$\alpha^2 M_5 L \frac{\pi k^2}{2b(k)} \approx \alpha^2 M_5 L \frac{\pi k^2}{2} \frac{-kL_1 \frac{1}{\pi}\left(\frac{2}{kL_1}\right)^2 + \frac{8\lambda\xi^2}{M_5 L}\frac{1}{\pi}\frac{2}{kL_1}}{\frac{8\lambda\xi^2}{M_5 L}\frac{kL_1}{2}}$$
$$= \alpha^2 M_5 L \frac{\pi k^2}{2}\left(-\frac{2}{\pi}\frac{M_5 L}{8\lambda\xi^2} + \frac{1}{\pi}\right)\left(\frac{2}{kL_1}\right)^2 \tag{4.65}$$
$$= 3\widetilde{N}_C \left(\frac{2}{L_1^2} - \frac{\widetilde{N}_C}{2\lambda\xi^2 L_1^2}\right)$$

and thus

$$\Pi_S(k^2) \approx 3\widetilde{N}_C \left(\frac{2}{L_1^2} - \frac{\widetilde{N}_C}{2\lambda\xi^2 L_1^2}\right) + \mathcal{O}(k^2) \tag{4.66}$$

is the two-point correlator in the limit of small momentum k.

Going back to the equation (4.58) we observe that for $b(k) = 0$ the correlator becomes infinite. This happens for infinitely many, discrete values of k, which correspond to the masses of scalar mesons.[11] The masses can be determined by finding the roots of $b(k)$. This has to be done numerically. We have to insert some values for our parameters. We found that $\frac{1}{L_1} \approx 320$ MeV and ξ should be taken as $\xi \approx 4$ [32]. λ is still undetermined. So we will have to see how the results behave in dependence of λ. For the mass of the first resonance we find that $M_{S_1} = 0$ MeV for $\lambda \to 0$ and $M_{S_1} = 1226$ MeV for $\lambda \to \infty$. The masses of the first two resonances are pictured in Figure 4.2. We can then compare them to experimental values of the scalar resonances, say $a_0(980)$ and $a_0(1450)$.

We can also derive an approximate analytic expression for the zeros of $b(k)$. First note that

$$xJ_2(x) + xJ_0(x) = 2J_1(x) \tag{4.67}$$

and so $b(k) = 0$ simplifies to

$$kL_1 J_0(kL_1) + \left(\frac{8\lambda\xi^2}{M_5 L} - 2\right) J_1(kL_1) = 0. \tag{4.68}$$

For not too small arguments ($x \gg \left|\beta^2 - \frac{1}{4}\right|$) the Bessel functions behave like

$$J_\beta(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - (2\beta+1)\frac{\pi}{4}\right) \tag{4.69}$$

and so we get

$$\frac{kL_1}{2 - \frac{8\lambda\xi^2}{M_5 L}} = \tan\left(kL_1 - \frac{\pi}{4}\right). \tag{4.70}$$

[11] We did not explicitly calculate the scalar meson modes in this Model, but we could in principle do this as we did in the soft-wall case. This would then lead to eigenmodes with exactly these masses as eigenvalues.

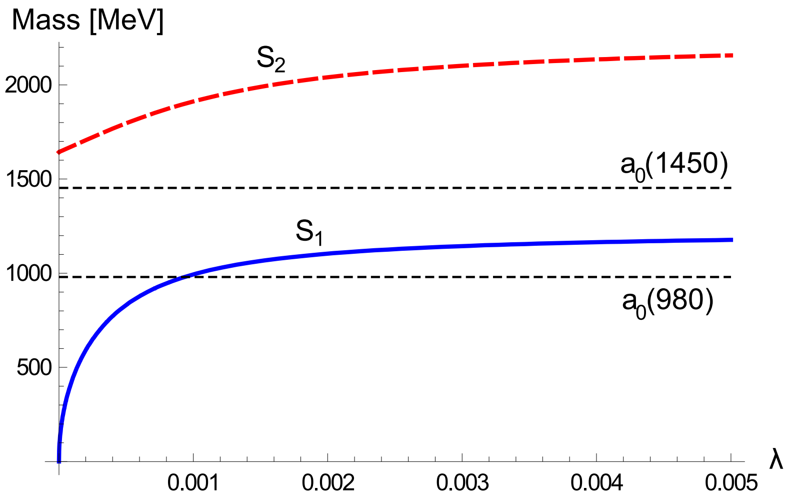

Figure 4.2: The first two resonances S_1 and S_2 as functions of λ with comparison to $a_0(980)$ and $a_0(1450)$.

The solutions of this transcendental equation can be approximated by

$$k_n = \left(n - \frac{1}{4}\right)\frac{\pi}{L_1}, \tag{4.71}$$

where the error goes to zero as $n \to \infty$. The constant $2 - \frac{8\lambda\xi^2}{M_5 L}$ does not appear in the solution, but influences how fast the errors become small. So, as for the other mesons in a hard-wall approach (Chapter 2), we get unphysical Regge trajectories of the form $m_n^2 \sim n^2$. Figure 4.3 shows $b(k)$ for some values of λ together with the approximate zeros of $b(p)$.

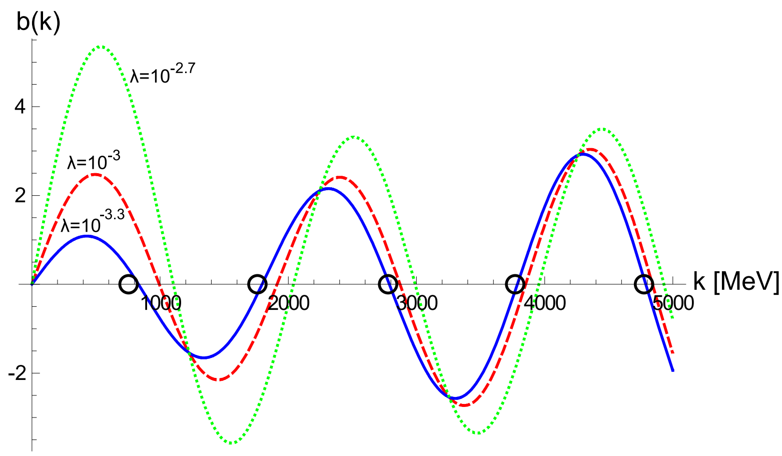

Figure 4.3: $b(k)$ for some values of λ. The black circles mark the approximate zeros.

5 Summary and Conclusions

We have treated the extension of the hard-wall AdS/QCD model to the case $N_f = 3$ with a broken flavour symmetry. We essentially confirmed the results of [2] concerning masses and decay constants of the ground states and the $K_{\ell 3}$ form factor. In addition, we calculated the pion form factor, which was a little bit higher than it is supposed to be, but not worse than other AdS/QCD results. All masses and decay constants were well within 20% of the experimental data.

We then worked in a soft-wall approach, first in an exact model for $N_f = 2$ and then in an approximate model for the flavour-asymmetric case. The main difference to the hard-wall model was that we were able get reasonably good results for higher radial modes. Again most of the masses and decay constants differed from the measured values by about 10% to 20%. The results for the decay constants in the approxmate model are questionable, but the results for the masses were good. Here, results could be improved by making a global fit as we did in the hard-wall case.

Alltogether, the results we obtained and those from other articles seem quite promising considering that this field of research is quite new. Many more observables can be and have been calculated with similarly good results.

Appendices

A.1 The 5D Action

We want to expand the 5D action

$$\mathcal{S} = \int \mathrm{d}^5x\sqrt{g}\,\mathrm{Tr}\left(D_M X^\dagger D^M X + \frac{3}{L^2}X^\dagger X - \frac{1}{4g_5^2}\left(F^L_{MN}F_L^{MN} + F^R_{MN}F_R^{MN}\right)\right) \tag{A.1}$$

up to quadratic order in the fields π, A, and V. We will consider the kinetic term $D_M X^\dagger D^M X$, the mass term $X^\dagger X$, and the field strength term $F^L_{MN}F_L^{MN} + F^R_{MN}F_R^{MN}$ separately.

A.1.1 The Kinetic Term

We want to expand

$$\begin{aligned} D_M X^\dagger D^M X = &\left(\partial_M X^\dagger + iX^\dagger L_M - iR_M X^\dagger\right)\left(\partial^M X - iL^M X + iXR^M\right) \\ = &\underbrace{\partial_M X^\dagger \partial^M X}_{=:(1)} \\ &\underbrace{-i\partial_M X^\dagger L^M X + i\partial_M X^\dagger X R^M + iX^\dagger L_M \partial^M X - iR_M X^\dagger \partial^M X}_{=:(2)} \\ &\underbrace{+X^\dagger L_M L^M X - X^\dagger L_M X R^M - R_M X^\dagger L^M X + R_M X^\dagger X R^M}_{=:(3)}. \end{aligned} \tag{A.2}$$

It is convenient to split up the first term into

$$(1) = \partial_M X^\dagger \partial^M X = \partial_\mu X^\dagger \partial^\mu X + \partial_z X^\dagger \partial^5 X. \tag{A.3}$$

To evaluate these terms we first have to expand $\partial_\mu X$ and $\partial_z X$. We get[12]

$$\begin{aligned} \partial_\mu X &= \partial_\mu\left(e^{i\pi}X_0 e^{i\pi}\right) = \partial_\mu\left(\left(I + i\pi - \frac{1}{2}\pi^2\right)X_0\left(I + i\pi - \frac{1}{2}\pi^2\right)\right) \\ &= \partial_\mu\left(X_0 + iX_0\pi + i\pi X_0 - \frac{1}{2}X_0\pi^2 - \pi X_0\pi - \frac{1}{2}\pi^2 X_0\right) \\ &= iX_0\partial_\mu\pi + i\partial_\mu\pi X_0 - \frac{1}{2}X_0\partial_\mu\pi\pi - \frac{1}{2}X_0\pi\partial_\mu\pi \\ &\quad - \partial_\mu\pi X_0\pi - \pi X_0\partial_\mu\pi - \frac{1}{2}\partial_\mu\pi\pi X_0 - \frac{1}{2}\pi\partial_\mu\pi X_0. \end{aligned} \tag{A.4}$$

[12] In the following calculations, i.e. throughout this section, we will use "=" as a symbol meaning that the trace of the left and the right hand side is identical up to second order in π, A, and V. Since this gives an equivalence relation, the use of the "="-sign is justified.

For $\partial_z X$ we get the same terms, but in addition also terms involving $\partial_z X_0$, namely

$$\begin{aligned}&\left(I + i\pi - \frac{1}{2}\pi^2\right)\partial_z X_0 \left(I + i\pi - \frac{1}{2}\pi^2\right)\\ &= \partial_z X_0 + i\partial_z X_0 \pi + i\pi\partial_z X_0 - \frac{1}{2}\partial_z X_0 \pi^2 - \pi\partial_z X_0 \pi - \frac{1}{2}\pi^2 \partial_z X_0.\end{aligned} \tag{A.5}$$

Together this gives

$$\begin{aligned}\partial_z X = {} & iX_0\partial_z\pi + i\partial_z\pi X_0 - \frac{1}{2}X_0\partial_z\pi\pi - \frac{1}{2}X_0\pi\partial_z\pi\\ & - \partial_z\pi X_0\pi - \pi X_0\partial_z\pi - \frac{1}{2}\partial_z\pi\pi X_0 - \frac{1}{2}\pi\partial_z\pi X_0\\ & + \partial_z X_0 + i\partial_z X_0\pi + i\pi\partial_z X_0 - \frac{1}{2}\partial_z X_0\pi^2 - \pi\partial_z X_0\pi - \frac{1}{2}\pi^2\partial_z X_0.\end{aligned} \tag{A.6}$$

Then we can calculate $\partial_\mu X^\dagger \partial^\mu X$. We get

$$\begin{aligned}\partial_\mu X^\dagger \partial^\mu X &= \left(-i\partial_\mu\pi X_0 - iX_0\partial_\mu\pi\right)\left(iX_0\partial^\mu\pi + i\partial^\mu\pi\right)\\ &= \partial_\mu\pi X_0^2\partial^\mu\pi + i\partial_\mu\pi X_0\partial^\mu\pi X_0 + X_0\partial_\mu\pi X_0\partial^\mu\pi + X_0\partial_\mu\pi\partial^\mu\pi X_0\\ &= \left\{\partial_\mu\pi, X_0\right\}\left\{\partial^\mu\pi, X_0\right\} = \{\partial_\mu\pi^a t^a, X_0\}\{\partial^\mu\pi^b t^b, X_0\}\\ &= \partial_\mu\pi^a\partial^\mu\pi^b\{t^a, X_0\}\{t^b, X_0\} = \sum_a \frac{M_A^{a\,2}}{2}\partial_\mu\pi^a\partial^\mu\pi^a.\end{aligned} \tag{A.7}$$

The expression for $\partial_z X^\dagger \partial^5 X$ is more complicated. We get the same terms we already had for $\partial_\mu X^\dagger \partial^\mu X$ (those quadratic in X_0) plus terms quadratic in $\partial_z X_0$ as well as mixed terms containing both X_0 and $\partial_z X_0$. The terms quadratic in X_0 are

$$\partial_z\pi^a\partial^5\pi^b\{t^a, X_0\}\{t^b, X_0\} = \sum_a \frac{M_A^{a\,2}}{2}\partial_z\pi^a\partial^5\pi^a. \tag{A.8}$$

The terms quadratic in $\partial_z X_0$ are

$$\begin{aligned}& \partial_z X_0\partial^5 X_0 + i\partial_z X_0\partial^5 X_0\pi + i\partial_z X_0\pi\partial^5 X_0 - i\pi\partial_z X_0\partial^5 X_0 - i\partial_z X_0\pi\partial^5 X_0\\ & + \pi\partial_z X_0\partial^5 X_0\pi + \pi\partial_z X_0\pi\partial^5 X_0 + \partial_z X_0\pi\partial^5 X_0\pi + \partial_z X_0\pi^2\partial^5 X_0\\ & - \frac{1}{2}\partial_z X_0\partial^5 X_0\pi^2 - \partial_z X_0\pi\partial^5 X_0\pi - \frac{1}{2}\partial_z X_0\pi^2\partial^5 X_0\\ & - \frac{1}{2}\pi^2\partial_z X_0\partial^5 X_0 - \pi\partial_z X_0\pi\partial^5 X_0 - \frac{1}{2}\partial_z X_0\pi^2\partial^5 X_0\\ = {} & \partial_z X_0\partial^5 X_0,\end{aligned} \tag{A.9}$$

where the equally coloured terms cancel each other. Finally, the mixed terms are

$$
\begin{aligned}
&- i\partial_z\pi X_0\partial^5 X_0 + \partial_z\pi X_0\partial^5 X_0\pi + \partial_z\pi X_0\pi\partial^5 X_0 \\
&- iX_0\partial_z\pi\partial^5 X_0 + X_0\partial_z\pi\partial^5 X_0\pi + X_0\partial_z\pi\pi\partial^5 X_0 \\
&+ i\partial_z X_0 X_0\partial^5\pi + \pi\partial_z X_0 X_0\partial^5\pi + \partial_z X_0\pi X_0\partial^5\pi \\
&+ i\partial_z X_0\partial^5\pi X_0 + \pi\partial_z X_0\partial^5\pi X_0 + \partial_z X_0\pi\partial^5\pi X_0 \\
&- \frac{1}{2}\pi\partial_z\pi X_0\partial^5 X_0 - \frac{1}{2}\partial_z\pi\pi X_0\partial^5 X_0 - \pi X_0\partial_z\pi\partial^5 X_0 - \partial_z\pi X_0\pi\partial^5 X_0 \\
&- \frac{1}{2}X_0\pi\partial_z\pi\partial^5 X_0 - \frac{1}{2}X_0\partial_z\pi\pi\partial^5 X_0 - \frac{1}{2}\partial_z X_0 X_0\partial^5\pi\pi - \frac{1}{2}\partial_z X_0 X_0\pi\partial^5\pi \\
&- \partial_z X_0\partial^5\pi X_0\pi - \partial_z X_0\pi X_0\partial^5\pi - \frac{1}{2}\partial_z X_0\partial^5\pi\pi X_0 - \frac{1}{2}\partial_z X_0\pi\partial^5\pi X_0 \\
= {}& 0.
\end{aligned}
\tag{A.10}
$$

We conclude that

$$
\partial_z X^\dagger\partial^5 X = \partial_z X_0\partial^5 X_0 + \sum_a \frac{M_A^{a\,2}}{2}\partial_z\pi^a\partial^5\pi^a. \tag{A.11}
$$

With this, the term $(1) = \partial_M X^\dagger\partial^M X$ is given by

$$
\begin{aligned}
\partial_M X^\dagger\partial^M X &= \sum_a \frac{M_A^{a\,2}}{2}\partial_\mu\pi^a\partial^\mu\pi^a + \partial_z X_0\partial^5 X_0 + \sum_a \frac{M_A^{a\,2}}{2}\partial_z\pi^a\partial^5\pi^a \\
&= \partial_z X_0\partial^5 X_0 + \sum_a \frac{M_A^{a\,2}}{2}\partial_M\pi^a\partial^M\pi^a.
\end{aligned}
\tag{A.12}
$$

Next, let us turn to the expression (2) in (A.2). We want to express the fields L and R by the fields V and A. One gets:

$$
\begin{aligned}
&- i\partial_M X^\dagger L^M X + i\partial_M X^\dagger X R^M + iX^\dagger L_M\partial^M X - iR_M X^\dagger\partial^M X \\
={}& -i\partial_M X^\dagger(V^M + A^M)X + i\partial_M X^\dagger X(V^M - A^M) \\
&+ iX^\dagger(V_M + A_M)\partial^M X - i(V_M - A_M)X^\dagger\partial^M X \\
={}& -i\partial_M X^\dagger V^M X + i\partial_M X^\dagger X V^M + iX^\dagger V_M\partial^M X - iV_M X^\dagger\partial^M X \\
&- i\partial_M X^\dagger A^M X - i\partial_M X^\dagger X A^M + iX^\dagger A_M\partial^M X + iA_M X^\dagger\partial^M X.
\end{aligned}
\tag{A.13}
$$

Let us first treat the terms involving the V-field. Again, we should split up ∂_M into ∂_μ and ∂_z. Beginning with ∂_μ, we get:

$$
\begin{aligned}
&- i\partial_\mu X^\dagger V^\mu X + i\partial_\mu X^\dagger X V^\mu + iX^\dagger V_\mu\partial^\mu X - iV_\mu X^\dagger\partial^\mu X \\
={}& -X_0\partial_\mu\pi V^\mu X_0 - \partial_\mu\pi X_0 V^\mu X_0 - X_0\partial_\mu\pi X_0 V^\mu + \partial_\mu X_0^2 V^\mu \\
&- X_0 V_\mu\partial^\mu\pi X_0 - X_0 V_\mu X_0\partial^\mu\pi + V_\mu X_0\partial^\mu\pi X_0 + V_\mu X_0^2\partial^\mu\pi \\
={}& 0.
\end{aligned}
\tag{A.14}
$$

Naturally, for ∂_z the expression is a bit more complicated. We get:

$$\begin{aligned}
&-i\partial_z X^\dagger V^5 X + i\partial_z X^\dagger X V^5 + iX^\dagger V_5 \partial^5 X - iV_5 X^\dagger \partial^5 X \\
&= -X_0\partial_z\pi V^5 X_0 - \partial_z X_0 \pi V^5 X_0 - \pi\partial_z X_0 V^5 X_0 - \partial_z \pi X_0 V^5 X_0 \\
&\quad - i\partial_z X_0 V^5 X_0 + \partial_z X_0 V^5 \pi X_0 + \partial_z X_0 V^5 X_0 \pi \\
&\quad + X_0 \partial_z \pi X_0 V^5 + \partial_z X_0 \pi X_0 V^5 + \pi \partial_z X_0 X_0 V^5 + \partial_z \pi X_0^2 V^5 \\
&\quad + i\partial_z X_0 X_0 V^5 - \partial_z X_0 \pi X_0 V^5 - \partial_z X_0 X_0 \pi V^5 \\
&\quad - X_0 V_5 \partial^5 \pi X_0 - X_0 V_5 \pi \partial^5 X_0 - X_0 V_5 \partial^5 X_0 \pi - X_0 V_5 X_0 \partial^5 \pi \\
&\quad + iX_0 V_5 \partial^5 X_0 + X_0 \pi V_5 \partial^5 X_0 + \pi X_0 V_5 \partial^5 X_0 \\
&\quad + V_5 X_0 \partial^5 \pi X_0 + V_5 X_0 \pi \partial^5 X_0 + V_5 X_0 \partial^5 X_0 \pi + V_5 X_0^2 \partial^5 \pi \\
&\quad - iV_5 X_0 \partial^5 X_0 - V_5 X_0 \pi \partial^5 X_0 - V_5 \pi X_0 \partial^5 X_0 \\
&= 0.
\end{aligned} \tag{A.15}$$

So altogether the terms involving V in (A.13) vanish. Let us then do the same calculations with A. Starting again with ∂_μ, we get:

$$\begin{aligned}
&-i\partial_\mu X^\dagger A^\mu X - i\partial_\mu X^\dagger X A^\mu + iX^\dagger A_\mu \partial^\mu X + iA_\mu X^\dagger \partial^\mu X \\
&= -X_0 \partial_\mu \pi A^\mu X_0 - \partial_\mu \pi X_0 A^\mu X_0 - X_0 \partial_\mu \pi X_0 A^\mu - \partial_\mu \pi X_0^2 A^\mu \\
&\quad - X_0 A_\mu \partial^\mu \pi X_0 - X_0 A_\mu X_0 \partial^\mu \pi - A_\mu X_0 \partial^\mu \pi X_0 - A_\mu X_0^2 \partial^\mu \pi \\
&= -2\partial_\mu \pi^a A^{\mu b} \left(X_0 t^a t^b X_0 + t^a X_0 t^b X_0 + X_0 t^a X_0 t^b + t^a X_0^2 t^b \right) \\
&= -2\partial_\mu \pi^a A^{\mu b} \{t^a, X_0\}\{t^b, X_0\} = -\sum_a {M_A^a}^2 \partial_\mu \pi^a A^{\mu a}.
\end{aligned} \tag{A.16}$$

For ∂_z we get:

$$\begin{aligned}
&-i\partial_z X^\dagger A^5 X - i\partial_z X^\dagger X A^5 + iX^\dagger A_5 \partial^5 X + iA_5 X^\dagger \partial^5 X \\
&= -X_0 \partial_z \pi A^5 X_0 - \partial_z X_0 \pi A^5 X_0 - \pi \partial_z X_0 A^5 X_0 - \partial_z \pi X_0 A^5 X_0 \\
&\quad - i\partial_z X_0 A^5 X_0 + \partial_z X_0 A^5 \pi X_0 + \partial_z X_0 A^5 X_0 \pi \\
&\quad - X_0 \partial_z \pi X_0 A^5 - \partial_z X_0 \pi X_0 A^5 - \pi \partial_z X_0 X_0 A^5 - \partial_z \pi X_0^2 A^5 \\
&\quad - i\partial_z X_0 X_0 A^5 + \partial_z X_0 \pi X_0 A^5 + \partial_z X_0 X_0 \pi A^5 \\
&\quad - X_0 A_5 \partial^5 \pi X_0 - X_0 A_5 \pi \partial^5 X_0 - X_0 A_5 \partial^5 X_0 \pi - X_0 A_5 X_0 \partial^5 \pi \\
&\quad + iX_0 A_5 \partial^5 X_0 + X_0 \pi A_5 \partial^5 X_0 + \pi X_0 A_5 \partial^5 X_0 \\
&\quad - A_5 X_0 \partial^5 \pi X_0 - A_5 X_0 \pi \partial^5 X_0 - A_5 X_0 \partial^5 X_0 \pi - A_5 X_0^2 \partial^5 \pi \\
&\quad + iA_5 X_0 \partial^5 X_0 + A_5 X_0 \pi \partial^5 X_0 + A_5 \pi X_0 \partial^5 X_0 \\
&= -2\partial_z \pi^a A^{5b} \left(X_0 t^a t^b X_0 + t^a X_0 t^b X_0 + X_0 t^a X_0 t^b + t^a X_0^2 t^b \right) \\
&= -2\partial_z \pi^a A^{5b} \{t^a, X_0\}\{t^b, X_0\} = -\sum_a {M_A^a}^2 \partial_z \pi^a A^{5a}.
\end{aligned} \tag{A.17}$$

So together the terms involving A in (A.13) are

$$-\sum_a M_A^{a\,2}\left(\partial_\mu \pi^a A^{\mu a} + \partial_z \pi^a A^{5a}\right) = -\sum_a M_A^{a\,2}\partial_M \pi^a A^{Ma} \tag{A.18}$$

which gives for expression (2) in (A.2)

$$(2) = -\sum_a M_A^{a\,2}\partial_M \pi^a A^{Ma}. \tag{A.19}$$

Finally, let us evaluate expression (3) in (A.2). It is given by

$$\begin{aligned}
&X^\dagger L_M L^M X - X^\dagger L_M X R^M - R_M X^\dagger L^M X + R_M X^\dagger X R^M \\
&= X_0 L_M L^M X_0 - X_0 L_M X_0 R^M - R_M X_0 L^M X_0 + R_M X_0 X_0 R^M \\
&= X_0 V_M V^M X_0 + X_0 V_M A^M X_0 + X_0 A_M V^M X_0 + X_0 A_M A^M X_0 \\
&\quad - X_0 V_M X_0 V^M + X_0 V_M X_0 A^M - X_0 A_M X_0 V^M + X_0 A_M X_0 A^M \\
&\quad - V_M X_0 V^M X_0 - V_M X_0 A^M X_0 + A_M X_0 V^M X_0 + A_M X_0 A^M X_0 \\
&\quad + V_M X_0^2 V^M - V_M X_0^2 A^M - A_M X_0^2 V^M + A_M X_0^2 A^M \\
&= -[V_M, X_0][V^M, X_0] + \{A_M, X_0\}\{A^M, X_0\} \\
&= -V_M^a V^{Mb}\{t^a, X_0\}\{t^b, X_0\} + A_M^a A^{Mb}[t^a, X_0][t^b, X_0] \\
&= \sum_a \left(\frac{M_V^{a\,2}}{2} V_M^a V^{Ma} + \frac{M_A^{a\,2}}{2} A_M^a A^{Ma}\right).
\end{aligned} \tag{A.20}$$

Now we can add up the terms (1) - (3) to get the whole expression for the kinetic term. We get

$$\begin{aligned}
D_M X^\dagger D^M X &= (1) + (2) + (3) = \partial_z X_0 \partial^5 X_0 + \sum_a \frac{M_A^{a\,2}}{2}\partial_M \pi^a \partial^M \pi^a \\
&\quad - \sum_a M_A^{a\,2}\partial_M \pi^a A^{Ma} + \sum_a \left(\frac{M_V^{a\,2}}{2} V_M^a V^{Ma} + \frac{M_A^{a\,2}}{2} A_M^a A^{Ma}\right) \\
&= \partial_z X_0 \partial^5 X_0 + \sum_a \left(\frac{M_A^{a\,2}}{2}(\partial_M \pi^a - A_M^a)(\partial^M \pi^a - A^{Ma}) + \frac{M_V^{a\,2}}{2} V_M^a V^{Ma}\right) \\
&= \partial_z X_0 \partial^5 X_0 + \sum_a \left(\frac{M_A^{a\,2} z^2}{2L^2}(\partial_M \pi^a - A_M^a)^2 + \frac{M_V^{a\,2} z^2}{2L^2} V_M^{a\,2}\right),
\end{aligned} \tag{A.21}$$

where the square implies contraction over η^{MN}.

A.1.2 The Mass Term

Let us see how the mass term $X^\dagger X$ in the action (A.1) can be simplified. We get

$$X^\dagger X = e^{-i\pi} X_0 e^{-i\pi} e^{i\pi} X_0 e^{i\pi} = e^{-i\pi} X_0^2 e^{i\pi} = X_0^2. \tag{A.22}$$

A.1.3 The Field Strength Term

The field strength term is

$$
\begin{aligned}
& F^L_{MN}F^{MN}_L + F^R_{MN}F^{MN}_R \\
&= (\partial_M L_N - \partial_N L_M - i[L_M, L_N])\left(\partial^M L^N - \partial^N L^M - i[L^M, L^N]\right) \\
&\quad + (\partial_M R_N - \partial_N R_M - i[R_M, R_N])\left(\partial^M R^N - \partial^N R^M - i[R^M, R^N]\right) \\
&= (\partial_M L_N - \partial_N L_M)\left(\partial^M L^N - \partial^N L^M\right) + (\partial_M R_N - \partial_N R_M)\left(\partial^M R^N - \partial^N R^M\right) \\
&= (\partial_M(V_N + A_N) - \partial_N(V_M + A_M))\left(\partial^M(V^N + A^N) - \partial^N(V^M + A^M)\right) \\
&\quad + (\partial_M(V_N - A_N) - \partial_N(V_M - A_M))\left(\partial^M(V^N - A^N) - \partial^N(R^M - A^M)\right) \\
&= \partial_M V_N \partial^M V^N + \partial_M V_N \partial^M A^N - \partial_M V_N \partial^N V^M - \partial_M V_N \partial^N A^M \\
&\quad + \partial_M A_N \partial^M V^N + \partial_M A_N \partial^M A^N - \partial_A V_N \partial^N V^M - \partial_M A_N \partial^N A^M \\
&\quad - \partial_N V_M \partial^M V^N - \partial_N V_M \partial^M A^N + \partial_N V_M \partial^N V^M + \partial_N V_M \partial^N A^M \\
&\quad - \partial_N A_M \partial^M V^N - \partial_N A_M \partial^M A^N + \partial_N A_M \partial^N V^M + \partial_N A_M \partial^N A^M \\
&\quad + \partial_M V_N \partial^M V^N - \partial_M V_N \partial^M A^N - \partial_M V_N \partial^N V^M + \partial_M V_N \partial^N A^M \\
&\quad - \partial_M A_N \partial^M V^N + \partial_M A_N \partial^M A^N + \partial_A V_N \partial^N V^M - \partial_M A_N \partial^N A^M \\
&\quad - \partial_N V_M \partial^M V^N + \partial_N V_M \partial^M A^N + \partial_N V_M \partial^N V^M - \partial_N V_M \partial^N A^M \\
&\quad + \partial_N A_M \partial^M V^N - \partial_N A_M \partial^M A^N - \partial_N A_M \partial^N V^M + \partial_N A_M \partial^N A^M \\
&= 2\left(\partial_M V_N \partial^M V^N - \partial_M V_N \partial^N V^M - \partial_N V_M \partial^M V^N + \partial_N V_M \partial^N V^M\right) \\
&\quad + 2\left(\partial_M A_N \partial^M A^N - \partial_M A_N \partial^N A^M - \partial_N A_M \partial^M A^N + \partial_N A_M \partial^N A^M\right) \\
&= 2(\partial_M V_N - \partial_N V_M)(\partial^M V^N - \partial^N V^M) \\
&\quad + 2(\partial_M A_N - \partial_N A_M)(\partial^M A^N - \partial^N A^M) \\
&= 2(\partial_M V^a_N - \partial_N V^a_M)(\partial^M V^{Nb} - \partial^N V^{Mb})t^a t^b \\
&\quad + 2(\partial_M A^a_N - \partial_N A^a_M)(\partial^M A^{Nb} - \partial^N A^{Mb})t^a t^b \\
&= (\partial_M V^a_N - \partial_N V^a_M)(\partial^M V^{Na} - \partial^N V^{Ma}) \\
&\quad + (\partial_M A^a_N - \partial_N A^a_M)(\partial^M A^{Na} - \partial^N A^{Ma}) \\
&= \frac{z^4}{L^4}(\partial_M V^a_N - \partial_N V^a_M)^2 + \frac{z^4}{L^4}(\partial_M A^a_N - \partial_N A^a_M)^2.
\end{aligned}
\tag{A.23}
$$

A.1.4 The Complete Action

Combining the results of the previous three subsections we are able to write down the expression for the action up to quadratic order in π, V, and A. We get

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^5x\sqrt{g}\,\mathrm{Tr}\left((D_MX)^\dagger(D^MX) + \frac{3}{L^2}X^\dagger X - \frac{1}{g_5^2}\left(F^L_{MN}F_L^{MN} + F^R_{MN}F_R^{MN}\right)\right)\\ &= \int \mathrm{d}^5x\left(\frac{L^5}{z^5}\,\mathrm{Tr}(\partial_zX_0\partial^5X_0) + \sum_a\left(\frac{M_A^{a\,2}L^3}{2z^3}(\partial_M\pi^a - A^a_M)^2 + \frac{M_V^{a\,2}L^3}{2z^3}V_M^{a\,2}\right)\right.\\ &\quad\left. + \frac{3L^3}{z^5}\,\mathrm{Tr}(X_0^2) - \sum_a\frac{L}{4zg_5^2}\left((\partial_MV^a_N - \partial_NV^a_M)^2 + (\partial_MA^a_N - \partial_NA^a_M)^2\right)\right).\end{aligned} \tag{A.24}$$

The terms that only involve X_0 or its derivative give (using the equation of motion)

$$\int \mathrm{d}^5x\,\mathrm{Tr}\left(-\frac{L^3}{z^3}\partial_zX_0\partial_zX_0 + \frac{3L^3}{z^5}X_0^2\right) = \int \mathrm{d}^5xL^3\,\mathrm{Tr}\left(\partial_z\left(\frac{1}{z^3}\partial_zX_0\right)X_0 + \frac{3}{z^5}X_0^2\right) = 0 \tag{A.25}$$

plus a constant boundary term. We can split the action into an axial (A-dependent) and a vector (V-dependent) part. This gives

$$\begin{aligned}\mathcal{S} = \int \mathrm{d}^5x\sum_a&\left(-\frac{L}{4g_5^2z}(\partial_MV^a_N - \partial_NV^a_M)^2 + \frac{M_V^{a\,2}L^3}{2z^3}V_M^{a\,2}\right.\\ &\left.-\frac{L}{4g_5^2z}(\partial_MA^a_N - \partial_NA^a_M)^2 + \frac{M_A^{a\,2}L^3}{2z^3}(\partial_M\pi^a - A^a_M)^2\right)\end{aligned} \tag{A.26}$$

A.2 Vector Equation of Motion

We want to derive the equation of motion for the vector sector with the action

$$\begin{aligned}\mathcal{S}_V = \int \mathrm{d}^5x\frac{L}{4g_5^2}\sum_a&\left(-\frac{1}{z}\left(\eta^{MM'}\eta^{NN'}(\partial_MV^a_N - \partial_NV^a_M)(\partial_{M'}V^a_{N'} - \partial_{N'}V^a_{M'})\right)\right.\\ &\left.+\frac{2\alpha^a(z)}{z}\eta^{MM'}V^a_MV^a_{M'}\right).\end{aligned} \tag{A.27}$$

Since the V^a are independent of each other for different a, we can look at each V^a separately. Furthermore, an overall constant factor in the action is irrelevant. So we have to look at

$$\begin{aligned}\mathcal{S}^a = \int \mathrm{d}^5x\frac{1}{z}&\left(-\eta^{MM'}\eta^{NN'}(\partial_MV^a_N - \partial_NV^a_M)(\partial_{M'}V^a_{N'} - \partial_{N'}V^a_{M'})\right.\\ &\left.+2\alpha^a(z)\eta^{MM'}V^a_MV^a_{M'}\right).\end{aligned} \tag{A.28}$$

Then if $\delta\mathcal{S} = \mathcal{S}[V+\delta V] - \mathcal{S}[V]$, we get (dropping quadratic terms in δV):

$$\begin{aligned}\delta\mathcal{S}^a =& \int \mathrm{d}^5x \frac{1}{z}\Big(-\eta^{MM'}\eta^{NN'}(\partial_M V_N^a - \partial_N V_M^a)(\partial_{M'}\delta V_{N'}^a - \partial_{N'}\delta V_{M'}^a) \\ &+ 2\alpha^a(z)\eta^{MM'}V_M^a\delta V_{M'}^a\Big) \\ &+ \int \mathrm{d}^5x \frac{1}{z}\Big(-\eta^{MM'}\eta^{NN'}(\partial_M \delta V_N^a - \partial_N \delta V_M^a)(\partial_{M'} V_{N'}^a - \partial_{N'} V_{M'}^a) \\ &+ 2\alpha^a(z)\eta^{MM'}\delta V_M^a V_{M'}^a\Big) \\ =& \, 2\int \mathrm{d}^5x \frac{1}{z}\Big(-\eta^{MM'}\eta^{NN'}(\partial_M V_N^a - \partial_N V_M^a)(\partial_{M'}\delta V_{N'}^a - \partial_{N'}\delta V_{M'}^a) \\ &+ 2\alpha^a(z)\eta^{MM'}V_M^a\delta V_{M'}^a\Big),\end{aligned} \tag{A.29}$$

where we used the symmetry of η^{MN} in the last step. We can expand the bracket term in the middle and do partial integration to remove the derivative from δV. This gives

$$\begin{aligned}\delta\mathcal{S}^a =& \, 2\int \mathrm{d}^5x \left(\eta^{MM'}\eta^{NN'}\left(\partial_{M'}\left(\frac{1}{z}(\partial_M V_N^a - \partial_N V_M^a)\right)\delta V_{N'}^a\right.\right. \\ &\left.\left. -\partial_{N'}\left(\frac{1}{z}(\partial_M V_N^a - \partial_N V_M^a)\right)\delta V_{M'}^a\right) + 2\frac{\alpha(z)^a}{z}\eta^{MM'}V_M^a\delta V_{M'}^a\right) \\ =& \, 4\int \mathrm{d}^5x \left(-\eta^{MM'}\eta^{NN'}\partial_{N'}\left(\frac{1}{z}(\partial_M V_N^a - \partial_N V_M^a)\right)\delta V_{M'}^a + \frac{\alpha(z)^a}{z}\eta^{MM'}V_M^a\delta V_{M'}^a\right) \\ =& \, 4\int \mathrm{d}^5x\eta^{MM'}\left(\eta^{NN'}\partial_{N'}\left(\frac{1}{z}(\partial_N V_M^a - \partial_M V_N^a)\right) + \frac{\alpha(z)^a}{z}V_M^a\right)\delta V_{M'}^a.\end{aligned} \tag{A.30}$$

We can then read of the equation of motion

$$\eta^{NN'}\partial_{N'}\left(\frac{1}{z}(\partial_N V_M^a - \partial_M V_N^a)\right) + \frac{\alpha(z)^a}{z}V_M^a = 0. \tag{A.31}$$

A.3 Orthogonality Relation for the $\eta_n^a(z)$

We want to show that

$$\int_{L_0}^{L_1} \mathrm{d}z \frac{z}{\beta^a(z)}\eta_n^a(z)\eta_m^a(z) = 0, \tag{A.32}$$

whenever $m \neq n$. We use partial integration and equation (2.96) to get

$$\begin{aligned}
\int_{L_0}^{L_1} \mathrm{d}z \frac{z}{\beta^a(z)} \eta_n^a(z) \eta_m^a(z) &= \underbrace{\frac{1}{m_n^{a\,2}} \left(\widehat{\phi}_n^a(z) - \frac{z}{\beta^a(z)} \partial_z \eta_n^a(z) \right) \eta_m^a(z) \Big|_{L_0}^{L_1}}_{=0} \\
&\quad - \int_{L_0}^{L_1} \mathrm{d}z \frac{1}{m_m^{a\,2}} \left(\widehat{\phi}_n^a(z) - \frac{z}{\beta^a(z)} \partial_z \eta_n^a(z) \right) \partial_z \eta_m^a(z) \\
&= -\frac{1}{m_n^{a\,2}} \left(\underbrace{\widehat{\phi}_n^a(z) \eta_m^a(z) \Big|_{L_0}^{L_1}}_{=0} - \int_{L_0}^{L_1} \mathrm{d}z \eta_n^a(z) \eta_m^a(z) - \int_{L_0}^{L_1} \mathrm{d}z \frac{z}{\beta^a(z)} \partial_z \eta_n^a(z) \partial_z \eta_m^a(z) \right) \\
&= \frac{1}{m_n^{a\,2}} \int_{L_0}^{L_1} \mathrm{d}z \left(\frac{z}{\beta^a(z)} \partial_z \eta_n^a(z) \partial_z \eta_m^a(z) + \eta_n^a(z) \eta_m^a(z) \right).
\end{aligned} \tag{A.33}$$

By symmetry we also get

$$\int_{L_0}^{L_1} \mathrm{d}z \frac{z}{\beta^a(z)} \eta_n^a(z) \eta_m^a(z) = \frac{1}{m_m^{a\,2}} \int_{L_0}^{L_1} \mathrm{d}z \left(\frac{z}{\beta^a(z)} \partial_z \eta_n^a(z) \partial_z \eta_m^a(z) + \eta_n^a(z) \eta_m^a(z) \right). \tag{A.34}$$

Since only the mass factor differs and $m_n^a \neq m_m^a$ for $m \neq n$, the integral has to be zero.

A.4 Writing y^a as a Sum over Meson Poles

We want to write $y^a(k^2, z)$ as a sum over meson poles. For this we make the general ansatz

$$y^a(k^2, z) = \sum_n c_n^a(k^2) \eta_n^a(z). \tag{A.35}$$

Multiplying by $\frac{z}{\beta^a(z)} \eta_m^a(z)$, integrating and using the orthogonality relation (2.98) one gets

$$\begin{aligned}
\frac{c_m^a(k^2)}{m_m^{a\,2}} &= \int_{L_0}^{L_1} \mathrm{d}z \frac{z}{\beta^a(z)} \eta_m^a(z) y^a(k^2, z) = \underbrace{\frac{1}{m_m^{a\,2}} \left(\widehat{\phi}_m^a(z) - \frac{z}{\beta^a(z)} \partial_z \eta_m^a(z) \right) y^a(k^2, z) \Big|_{L_0}^{L_1}}_{=0} \\
&\quad - \int_{L_0}^{L_1} \mathrm{d}z \frac{1}{m_m^{a\,2}} \left(\widehat{\phi}_m^a(z) - \frac{z}{\beta^a(z)} \partial_z \eta_m^a(z) \right) \partial_z y^a(k^2, z) \\
&= -\frac{1}{m_m^{a\,2}} \int_{L_0}^{L_1} \mathrm{d}z \widehat{\phi}_m^a(z) \left(\frac{1}{z} \partial_z \widehat{\phi}^a(z) \right) + \frac{1}{m_m^{a\,2}} \int_{L_0}^{L_1} \mathrm{d}z \frac{z}{\beta^a(z)} \partial_z \eta_m^a(z) \partial_z y^a(k^2, z).
\end{aligned} \tag{A.36}$$

The second term gives

$$
\begin{aligned}
&\int_{L_0}^{L_1} \mathrm{d}z \frac{z}{\beta^a(z)} \partial_z \eta_m^a(z) \partial_z y^a(k^2,z) \\
&= \frac{z}{\beta^a(z)} \partial_z y^a(z) \eta_m^a(k^2,z) \Big|_{L_0}^{L_1} + \int_{L_0}^{L_1} \mathrm{d}z z \left(\frac{k^2}{\beta^a(z)} - 1 \right) y^a(k^2,z) \eta_m^a(z) \\
&= \frac{z}{\beta^a(z)} \partial_z y^a(z) \eta_m^a(k^2,z) \Big|_{L_0}^{L_1} + k^2 \frac{c_m^a(k^2)}{{m_m^a}^2} - \int_{L_0}^{L_1} \mathrm{d}z \partial_z \widehat{\phi}^a(k^2,z) \frac{1}{z} \widehat{\phi}_m^a(z) \\
&= \frac{z}{\beta^a(z)} \partial_z y^a(z) \eta_m^a(k^2,z) \Big|_{L_0}^{L_1} + k^2 \frac{c_m^a(k^2)}{{m_m^a}^2} - \underbrace{\widehat{\phi}_m^a(z) \frac{1}{z} \partial_z \widehat{\phi}^a(k^2,z) \Big|_{L_0}^{L_1}}_{=0} \\
&\quad + \int_{L_0}^{L_1} \mathrm{d}z \widehat{\phi}_m^a(z) \partial_z \left(\frac{1}{z} \widehat{\phi}^a(k^2,z) \right).
\end{aligned}
\tag{A.37}
$$

On sees that the third term and the first term in the last row of (A.36) cancel and one gets

$$
\frac{c_m^a(k^2)}{{m_m^a}^2} = \frac{1}{{m_m^a}^2} \frac{z}{\beta^a(z)} \partial_z y^a(z) \eta_m^a(k^2,z) \Big|_{L_0}^{L_1} + \frac{k^2}{{m_m^a}^2} \frac{c_m^a(k^2)}{{m_n^a}^2}
\tag{A.38}
$$

or

$$
c_m^a(k^2) = \frac{{m_m^a}^2}{{m_m^a}^2 - k^2} \frac{z}{\beta^a(z)} \partial_z y^a(z) \eta_m^a(k^2,z) \Big|_{L_0}^{L_1} = \frac{{m_m^a}^2}{k^2 - {m_m^a}^2} \eta_m^a(L_0).
\tag{A.39}
$$

This result yields

$$
y^a(k^2,z) = \sum_n \frac{{m_n^a}^2 \eta_n^a(L_0) \eta_n^a(z)}{k^2 - {m_n^a}^2}.
\tag{A.40}
$$

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