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Leading logarithms in N-flavour mesonic Chiral Perturbation Theory

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Abstract

We extend earlier work on leading logarithms in the massive nonlinear $O(n)$ sigma model to the case of $SU(N) \times SU(N)/SU(N)$ which coincides with mesonic chiral perturbation theory for N flavours of light quarks. We discuss the leading logarithms for the mass and decay constant to six loops and for the vacuum expectation value $\langle \bar{q}q \rangle$ to seven loops. For dynamical quantities the expressions grow extremely large much faster such that we only quote the leading logarithms to five loops for the vector and scalar form factor and for meson-meson scattering. The last quantity we consider is the vector-vector to meson-meson amplitude where we quote results up to four loops for a subset of quantities, in particular for the pion polarizabilities. As a side result we provide an elementary proof that the factors of N appearing at each loop order are odd or even depending on the order and the remaining traces over external flavours.

Contents

1 Introduction

The calculation of higher loop corrections is an important problem in all areas of particle physics. The leading logarithms in a renormalizable field theory can be calculated to all orders by simply using the renormalization group. In nonrenormalizable effective field theories like Chiral Perturbation Theory (ChPT) [1–3], the recursive argument underlying the renormalization group does not work since one has a new Lagrangian at each order. Weinberg [1] showed that using the requirement that all nonlocal divergences cancel, one could obtain the leading logarithms (LL) at two-loop order with only one-loop calculations. This method has then been applied to various processes at the two-loop level [4]. That it works to all orders was later proven using beta-functions [5] and also with a more diagrammatic method [6].

Using this method, [7–10] found recursion relations valid in the massless limit and applied them to a number of processes. Away from the massless limit the tadpoles do not vanish and this causes the number of needed one-loop diagrams at every order to increase considerably. A systematic method to automatize the calculational process was found in [6] and then applied to a number of processes in the normal [6, 11] and abnormal or anomalous intrinsic parity [12] sector of the massive $O(n)$ nonlinear sigma model. In

the present paper we extend the calculations in the even sector to the symmetry breaking pattern of $SU(N) \times SU(N)/SU(N)$. All results are for the case of equal masses.

We discuss the leading logarithm contribution to the mass, decay constant, and vacuum expectation value to sixth or seventh order. Numerical results are discussed for the two physical cases $N = 2, 3$. For the vector and scalar form factors we give expressions for the full results and for the radius and curvature. We present no numerical results, but some discussion of numerics for the scalar form factor for $N = 2$ can be found in [11]. For meson-meson scattering we present analytic results for the amplitude and the scattering lengths up to fifth order. We show numerical results only for the singlet scattering length for $N = 3$, which we compare with the full two-loop calculation as well. For $\gamma\gamma \to \pi\pi$ we give analytic results for the full amplitude for general N and for the polarizabilities for $N = 2$. For the latter we also present numerical results.

We provide some references to the $N = 2$ and $N = 3$ cases at the two-loop level where the general-N case is not known to that level. An extensive discussion of the corresponding literature can be found in the review [13].

In Sect. 2 we present the model and the different parametrizations we use. Sect. 3 describes the changes needed compared to the $O(n)$ work and provides the necessary definitions such that the formulas in this paper are self-contained. We do however not discuss in detail the methods used. The remaining sections present results for the mass (Sect. 4), decay constant (Sect. 5), vacuum expectation value (Sect. 6), vector form factor (Sect. 7), scalar form factor (Sect. 8), meson-meson scattering (Sect. 9) and vector-vector to meson-meson scattering (Sect. 10). In addition we prove in Appendix A that only certain powers of N can show up at each order.

2 N-flavour mesonic Chiral Perturbation Theory

The Lagrangian of the massive nonlinear $SU(N) \times SU(N)/SU(N)$ sigma model or N-flavour mesonic ChPT at lowest order is given by

$$\mathcal{L} = \frac{F^2}{4} \langle D_\mu U D^\mu U^\dagger \rangle + \frac{F^2}{4} \langle \chi U^\dagger + U \chi^\dagger \rangle \,, \tag{1}$$

where U is a special unitary $N \times N$ matrix, which contains $N^2 - 1$ degrees of freedom. $\langle A \rangle = \mathrm{tr}(A)$. The interaction with external axial-vector and vector fields enters through the covariant derivative

$$D_\mu U = \partial_\mu U - \frac{i}{2}[v_\mu, U] - \frac{i}{2}\{a_\mu, U\} \,, \tag{2}$$

while the explicitly chiral symmetry breaking terms as well as the scalar and pseudoscalar external sources are contained in

$$\chi = 2B(s + ip) + M^2 \, \mathbf{1} \,. \tag{3}$$

The chiral $SU(N) \times SU(N)$ symmetry is broken spontaneously to $SU(N)$ by the vacuum expectation value $\langle 0|U|0 \rangle = \mathbf{1}$, where $\mathbf{1}$ is the $N \times N$ unit matrix. This leads to the appearance of $N^2 - 1$ Goldstone bosons, which correspond to the degrees of freedom contained in the matrix field U. The term proportional to M^2 breaks the symmetry explicitly and causes the Goldstone bosons to pick up a mass which, at tree level, is equal to M. In terms of equal quark masses $\hat{m}$ we have $M^2 = 2B\hat{m}$.

The Lagrangian (1) coincides with ChPT and therefore constitutes an effective Lagrangian for two- and three-flavour QCD for $N = 2$ and $N = 3$, respectively. Note, however, that in the case considered here, all mesons have the same mass. How this corresponds to a theory formulated in terms of quarks can be found in more detail in, e.g., [14]. Below we occasionally use a vector notation for quarks q with $q^T = (q_1, \ldots, q_N)$ where the subscript denotes the flavour.

In previous publications [6, 11, 12], the chiral logarithms of the massive nonlinear $O(n + 1)/O(n)$ model have been considered. The two models coincide for $N = 2$ and $n = 3$, such that the corresponding results can be used as a check.

There are many ways the special unitary matrix U can be parametrized in terms of the meson matrix $\phi = \phi^a T^a$, where T^a are the generators of $SU(N)$ normalized as $\langle T^a T^b \rangle = \delta^{ab}$. Physical results are independent of this choice. As in the earlier work on the massive $O(n)$ model, one can therefore use different parametrizations to obtain a thorough check of the calculation. The four parametrizations we have used are

$$U_1 = \exp\left(\frac{i\sqrt{2}}{F}\phi\right) , \qquad U_2 = \frac{\mathbf{1} + i\left(\beta_2 + \frac{1}{\sqrt{2}F}\phi\right)}{\mathbf{1} - i\left(\beta_2 + \frac{1}{\sqrt{2}F}\phi\right)} ,$$

$$U_3 = e^{i\beta_3}\frac{\mathbf{1} + \frac{i}{\sqrt{2}F}\phi}{\mathbf{1} - \frac{i}{\sqrt{2}F}\phi} , \qquad U_4 = e^{i\beta_4}\left(\sqrt{1 - \frac{2}{F^2}\phi^2} + i\frac{\sqrt{2}}{F}\phi\right) . \qquad (4)$$

The matrices must be special, i.e., $\det U_i = 1$ which for U_1 is an automatic consequence of $\langle \phi \rangle = 0$. For the other cases one has to solve for β_i in terms of ϕ. Using $\langle \log U_i \rangle = \log \det U_i = 0$, one finds that the β_i start at order ϕ^3. For β_2 we can then solve the resulting equation perturbatively while β_3 and β_4 can be written explicitly in terms of ϕ as

$$\beta_3 = \frac{i}{N}\left\langle \log\left(\mathbf{1} + \frac{i\phi}{\sqrt{2}F}\right) - \log\left(\mathbf{1} - \frac{i\phi}{\sqrt{2}F}\right) \right\rangle = -\frac{2}{N}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\left\langle \left(\frac{\phi}{\sqrt{2}F}\right)^{2n+1} \right\rangle ,$$

$$\beta_4 = -\frac{1}{N}\left\langle \arcsin\left(\frac{\sqrt{2}\phi}{F}\right) \right\rangle = -\frac{1}{N}\sum_{n=1}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}\left\langle \left(\frac{\sqrt{2}\phi}{F}\right)^{2n+1} \right\rangle . \qquad (5)$$

Note that the $n = 0$ term vanishes in both sums since $\langle \phi \rangle = 0$. We could have added a fifth parametrization by adding a singlet component to ϕ in U_4 as was done for parametrization 2.

It is also possible to treat $U(N) \times U(N)/U(N)$ by simply allowing ϕ to have a singlet component and removing the β_i. We do not discuss this case.

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3 Leading logarithms

The method used here is entirely analogous to the work in [6, 11, 12] but with ϕ a traceless $N \times N$ Hermitian matrix instead of an n-dimensional vector. The calculations are done schematically as follows: first we generate all needed one-loop diagrams with a C++ program. The diagrams at each order are then evaluated using FORM [15]. The integrals are performed using a recursive method. The results are then combined to provide the needed Lagrangians at the next order. A more detailed discussion of the method and the underlying principles can be found in [6, 11].

Flavour sums in the earlier work were rather trivial to perform. Here, keeping track of the different terms is somewhat more tricky but all the flavour sums can be performed using the methods of [14]. The underlying idea is to use

$$\langle T^a A T^a B \rangle = \langle A \rangle \langle B \rangle - \frac{1}{N} \langle AB \rangle \,, \qquad \langle T^a A \rangle \langle T^a B \rangle = \langle AB \rangle - \frac{1}{N} \langle A \rangle \langle B \rangle \,, \qquad (6)$$

for the sums over the generators T^a.

In the following, we will present our results for the coefficients of the leading logarithm contribution to several physical quantities. In all cases, we have the choice of expressing these in terms of lowest order or physical parameters, which can have quite a substantial effect on the convergence of the series. Following the definitions in [11] we expand a given observable O_{phys} as

$$\begin{aligned} O_{\text{phys}} &= O_0 \left(1 + a_1 L + a_2 L^2 + \cdots \right) \,, & (7) \\ O_{\text{phys}} &= O_0 \left(1 + c_1 L_{\text{phys}} + c_2 L_{\text{phys}}^2 + \cdots \right) \,, & (8) \end{aligned}$$

where the chiral logarithms are defined either from the lowest-order parameters M and F as

$$L \equiv \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2} \,, \qquad (9)$$

or from the physical mass M_π and decay constant F_π as

$$L_{\text{phys}} \equiv \frac{M_\pi^2}{16\pi^2 F_\pi^2} \log \frac{\mu^2}{M_\pi^2} \,. \qquad (10)$$

These are relevant for the static quantities where the mass is the only dimensionful parameter. In general the argument of the logarithm is not uniquely determined at the level of leading logarithms. For the cases with more dimensionful quantities we usually use the more general

$$L_{\mathcal{M}} \equiv \frac{M_\pi^2}{16\pi^2 F_\pi^2} \log \frac{\mu^2}{\mathcal{M}^2} \,, \qquad (11)$$

where $\mathcal{M}$ is some combination of the relevant dimensionful quantities.

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i	a_i for $N=2$	a_i for $N=3$	a_i for general N
1	$-1/2$	$-1/3$	$-N^{-1}$
2	$17/8$	$27/8$	$9/2\,N^{-2} - 1/2 + 3/8\,N^2$
3	$-103/24$	$-3799/648$	$-89/3\,N^{-3} + 19/3\,N^{-1} - 37/24\,N - 1/12\,N^3$
4	$24367/1152$	$146657/2592$	$2015/8\,N^{-4} - 773/12\,N^{-2} + 193/18 + 121/288\,N^2$ $+41/72\,N^4$
5	$-8821/144$	$-27470059/186624$	$-38684/15\,N^{-5} + 6633/10\,N^{-3} - 59303/1080\,N^{-1}$ $-5077/1440\,N - 11327/4320\,N^3 - 8743/34560\,N^5$
6^*	$\frac{1922964667}{6220800}$	$\frac{12902773163}{9331200}$	$7329919/240\,N^{-6} - 1652293/240\,N^{-4}$ $-4910303/15552\,N^{-2} + 205365409/972000$ $-69368761/7776000\,N^2 + 14222209/2592000\,N^4$ $+3778133/3110400\,N^6$

Table 1: The coefficients a_i of the leading logarithm L^i up to $i=6$ for the physical meson mass for the physical cases $N=2$ and $N=3$ as well as for general N.

i	c_i for $N=2$	c_i for $N=3$	c_i for general N
1	$-1/2$	$-1/3$	$-N^{-1}$
2	$7/8$	$163/72$	$7/2\,N^{-2} - 3/2 + 3/8\,N^2$
3	$211/48$	$9329/648$	$-109/6\,N^{-3} + 71/6\,N^{-1} - 55/24\,N + 2/3\,N^3$
4	$21547/1152$	$638861/7776$	$3185/24\,N^{-4} - 1067/12\,N^{-2} + 1795/72$ $-677/288\,N^2 + 77/72\,N^4$
5	$179341/2304$	$\frac{86965829}{186624}$	$-150877/120\,N^{-5} + 47473/60\,N^{-3} - 23407/120\,N^{-1}$ $+41713/1440\,N + 4891/8640\,N^3 + 57557/34560\,N^5$
6^*	$\frac{2086024177}{6220800}$	$\frac{38355806767}{13996800}$	$229179/16\,N^{-6} - 392659/48\,N^{-4} + 106873049/77760\,N^{-2}$ $+2800699/60750 - 277103161/7776000\,N^2$ $+68361593/5184000\,N^4 + 8001833/3110400\,N^6$

Table 2: The coefficients c_i of the leading logarithm L^i_{phys} up to $i=6$ for the physical meson mass for the physical cases $N=2$ and $N=3$ as well as for general N.

4 Mass

The mass is known fully to one [16] and two loops [14]. We have calculated the leading logarithms to six-loop order here. For this we have computed the generic two-point function of the ϕ fields in all four parametrizations and extracted the mass as well as the wave function renormalization, which will be needed later. The result is expressed in the form of (7) and (8) with $O_{\text{phys}} = M_\pi^2$ and $O_0 = M^2$. The first six coefficients a_i and c_i of the expansions of the physical mass are listed in Tables 1 and 2. The coefficients for $N = 2$ agree with the results from [6, 11, 12] and with the one- and two-loop results from [14, 16].

Since the calculation is very time consuming, the sixth order has only been checked with two of the four parametrizations in (4). Throughout the paper, results with this limitation are marked by an asterisk next to the number that labels the order.

It is rather clear from the expressions that there is a pattern in the powers of N that appear. They always jump by powers of 2. Similar steps can be seen in all results quoted in this paper. This is due to the $SU(N)$ group structure of all flavour traces that need to be evaluated as is proven in general in Appendix A.

We can use our results to check the convergence of the two expansions. In Fig. 1 for $N = 2$ and Fig. 2 for $N = 3$, the input values chosen are $F = 0.090$ GeV for the expansion in terms of L and $F_\pi = 0.0922$ GeV for the expansion in L_{phys} as well as $\mu = 0.77$ GeV. The convergence is somewhat worse for $N = 3$ than for $N = 2$.

5 Decay constant

The decay constant F_π is defined by

$$\langle 0 | j_{A,\mu}^b | \phi^a(p) \rangle = i\sqrt{2} F_\pi p_\mu \delta^{ab} \tag{12}$$

for a meson corresponding to the quark flavour combination $i\bar{q}\gamma_5 T^a q$ and the axial current $\bar{q}\gamma_\mu\gamma_5 T^b q$. Note that F_π is equal for all mesons since we are in the equal mass limit. The decay constant is known fully to one [16] and two loops [14]. Here, we evaluate the leading logarithms to six loops.

We need to evaluate a matrix-element with one external axial current and one incoming meson. The diagrams required for the wave function renormalization were already done in the mass calculation in the previous section. We thus need to evaluate all relevant one-particle-irreducible (1PI) diagrams with an external a_μ^a. Up to the more complicated group theory the calculation is the same as in our earlier work.

We give the first six coefficients for both leading logarithm series with $O_{\text{phys}} = F_\pi$ and $O_0 = F$ in Tables 3 and 4. Note that once F_π is known as a function of F, we can express all observables as a function of the physical M_π^2 and F_π. We already used this to get the coefficients c_i in Tables 2 and 4 from the corresponding a_i.

We have plotted in Figs. 3 and 4 the expansion in terms of the unrenormalized quantities and in terms of the physical quantities for $N = 2$ and $N = 3$ respectively. In both cases
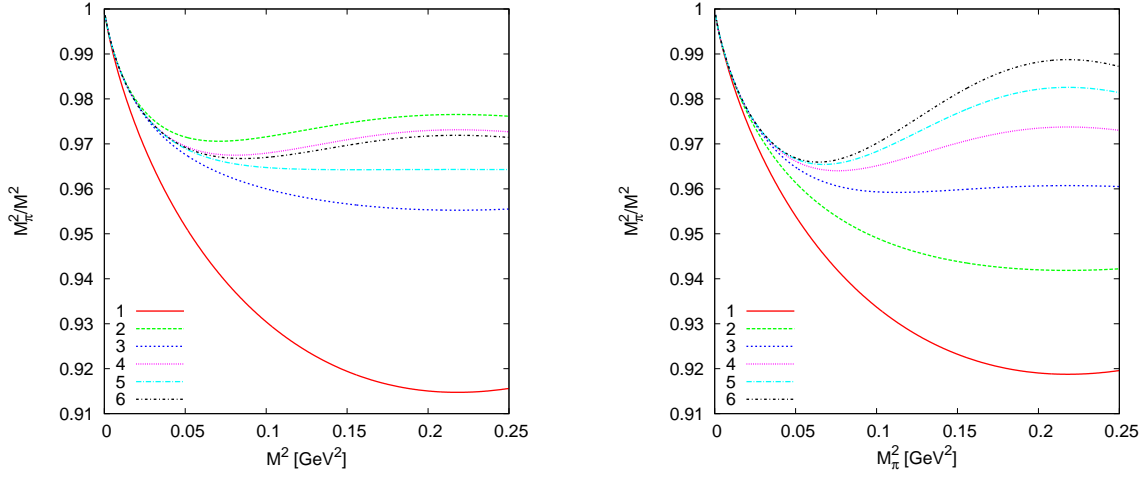
Figure 1: The contribution of the leading logarithms to M_π^2/M^2 order by order for $F = 0.090$ GeV, $F_\pi = 0.0922$ GeV, $\mu = 0.77$ GeV and $N = 2$. The left panel shows the expansion in L keeping F fixed, the right panel the expansion in L_{phys} keeping F_π fixed. Plots similar to Fig. 1 in [12].
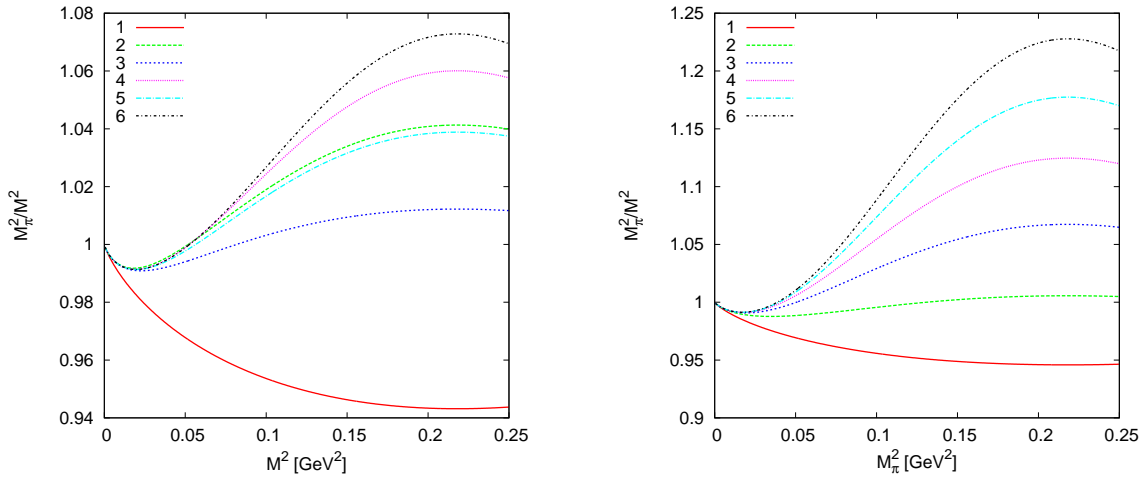

Figure 2: The contribution of the leading logarithms to M_π^2/M^2 order by order for $F = 0.090$ GeV, $F_\pi = 0.0922$ GeV, $\mu = 0.77$ GeV and $N = 3$. The left panel shows the expansion in L keeping F fixed, the right panel the expansion in L_{phys} keeping F_π fixed.

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i	a_i for $N=2$	a_i for $N=3$	a_i for general N
1	1	$3/2$	$1/2\,N$
2	$-5/4$	$-35/16$	$-1/2 - 3/16\,N^2$
3	$83/24$	$293/36$	$23/12\,N^{-1} + 1/4\,N + 1/4\,N^3$
4	$-3013/288$	$-\frac{413359}{13824}$	$-139/12\,N^{-2} + 7/54 - 523/576\,N^2 - 3511/13824\,N^4$
5	$\frac{2060147}{51840}$	$\frac{96197471}{622080}$	$22357/240\,N^{-3} - 5063/648\,N^{-1} + 16157/5184\,N$ $+26237/17280\,N^3 + 5885/13824\,N^5$
6^*	$-\frac{69228787}{466560}$	$-\frac{932532830269}{1343692800}$	$-41296/45\,N^{-4} + 5690093/58320\,N^{-2}$ $+14622631/9331200 - 1944182341/186624000\,N^2$ $-945730747/373248000\,N^4 - 81119291/149299200\,N^6$

Table 3: The coefficients a_i of the leading logarithm L^i up to $i=6$ for the decay constant F_π for the physical cases $N=2$ and $N=3$ as well as for general N.

i	c_i for $N=2$	c_i for $N=3$	c_i for general N
1	1	$3/2$	$1/2\,N$
2	$5/4$	$45/16$	$5/16\,N^2$
3	$13/12$	$131/36$	$-1/3\,N^{-1} + 1/8\,N + 1/8\,N^3$
4	$-577/288$	$-\frac{113471}{13824}$	$9/4\,N^{-2} + 209/576\,N^2 - 229/108 - 1639/13824\,N^4$
5	$-14137/810$	$-\frac{65712649}{622080}$	$-1097/60\,N^{-3} + 11095/648\,N^{-1} - 40225/5184\,N$ $+2137/2880\,N^3 - 679/1536N^5$
6^*	$-\frac{37737751}{466560}$	$-\frac{889506647989}{1343692800}$	$6745/36\,N^{-4} - 9274909/58320\,N^{-2} + 611736991/9331200$ $-3858946741/186624000N^2 + 440983853/373248000\,N^4$ $-127342211/149299200\,N^6$

Table 4: The coefficients c_i of the leading logarithm L^i_{phys} up to $i=6$ for the decay constant F_π for the physical cases $N=2$ and $N=3$ as well as for general N.

we get convergence but it is better for the expansion in physical quantities. It is also much better for $N=2$ than for $N=3$.

6 Vacuum expectation value

The expression for the leading logarithms of the vacuum expectation value (VEV) follows from the definition

$$V_{\text{phys}} = \langle 0| - j_0^s |0 \rangle , \tag{13}$$

where j_0^s is the QCD current associated with the scalar external source s introduced in (3) with the singlet generator normalized to 1. In terms of quarks the definition is

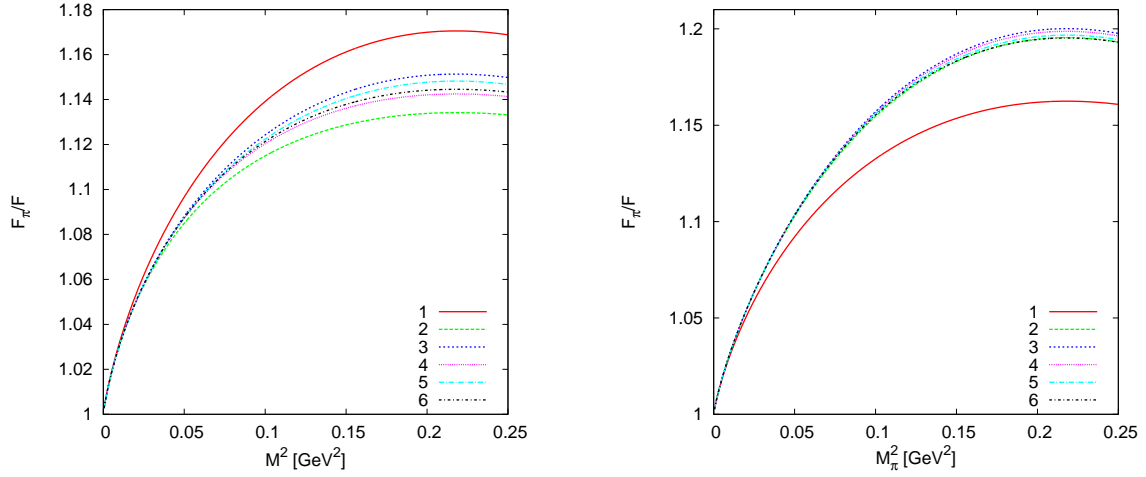$$\langle 0|\bar{q}_i q_j|0 \rangle = V_{\text{phys}} \delta_{ij} . \tag{14}$$

Figure 3: The contribution of the leading logarithms to F_π/F order by order for $F = 0.090$ GeV, $F_\pi = 0.0922$ GeV, $\mu = 0.77$ GeV and $N = 2$. The left panel shows the expansion in L keeping F fixed, the right panel the expansion in L_{phys} keeping F_π fixed. Plots similar to Fig. 2 in [12].
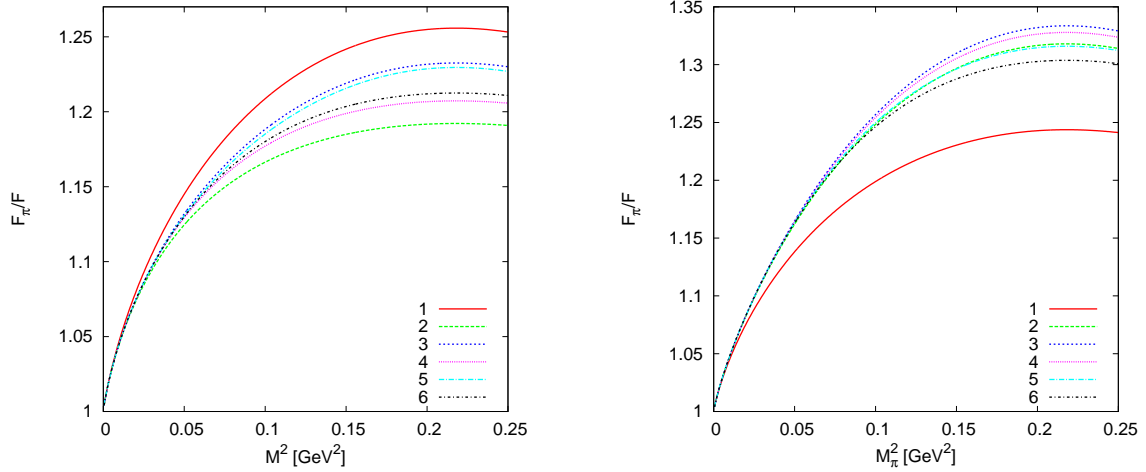

Figure 4: The contribution of the leading logarithms to F_π/F order by order for $F = 0.090$ GeV, $F_\pi = 0.0922$ GeV, $\mu = 0.77$ GeV and $N = 3$. The left panel shows the expansion in L keeping F fixed, the right panel the expansion in L_{phys} keeping F_π fixed.

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At lowest order, $V_{\text{phys}} \equiv V_0 = -2BF^2$. The VEV is known fully to one [16] and two loops [14]. Here we evaluate the leading logarithms to seven loops.

The first seven coefficients of the expansions in (7) and (8) for $O_{\text{phys}} = V_{\text{phys}}$ and $O_0 = V_0$ are given in Tables 5 and 6, respectively.

We have plotted in Figures 5 and 6 the expansion in terms of the unrenormalized quantities and in terms of the physical quantities for $N = 2$ and $N = 3$, respectively. In both cases we get a good convergence but it is excellent for the expansion in physical quantities.

7   Vector form factor

We turn now to the vector form factor which is defined by

$$\langle \phi^a(p_f)|j^c_{V,\mu}|\phi^b(p_i)\rangle = \left\langle T^c\left(T^bT^a - T^aT^b\right)\right\rangle (p_f + p_i)_\mu F_V\left[(p_f - p_i)^2\right] . \qquad (15)$$

The vector current is $\bar{q}\gamma_\mu T^c q$. It is known fully in two- and three-flavour ChPT to one [2, 17] and two loops [18, 19]. Here we calculate the leading logarithms in the equal mass case to five loops.

The procedure to find the leading logarithms is entirely the same as in the earlier work [11, 12] with the modifications needed for the more complicated flavour structure. We express the result in terms of $\tilde{t} = t/M_\pi^2$ and the logarithm (11) with a scale $\mathcal{M}^2$ that is some combination of t and M_π^2. To fifth order we find

$$
\begin{aligned}
F_V(t) = {}& 1 + L_\mathcal{M}\left[N/12\,\tilde{t}\right] + L_\mathcal{M}^2\left[\tilde{t}\,(1/12 + N^2/16) + \tilde{t}^2\,(N^2/288)\right] \\
& + L_\mathcal{M}^3\left[\tilde{t}\,(-1/12\,N^{-1} + 29/72\,N + 179/2592\,N^3) + \tilde{t}^2\,(-137/648\,N - 5/2592\,N^3) \right.\\
& \left. + \tilde{t}^3\,(N/32 + 5/5184\,N^3)\right] \\
& + L_\mathcal{M}^4\left[\tilde{t}\,(17/18\,N^{-2} - 617/576 + 1111/768\,N^2 + 12011/124416\,N^4)\right. \\
& + \tilde{t}^2\,(445/7776 - 79861/103680\,N^2 - 5519/414720\,N^4) \\
& \left. + \tilde{t}^3\,(79459/777600\,N^2 + 407/172800\,N^4) + \tilde{t}^4\,(-N^2/345600 + N^4/518400)\right] \\
& + L_\mathcal{M}^5\left[\tilde{t}\,(-391/45\,N^{-3} + 134797/12960\,N^{-1} - 5429753/933120\,N + 5232541/1244160\,N^3\right.\\
& + 14129/93312\,N^5) + \tilde{t}^2\,(-181271/233280\,N^{-1} + 194389/405000\,N \\
& - 262760809/116640000\,N^3 - 61724321/1866240000\,N^5) + \tilde{t}^3\,(35/384\,N^{-1} \\
& + 1261489/23328000\,N + 35449669/90720000\,N^3 + 129189077/26127360000\,N^5) \\
& + \tilde{t}^4\,(-226531/4320000\,N - 58095211/1632960000\,N^3 - 935713/4354560000\,N^5) \\
& \left. + \tilde{t}^5\,(871/115200\,N + 545009/181440000\,N^3 + 126059/6531840000\,N^5)\right] . \qquad (16)
\end{aligned}
$$

Note that $F_V(0) = 1$ as it should be.

i	a_i for $N=2$	a_i for $N=3$	a_i for general N
1	$3/2$	$8/3$	$-N^{-1} + N$
2	$-9/8$	$-4/3$	$3/2\,N^{-2} - 3/2$
3	$9/2$	$988/81$	$-20/3\,N^{-3} + 22/3\,N^{-1} - 7/6\,N + 1/2\,N^3$
4	$-1285/128$	$-5660/243$	$1025/24\,N^{-4} - 205/4\,N^{-2} + 175/16 - 55/24\,N^2 - 5/48\,N^4$
5	46	$\frac{399563}{1944}$	$-350N^{-5} + 2188/5\,N^{-3} - 12539/120\,N^{-1} + 1321/80\,N$
			$-373/960\,N^3 + 737/960\,N^5$
6^*	$-\frac{1305605}{9216}$	$-\frac{242777185}{419904}$	$2490019/720\,N^{-6} - 3137701/720\,N^{-4}$
			$+12971623/12960\,N^{-2} - 1295581/12960$
			$+154399/51840\,N^2 - 277697/69120\,N^4 - 68761/207360\,N^6$
7^*	$\frac{153149887}{226800}$	$\frac{137725650367}{27556200}$	$-12489752/315\,N^{-7} + 15424312/315\,N^{-5}$
			$-79037542/8505N^{-3} - 129606331/567000\,N^{-1}$
			$+66338023/324000\,N - 19464419/3402000\,N^3$
			$+67022189/13608000\,N^5 + 4453133/2721600\,N^7$

Table 5: The coefficients a_i of the leading logarithm L^i up to $i = 7$ for the vacuum expectation value V_{phys} for the physical cases $N = 2$ and $N = 3$ as well as for general N.

i	c_i for $N=2$	c_i for $N=3$	c_i for general N
1	$3/2$	$8/3$	$-N^{-1} + N$
2	$21/8$	$68/9$	$1/2\,N^{-2} - 3/2 + N^2$
3	$75/16$	$1720/81$	$-7/6\,N^{-3} + 7/3\,N^{-1} - 13/6\,N + N^3$
4	$1023/128$	$26881/486$	$109/24\,N^{-4} - 103/12\,N^{-2} + 277/48$
			$-127/48\,N^2 + 11/12\,N^4$
5	$2669/256$	$82861/729$	$-637/24\,N^{-5} + 5587/120\,N^{-3} - 57887/2160\,N^{-1}$
			$+9241/1080\,N - 5263/2160N^3 + 179/270\,N^5$
6^*	$-\frac{48029}{138240}$	$\frac{67564919}{1049760}$	$150877/720\,N^{-6} - 49505/144\,N^{-4} + 46879/288\,N^{-2}$
			$-378373/12960 + 9427/10368\,N^2 - 2741/6912\,N^4$
			$+14701/103680\,N^6$
7^*	$-\frac{6924628769}{87091200}$	$-\frac{966193799261}{881798400}$	$-229179/112\,N^{-7} + 440981/140\,N^{-5}$
			$-967456169/816480N^{-3} - 986277601/163296000\,N^{-1}$
			$+3916037663/31104000\,N - 56820907057/1306368000\,N^3$
			$+7411227769/1306368000\,N^5 - 19637251/26127360\,N^7$

Table 6: The coefficients c_i of the leading logarithm L^i_{phys} up to $i = 7$ for the vacuum expectation value V_{phys} for the physical cases $N = 2$ and $N = 3$ as well as for general N.
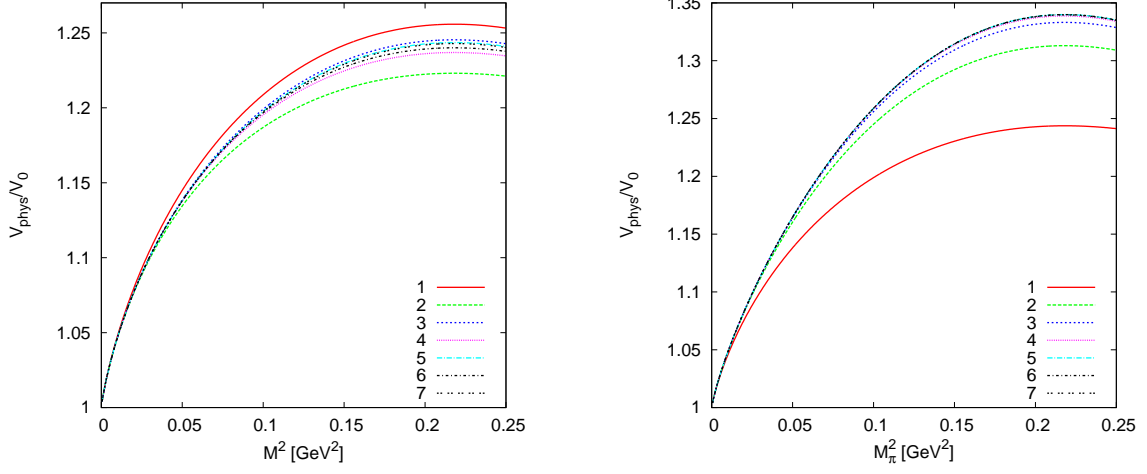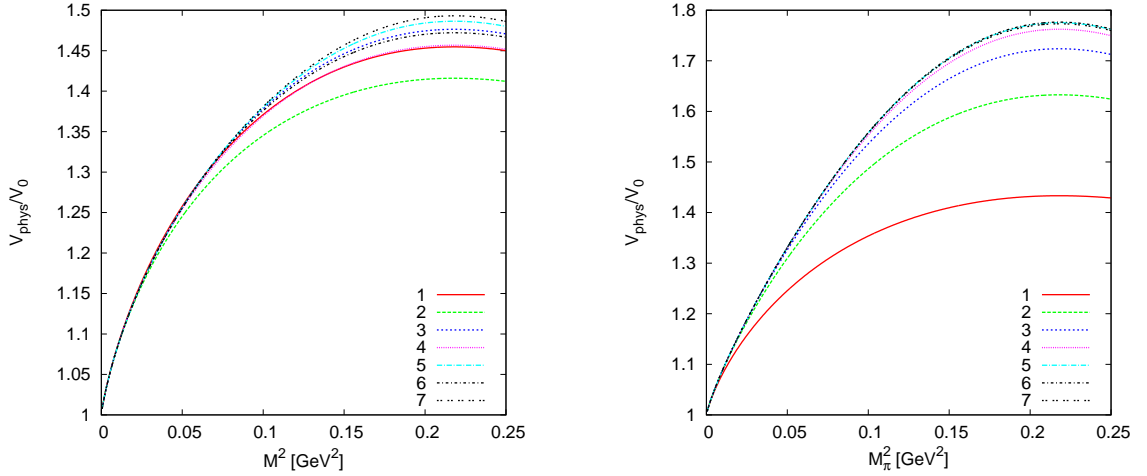
Figure 5: The contribution of the leading logarithms to V_{phys}/V_0 order by order for $F = 0.090$ GeV, $F_\pi = 0.0922$ GeV, $\mu = 0.77$ GeV and $N = 2$. The left panel shows the expansion in L keeping F fixed, the right panel the expansion in L_{phys} keeping F_π fixed.

Figure 6: The contribution of the leading logarithms to V_{phys}/V_0 order by order for $F = 0.090$ GeV, $F_\pi = 0.0922$ GeV, $\mu = 0.77$ GeV and $N = 3$. The left panel shows the expansion in L keeping F fixed, the right panel the expansion in L_{phys} keeping F_π fixed.

i	c_i for $N=2$	c_i for $N=3$	c_i for general N
1	1	3/2	$N/2$
2	2	31/8	$1/2 + 3/8\,N^2$
3	853/108	877/48	$-1/2\,N^{-1} + 29/12\,N + 179/432\,N^3$
4	50513/1296	824171/6912	$17/3\,N^{-2} - 617/96 + 1111/128\,N^2 + 12011/20736\,N^4$
5	120401/648	33850135/41472	$-782/15\,N^{-3} + 134797/2160\,N^{-1} - 5429753/155520\,N$ $+5232541/207360\,N^3 + 14129/15552\,N^5$

Table 7: The coefficients c_i of the leading logarithm L_{phys}^i up to $i=5$ for the vector radius $\langle r^2 \rangle_V$ for the physical cases $N=2$ and $N=3$ as well as for general N in units of M_π^2.

i	c_i for $N=2$	c_i for $N=3$	c_i for general N
1	0	0	0
2	1/72	1/32	$N^2/288$
3	−71/162	−593/864	$-137/648\,N - 5/2592\,N^3$
4	−25169/7776	−1978981/248832	$445/7776 - 79861/103680\,N^2 - 5519/414720\,N^4$
5	−1349303/72900	$-\dfrac{75784701173}{1119744000}$	$-181271/233280\,N^{-1} + 194389/405000\,N$ $-262760809/116640000\,N^3$ $-61724321/1866240000\,N^5$

Table 8: The coefficients c_i of the leading logarithm L_{phys}^i up to $i=5$ for the curvature c_V for the physical cases $N=2$ and $N=3$ as well as for general N in units of M_π^2.

The formula in the massless case is much simpler. The logarithm is now a bit more unique. We replace $\mathcal{M}^2$ by $-t$ and define

$$K_t \equiv \frac{t}{16\pi^2 F^2} \log\left(-\frac{\mu^2}{t}\right). \tag{17}$$

Taking the limit $M_\pi^2 \to 0$ (which implies $F_\pi \to F$), we get from (16):

$$
\begin{aligned}
F_V^0(t) = {} & 1 + K_t(N/12) + K_t^2(N^2/288) + K_t^3(N/32 + 5/5184\,N^3) \\
& + K_t^4(-N^2/345600 + N^4/518400) \\
& + K_t^5(871/115200\,N + 545009/181440000\,N^3 + 126059/6531840000\,N^5).
\end{aligned} \tag{18}
$$

We close this section with giving the expansion for the radius and curvature of the vector form factor defined by

$$F_V(t) = 1 + \frac{1}{6}\langle r^2 \rangle_V t + c_V t^2 + \cdots. \tag{19}$$

The coefficients c_i for the expansion in physical quantities are given in Tables 7 and 8 in units of M_π^2. The result up to two-loop order agrees with the LL extracted from the full

13

i	c_i for $N=2$	c_i for $N=3$	c_i for general N
1	-1	$-2/3$	$-2N^{-1}$
2	$31/8$	$569/72$	$23/2\,N^{-2} - 7/2 + 9/8\,N^2$
3	$65/6$	$3205/81$	$-260/3\,N^{-3} + 133/3\,N^{-1} - 95/12\,N + 23/12\,N^3$
4	$76307/1152$	$2330311/7776$	$19801/24\,N^{-4} - 5731/12\,N^{-2} + 1033/9$ $-3331/288\,N^2 + 295/72\,N^4$
5	$375263/1152$	$65426359/31104$	$-189077/20\,N^{-5} + 82642/15\,N^{-3} - 1352663/1080\,N^{-1}$ $+63967/360\,N - 35647/4320\,N^3 + 48487/5760\,N^5$

Table 9: The coefficients c_i of the leading logarithm L_{phys}^i up to $i=5$ for the scalar form factor at zero momentum transfer $F_S(0)$ for the physical cases $N=2$ and $N=3$ as well as for general N.

two-loop calculation [18]. We do not present numerical results for the vector form factor since these are dominated by large higher-order contributions, see, e.g., [2, 18].

All the results presented in this section agree for $N=2$ up to fifth order with the findings of [12].

8 Scalar form factor

The (singlet) scalar form factor is defined by

$$\langle \phi^a(p_f)| - j_0^s|\phi^a(p_i)\rangle = F_S\left[(p_f - p_i)^2\right] , \tag{20}$$

where again we have normalized the scalar current generator to one. It is known fully in two- and three-flavour ChPT to one [2, 17] and two loops [18, 20]. Here we calculate the leading logarithms in the equal mass case to five loops.

As opposed to the vector case, the scalar form factor is not normalized to one, such that we also need to specify the LL expansion of $F_S(0)$. The coefficients c_i for the expansion in terms of physical logarithms for $O_0 = 2B$ and $O_{\text{phys}} = F_S(0)$ are given in Table 9. The momentum dependent part, $\tilde{F}_S(t) \equiv F_S(t)/F_S(0)$, can again be expressed in terms of $\tilde{t} = t/M_\pi^2$ and the logarithm defined in (11). To fifth order we find

$$\tilde{F}_S(t) = 1 + L_{\mathcal{M}}\left[N/2\,\tilde{t}\right] + L_{\mathcal{M}}^2\left[\tilde{t}\,(-1/2 - 5/18\,N^2) + \tilde{t}^2\,43/144\,N^2\right]$$

$$+ L_{\mathcal{M}}^3\left[\tilde{t}\,(5/6\,N^{-1} - 3/8\,N - 473/2592\,N^3) + \tilde{t}^2\,(-91/216\,N - 227/1296\,N^3) + \tilde{t}^3\,143/864\,N^3\right]$$

$$+ L_{\mathcal{M}}^4\left[\tilde{t}\,(-13/6\,N^{-2} - 1751/1944 - 10529/5184\,N^2 - 2117/3456\,N^4)\right.$$

$$+ \tilde{t}^2\,(4645/3888 + 245537/388800\,N^2 + 27103/259200\,N^4)$$

$$\left. + \tilde{t}^3\,(-196121/388800\,N^2 - 57061/388800\,N^4) + \tilde{t}^4\,(1129/57600\,N^2 + 580837/6220800\,N^4)\right]$$

i	c_i for $N = 2$	c_i for $N = 3$	c_i for general N
1	6	9	$3N$
2	$-29/3$	-18	$-3 - 5/3\,N^2$
3	$-581/54$	$-1663/48$	$5N^{-1} - 9/4\,N - 473/432\,N^3$
4	$-75301/648$	$-2147363/5184$	$-13N^{-2} - 1751/324 - 10529/864\,N^2 - 2117/576\,N^4$
5	$-\frac{5482247}{9720}$	$-\frac{535098163}{186624}$	$417/5\,N^{-3} + 327877/4860\,N^{-1} - 1550429/38880\,N$ $-28557851/777600\,N^3 - 3800759/518400\,N^5$

Table 10: The coefficients c_i of the leading logarithm L^i_{phys} up to $i = 5$ for the the scalar radius $\langle r^2 \rangle_S$ for the physical cases $N = 2$ and $N = 3$ as well as for general N.

i	c_i for $N = 2$	c_i for $N = 3$	c_i for general N
1	0	0	0
2	$43/36$	$43/16$	$43/144\,N^2$
3	$-727/324$	$-863/144$	$-91/216\,N - 227/1296\,N^3$
4	$4369/810$	$2386939/155520$	$4645/3888 + 245537/388800\,N^2 + 27103/259200\,N^4$
5	$\frac{16871641}{2916000}$	$\frac{1130937893}{55987200}$	$-222149/58320\,N^{-1} - 858337/466560\,N$ $+410235883/233280000\,N^3$ $-39345049/466560000\,N^5$

Table 11: The coefficients c_i of the leading logarithm L^i_{phys} up to $i = 5$ for the curvature c_S for the physical cases $N = 2$ and $N = 3$ as well as for general N.

$$
\begin{aligned}
&+ L^5_{\mathcal{M}} \Big[\tilde{t}\,(139/10\,N^{-3} + 327877/29160\,N^{-1} - 1550429/233280\,N - 28557851/4665600\,N^3 \\
&- 3800759/3110400\,N^5) + \tilde{t}^2\,(-222149/58320\,N^{-1} - 858337/466560\,N \\
&+ 410235883/233280000\,N^3 - 39345049/466560000\,N^5) \\
&+ \tilde{t}^3\,(324253/233280\,N - 7699463/38880000\,N^3 + 26029871/311040000\,N^5) \\
&+ \tilde{t}^4\,(-1129/144000\,N - 357457/1296000\,N^3 - 18692191/186624000\,N^5) \\
&+ \tilde{t}^5\,(315439/25920000\,N^3 + 48727189/933120000\,N^5) \Big] \,.
\end{aligned}
\tag{21}
$$

As for the vector form factor, we can also give our result for the radius and the curvature, which are defined as

$$
\tilde{F}_S(t) = 1 + \frac{1}{6} \langle r^2 \rangle_S t + c_S t^2 + \cdots .
\tag{22}
$$

The coefficients c_i for the expansion in physical quantities are given in Tables 10 and 11.

All the results presented in this section agree for $N = 2$ up to fourth order with the findings of [11].

9 Meson-meson scattering

The amplitude for general meson-meson scattering is defined from

$$\langle \phi^c(p_3)\phi^d(p_4)\text{out}|\phi^a(p_1)\phi^b(p_2)\text{in}\rangle = i(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)M(s,t,u)\,, \qquad (23)$$

with the Mandelstam variables

$$s=(p_1+p_2)^2\,, \qquad t=(p_1-p_3)^2\,, \qquad u=(p_1-p_4)^2\,. \qquad (24)$$

It has been calculated at one-loop order in [21]. The two-loop calculation has been performed together with two other symmetry breaking patterns in [22].

The structure of the $SU(N)$ meson-meson scattering amplitude has been derived in full generality in [21, 22]. It can be expressed in terms of two invariant amplitudes $B(s,t,u)$ and $C(s,t,u)$ as

$$
\begin{aligned}
M(s,t,u) = {}& \left[\langle T^aT^bT^cT^d\rangle + \langle T^aT^dT^cT^b\rangle\right]B(s,t,u) \\
& + \left[\langle T^aT^cT^dT^b\rangle + \langle T^aT^bT^dT^c\rangle\right]B(t,u,s) \\
& + \left[\langle T^aT^dT^bT^c\rangle + \langle T^aT^cT^bT^d\rangle\right]B(u,s,t) \\
& + \delta^{ab}\delta^{cd}C(s,t,u) + \delta^{ac}\delta^{bd}C(t,u,s) + \delta^{ad}\delta^{bc}C(u,s,t)\,.
\end{aligned}
\qquad (25)
$$

The T^a are the generators of $SU(N)$ normalized as $\langle T^aT^b\rangle = \delta^{ab}$. Crossing symmetry implies

$$B(s,t,u) = B(u,t,s)\,, \qquad C(s,t,u) = C(s,u,t)\,. \qquad (26)$$

In the case of $N=2$ the traces over four generators evaluate to products of Kronecker deltas such that the structure of the amplitude is reduced to the well-known expression

$$M(s,t,u) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,u,s) + \delta^{ad}\delta^{bc}A(u,s,t)\,, \qquad (27)$$

with

$$A(s,t,u) = C(s,t,u) + B(s,t,u) + B(t,u,s) - B(u,s,t)\,. \qquad (28)$$

This is of course the structure of the $\pi\pi$ scattering amplitude in two-flavour ChPT.

We have calculated the LL contribution to the two invariant amplitudes to fifth order. In order to make the symmetries (26) explicit, we have expressed $B(s,t,u)$ in terms of

$$\tilde{t} = t/M_\pi^2 \quad \text{and} \quad \tilde{\Delta}_{su} = (s-u)/M_\pi^2\,, \qquad (29)$$

and $C(s,t,u)$ in terms of

$$\tilde{s} = s/M_\pi^2 \quad \text{and} \quad \tilde{\Delta}_{ut} = (u-t)/M_\pi^2\,. \qquad (30)$$

We also express it in terms of the more general logarithm (11). Our result for general N reads

$$\frac{F_\pi^2}{M_\pi^2}B(s,t,u) = 1 - \tilde{t}/2 + L_\mathcal{M}[-2/3(3N^{-1}-N) - 5/12\,N\tilde{t} + 1/16\,N\tilde{t}^2 + 1/48\,N\tilde{\Delta}_{su}^2]$$

$$+ L_\mathcal{M}^2[1/36(684N^{-2}+12+29N^2) - 1/36(54N^{-2}+156+17N^2)\tilde{t}$$
$$+ 1/288(486+29N^2)\tilde{t}^2 - 1/288(42-11N^2)\tilde{\Delta}_{su}^2 - 5/1152(54+N^2)\tilde{t}^3$$
$$- 5/384(6+N^2)\tilde{t}\tilde{\Delta}_{su}^2]$$

$$+ L_\mathcal{M}^3[-1/6480(1188000N^{-3}-224160N^{-1}-54388N-7461N^3)$$
$$+ 1/12960(90720N^{-3}+40200N^{-1}-192278N-7391N^3)\tilde{t}$$
$$- 1/51840(33840N^{-1}-317952N-7127N^3)\tilde{t}^2$$
$$+ 1/51840(20400N^{-1}-17888N+3219N^3)\tilde{\Delta}_{su}^2$$
$$- 1/51840(56769N+502N^3)\tilde{t}^3 - 1/51840(9403N+1646N^3)\tilde{t}\tilde{\Delta}_{su}^2$$
$$+ 17/207360(711N+8N^3)\tilde{t}^4 + 1/207360(4122N+653N^3)\tilde{t}^2\tilde{\Delta}_{su}^2$$
$$- 1/69120(43N-29N^3)\tilde{\Delta}_{su}^4]$$

$$+ L_\mathcal{M}^4[-1/777600(-1710720000N^{-4}+272984400N^{-2}+7733020-24562906N^2$$
$$- 1393823N^4) + 1/3110400(-208396800N^{-4}-216817200N^{-2}+32764628$$
$$- 120616232N^2-2240271N^4)\tilde{t} + 1/777600(10376100N^{-2}+11270867$$
$$+ 12959626N^2+141465N^4)\tilde{t}^2 + 1/3110400(-9954000N^{-2}+4785940-1937608N^2$$
$$+ 310421N^4)\tilde{\Delta}_{su}^2 - 1/12441600(14175000N^{-2}+90102306+45489100N^2$$
$$+ 198731N^4)\tilde{t}^3 - 1/12441600(4725000N^{-2}+4860662+4068564N^2$$
$$+ 725363N^4)\tilde{t}\tilde{\Delta}_{su}^2 + 1/24883200(33470280+9600665N^2+52084N^4)\tilde{t}^4$$
$$- 1/2764800(281520-173014N^2-27275N^4)\tilde{t}^2\tilde{\Delta}_{su}^2 - 1/24883200(528120$$
$$+ 125223N^2-33181N^4)\tilde{\Delta}_{su}^4 - 1/24883200(2694708+395685N^2+1270N^4)\tilde{t}^5$$
$$- 1/12441600(418230+63783N^2+7363N^4)\tilde{t}^3\tilde{\Delta}_{su}^2$$
$$- 1/8294400(135360+25767N^2+2642N^4)\tilde{t}\tilde{\Delta}_{su}^4]$$

$$+ L_\mathcal{M}^5[-1/233280000(7018928640000N^{-5}-894502656000N^{-3}-283189944960N^{-1}$$
$$+ 72507663308N-22041000184N^3-690252879N^5) + 1/3265920000(2508879744000N^{-5}$$
$$+ 2387314944000N^{-3}-1788779140400N^{-1}+55061393444N-302932752254N^3$$
$$- 3125842333N^5)\tilde{t} + 1/1866240000(54336096000N^{-3}-23570392560N^{-1}+4485329438N$$
$$- 1556519119N^3+304085556N^5)\tilde{\Delta}_{su}^2 - 1/13063680000(1174116384000N^{-3}$$
$$- 743306538800N^{-1}-1129493851946N-538364942807N^3-3202580448N^5)\tilde{t}^2$$
$$+ 1/26127360000(40960080000N^{-3}+14810807200N^{-1}-46526628758N$$
$$- 14697445505N^3-2544291752N^5)\tilde{t}\tilde{\Delta}_{su}^2 + 1/26127360000(122880240000N^{-3}$$
$$- 16992114720N^{-1}-1115282964722N-281556881291N^3-615480176N^5)\tilde{t}^3$$

$$+ 1/14929920000(529874640N^{-1} - 971377232N - 249290704N^3 + 44246931N^5)\tilde{\Delta}_{su}^4$$

$$+ 1/52254720000(14905099440N^{-1} - 17891241888N + 7135232732N^3$$

$$+ 1114519563N^5)\tilde{t}^2\tilde{\Delta}_{su}^2 - 1/104509440000(17859033360N^{-1} - 964620068912N$$

$$- 176725206936N^3 - 477061005N^5)\tilde{t}^4 - 1/52254720000(3722683128N + 675880271N^3$$

$$+ 64699403N^5)\tilde{t}\tilde{\Delta}_{su}^4 - 1/26127360000(4775058204N + 535939563N^3 + 59878229N^5)\tilde{t}^3\tilde{\Delta}_{su}^2$$

$$- 1/52254720000(53926623792N + 7406736715N^3 + 10009775N^5)\tilde{t}^5$$

$$- 1/5971968000(1608228N + 172954N^3 - 42207N^5)\tilde{\Delta}_{su}^6 + 1/41803776000(35608896N$$

$$+ 27012324N^3 + 4741685N^5)\tilde{t}^2\tilde{\Delta}_{su}^4 + 1/13934592000(86939676N + 29291830N^3$$

$$+ 1761661N^5)\tilde{t}^4\tilde{\Delta}_{su}^2 + 1/209018880000(7967262600N + 1179147888N^3 + 2029543N^5)\tilde{t}^6]$$

$$\tag{31}$$

and

$$\frac{F_\pi^2}{M_\pi^2}C(s,t,u) =$$

$$+ L_\mathcal{M}[2N^{-2} + 3/8\,\tilde{s}^2 + 1/8\,\tilde{\Delta}_{ut}^2]$$

$$+ L_\mathcal{M}^2[-2(7N^{-3} - N^{-1}) - 1/36(108N^{-1} - 37N)\tilde{s} - 13/36\,N\tilde{s}^2$$

$$+ 5/24\,N\tilde{\Delta}_{ut}^2 + 55/192\,N\tilde{s}^3 + 5/192\,N\tilde{s}\tilde{\Delta}_{ut}^2]$$

$$+ L_\mathcal{M}^3[1/810(112860N^{-4} + 5580N^{-2} - 1173 + 1022N^2)$$

$$+ 1/3240(66240N^{-2} - 35174 + 1841N^2)\tilde{s} + 1/2592(4104N^{-2} + 9256 + 1661N^2)\tilde{s}^2$$

$$+ 1/12960(6840N^{-2} + 3936 + 3811N^2)\tilde{\Delta}_{ut}^2 - 1/12960(19044 + 3929N^2)\tilde{s}^3$$

$$+ 1/6480(1998 + 353N^2)\tilde{s}\tilde{\Delta}_{ut}^2 + 1/23040(3762 + 4177N^2)\tilde{s}^4$$

$$+ 1/11520(1254 + 49N^2)\tilde{s}^2\tilde{\Delta}_{ut}^2 + 1/23040(418 + 13N^2)\tilde{\Delta}_{ut}^4]$$

$$+ L_\mathcal{M}^4[-1/25920(41990400N^{-5} + 6609600N^{-3} - 4545600N^{-1} + 365524N - 112385N^3)$$

$$- 1/1555200(342921600N^{-3} - 85806200N^{-1} + 26129734N - 1244537N^3)\tilde{s}$$

$$- 1/3110400(21772800N^{-3} - 31914800N^{-1} - 11247384N - 615995N^3)\tilde{s}^2$$

$$- 1/622080(1451520N^{-3} + 90000N^{-1} - 1035376N - 245073N^3)\tilde{\Delta}_{ut}^2$$

$$+ 1/6220800(12042900N^{-1} + 1797031N + 2858510N^3)\tilde{s}^3$$

$$- 1/2073600(864300N^{-1} - 2339087N - 167184N^3)\tilde{s}\tilde{\Delta}_{ut}^2$$

$$- 1/12441600(10710972N + 2610871N^3)\tilde{s}^4 + 1/103680(46775N + 1366N^3)\tilde{s}^2\tilde{\Delta}_{ut}^2$$

$$+ 1/2488320(197684N + 4431N^3)\tilde{\Delta}_{ut}^4 + 1/33177600(3838644N + 3531083N^3)\tilde{s}^5$$

$$+ 1/19906560(275724N + 4553N^3)\tilde{s}\tilde{\Delta}_{ut}^4 + 1/9953280(285444N + 5093N^3)\tilde{s}^3\tilde{\Delta}_{ut}^2]$$

$$+ L_\mathcal{M}^5[1/81648000(1754648179200N^{-6} + 506658499200N^{-4} - 280670106528N^{-2}$$

$$+ 54081789480 - 2534550216N^2 + 999897691N^4) + 1/1632960000(4317509952000N^{-4}$$

$$- 1246702968000N^{-2} + 33240243520 - 42063222570N^2 + 13158613N^4)\tilde{s}$$

$$+ 1/6531840000(147311136000N^{-4} + 93672190080N^{-2} - 67579015880 + 41982479569N^2$$

$$
\begin{aligned}
&+ 3435837529N^4)\tilde{\Delta}_{ut}^2 + 1/6531840000(441933408000N^{-4} - 524994234240N^{-2} \\
&+ 414501298952 + 85767984257N^2 + 6718221609N^4)\tilde{s}^2 + 1/4354560000(21832020000N^{-2} \\
&- 10349834500 + 12661599885N^2 + 429983342N^4)\tilde{s}\tilde{\Delta}_{ut}^2 \\
&- 1/13063680000(238082755680N^{-2} + 224361657012 + 45828303299N^2 \\
&+ 836998070N^4)\tilde{s}^3 + 1/26127360000(2833329240N^{-2} + 1543460180 + 5975744498N^2 \\
&+ 97700803N^4)\tilde{\Delta}_{ut}^4 + 1/26127360000(16999975440N^{-2} + 11020206360 + 32781452788N^2 \\
&+ 783206303N^4)\tilde{s}^2\tilde{\Delta}_{ut}^2 + 1/13063680000(12749981580N^{-2} + 117034928874 \\
&+ 16745354989N^2 + 4551556991N^4)\tilde{s}^4 + 1/10450944000(1480020120 + 1322536120N^2 \\
&+ 14756647N^4)\tilde{s}^3\tilde{\Delta}_{ut}^2 + 1/20901888000(1604443896 + 1525971760N^2 + 18544639N^4)\tilde{s}\tilde{\Delta}_{ut}^4 \\
&- 1/34836480000(37975069080 + 25358850464N^2 + 4923041771N^4)\tilde{s}^5 \\
&+ 1/580608000(20634624 + 4848765N^2 + 66668N^4)\tilde{s}^4\tilde{\Delta}_{ut}^2 + 1/2322432000(77354172 \\
&+ 15213300N^2 + 129899N^4)\tilde{s}^2\tilde{\Delta}_{ut}^4 + 1/52254720000(118623420 + 27848862N^2 \\
&+ 216847N^4)\tilde{\Delta}_{ut}^6 + 1/34836480000(2583793620 + 2952657684N^2 + 2122427329N^4)\tilde{s}^6] \, .
\end{aligned}
\tag{32}
$$

The five-loop contribution is of the same calculational complexity as the mass to the sixth order and as the latter, has only been checked in two of the four parametrizations in (4). Up to fourth order and for $N = 2$, the amplitude agrees with $A(s, t, u)$ from [11].

It is well known that for $SU(2)$ $\pi\pi$ scattering, the amplitude can be decomposed into three amplitudes corresponding to intermediate states of fixed isospin 0, 1, or 2. This decomposition can be generalized to arbitrary values of N, where one finds seven different intermediate states. The corresponding amplitudes are obtained from the above invariant amplitudes as [22]

$$
\begin{aligned}
T_I &= 2\left(N - \frac{1}{N}\right)[B(s, t, u) + B(t, u, s)] - \frac{2}{N}B(u, s, t) \\
&\quad + (N^2 - 1)C(s, t, u) + C(t, u, s) + C(u, s, t) \, , \\
T_S &= \left(N - \frac{4}{N}\right)[B(s, t, u) + B(t, u, s)] - \frac{4}{N}B(u, s, t) \\
&\quad + C(t, u, s) + C(u, s, t) \, , \\
T_A &= N[-B(s, t, u) + B(t, u, s)] + C(t, u, s) - C(u, s, t) \, , \\
T_{SA} &= C(t, u, s) - C(u, s, t) \, , \\
T_{AS} &= C(t, u, s) - C(u, s, t) \, , \\
T_{SS} &= 2B(u, s, t) + C(t, u, s) + C(u, s, t) \, , \\
T_{AA} &= -2B(u, s, t) + C(t, u, s) + C(u, s, t) \, .
\end{aligned}
\tag{33}
$$

The full scattering amplitude is then built up from these as

$$
M(s, t, u) = \sum_J T_J(s, t, u) P_J \, ,
\tag{34}
$$

i	a_0^I for $N=2$	a_0^I for $N=3$	a_0^S for $N=3$	a_0^{SS} for $N=2$	a_0^{SS} for $N=3$
1	$9/2$	$358/51$	$59/21$	$-3/2$	$-14/9$
2	$857/42$	$28487/612$	$3505/252$	$-31/6$	$-955/108$
3	$153211/1512$	$7143269/22032$	$751735/9072$	$-7103/216$	$-255265/3888$
4	$41581/84$	$98674513/44064$	$26921179/51840$	$-7802/45$	$-6097649/12960$
5*	$\frac{139816697}{56700}$	$\frac{165016031929}{10575360}$	$\frac{228658804229}{65318400}$	$-\frac{3326573}{3375}$	$-\frac{83190853}{23040}$

Table 12: The coefficients c_i of the leading logarithm L_{phys}^i up to $i=5$ for the s-wave scattering lengths for the physical cases $N=2$ and $N=3$.

i	a_1^A for $N=2$	a_1^A for $N=3$	$a_1^{SA}=a_1^{AS}$ for $N=3$
1	2	$13/6$	$2/3$
2	$791/36$	$941/36$	$50/9$
3	$8528/81$	$665171/3888$	$25481/972$
4	$2291903/3888$	$678064381/559872$	$51822143/279936$
5*	$894986647/291600$	$234732737339/27993600$	$3480221279/2799360$

Table 13: The coefficients c_i of the leading logarithm L_{phys}^i up to $i=5$ for the p-wave scattering lengths for the physical cases $N=2$ and $N=3$.

where P_J are the respective projection operators. Since their explicit form is rather lengthy, we do not reproduce it here and refer the interested reader to [22]. For $N=2$, the channels with $J=S, SA, AS, AA$ do not exist and for $N=3$ the channel with $J=AA$ is not present. Each channel can be projected on partial waves by

$$T_\ell^J(s) = \frac{1}{64\pi} \int_{-1}^{1} d(\cos\theta) P_\ell(\cos\theta) T_J(s,t,u) \,. \tag{35}$$

Expanding these around threshold in powers of $q^2 = s/4 - M_\pi^2$ leads to the definition of the threshold parameters:

$$\mathrm{Re}\, T_\ell^J(s) = q^{2\ell}(a_\ell^J + q^2 b_\ell^J + q^4 c_\ell^J + \cdots)\,, \tag{36}$$

where a_ℓ^J are the scattering lengths and b_ℓ^J the slope parameters. We have calculated all the s- and p-wave scattering lengths. Note that for each channel, only one of the two partial waves is non-zero. For $N=2$, only three channels contribute and the scattering length are more commonly denoted by

$$a_0^0 \equiv a_0^I\,, \qquad a_1^1 \equiv a_1^A\,, \qquad a_0^2 \equiv a_0^{SS}\,. \tag{37}$$

For $N=3$, T^{AA} is the only channel that vanishes, such that there are six scattering lengths. The tree-level expressions for general N as well as for the physical cases $N=2$ and $N=3$
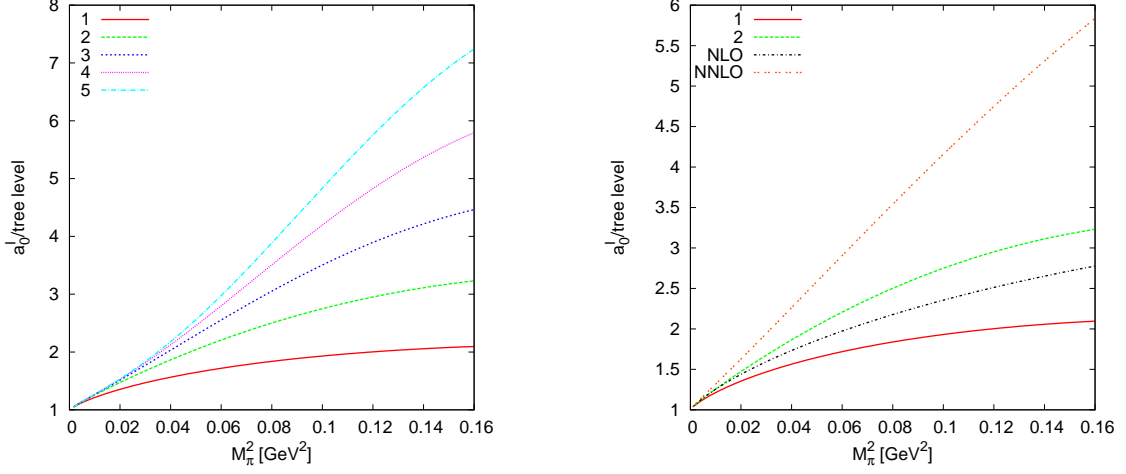
20

Figure 7: The contribution of the leading logarithms to $a_0^I/a_0^{I,tree}$ order by order for $F_\pi = 0.0922$ GeV, $\mu = 0.77$ GeV and $N = 3$. The left panel shows the expansion in L_{phys} keeping F_π fixed. The right panel shows the first two orders together with with the equivalent full results of [22], labeled NLO and NNLO.

are given by [21, 22].

$$
\begin{aligned}
a_0^{I,\text{tree}} &= \frac{M_\pi^2}{16\pi F_\pi^2}\left(2N - \frac{1}{N}\right) & \overset{N=2}{\Longrightarrow} \frac{7M_\pi^2}{32\pi F_\pi^2} & \overset{N=3}{\Longrightarrow} \frac{17M_\pi^2}{48\pi F_\pi^2}\,, \\
a_0^{S,\text{tree}} &= \frac{M_\pi^2}{16\pi F_\pi^2}\left(N - \frac{2}{N}\right) & & \overset{N=3}{\Longrightarrow} \frac{7M_\pi^2}{48\pi F_\pi^2}\,, \\
a_1^{A,\text{tree}} &= \frac{M_\pi^2}{48\pi F_\pi^2}N & \overset{N=2}{\Longrightarrow} \frac{M_\pi^2}{24\pi F_\pi^2} & \overset{N=3}{\Longrightarrow} \frac{M_\pi^2}{16\pi F_\pi^2}\,, \\
a_1^{SA,\text{tree}} &= a_1^{AS,\text{tree}} = 0 & & \overset{N=3}{\Longrightarrow} 0\,, \\
a_0^{SS,\text{tree}} &= -\frac{M_\pi^2}{16\pi F_\pi^2} & \overset{N=2}{\Longrightarrow} -\frac{M_\pi^2}{16\pi F_\pi^2} & \overset{N=3}{\Longrightarrow} -\frac{M_\pi^2}{16\pi F_\pi^2}\,, \\
a_0^{AA,\text{tree}} &= \frac{M_\pi^2}{16\pi F_\pi^2}\,.
\end{aligned}
\tag{38}
$$

The scattering lengths from channels that do not contribute for $N = 2$ or $N = 3$ have been omitted. $a_1^{SA,\text{tree}}$ and $a_1^{SA,\text{tree}}$ vanish at tree level, but the higher-order contributions are non-zero.

As usual, the LL expansion can be written in the form of (8), where now $O_0 = a_\ell^{J,\text{tree}}$. There is, however, an exception: since the tree-level contribution to a_1^{SA} and a_1^{AS} vanishes, the series is written in these cases as

$$
a_1^{SA} = a_1^{AS} = \frac{M_\pi^2}{16\pi F_\pi^2}(c_1 L_{\text{phys}} + c_2 L_{\text{phys}}^2 + \cdots)\,.
\tag{39}
$$

The corresponding coefficients for the two physical cases are listed in Table 12 for the s-wave and in Table 13 for the p-wave scattering lengths.

Our results for the scattering lengths agree up to two-loops with [22]. Furthermore, for $N = 2$ the LL contributions to a_0^I and a_0^{SS} have been calculated up to fourth order in [11] and are in agreement with the present results.

We show the convergence of the leading logarithm part of the scattering length a_0^I for $N = 3$ in the left panel of Fig. 7. On the right side, we show the one- and two-loop LL compared with the full calculation of [22]. One can see that the leading logarithms are about half of the full correction.

10 $\gamma\gamma \to \pi\pi$ and pion polarizabilities

The process $\gamma\gamma \to \pi\pi$ has been calculated in ChPT to one loop in [23, 24]. Already before the p^6 Lagrangian was explicitly known, $\gamma\gamma \to \pi^0\pi^0$ has been worked out to two loops in [25] and $\gamma\gamma \to \pi^+\pi^-$ in [26, 27]. Both calculations were redone and the $\mathcal{O}(p^6)$ counter terms added explicitly in [28, 29]. While the leading contribution to the charged process comes from tree-level diagrams, the neutral process only starts at the one-loop level. In that case, knowing at least the leading logarithms at higher orders is a particularly welcome check for the convergence of the chiral expansion. Starting at the four-loop order, there is also a doubly anomalous contribution to the amplitude. However, it only affects sub-leading logarithms and is therefore not relevant in the present context.

We have calculated the leading logarithms for the amplitude with two different vectors to two different mesons in general but the expressions are extremely lengthy. We thus restrict ourselves to the simpler case where both vectors are coupling to the current $\bar{q}\gamma_\mu T^c q$ and denote them as γ^c. For the numerical results we treat only the case where the vectors correspond to photons and are on-shell.

The $\gamma^c\gamma^c \to \pi^a\pi^b$ scattering amplitude is defined from the matrix element

$$\langle \pi^a(p_1)\pi^b(p_2)\text{out}|\gamma^c(k_1)\gamma^c(k_2)\text{in}\rangle = i(2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2)T^{abc}(s,t,u)\,, \tag{40}$$

with

$$s = (p_1 + p_2)^2\,, \qquad t = (p_1 - k_1)^2\,, \qquad u = (p_1 - k_2)^2\,, \tag{41}$$

and

$$T^{abc}(s,t,u) = e^2\epsilon_1^\mu\epsilon_2^\nu V_{\mu\nu}(s,t,u)\,. \tag{42}$$

The polarization vectors for the external vector are ϵ_1, ϵ_2 and we have added an overall coupling constant e to the vectors. We will consider the process for both vectors off-shell such that the amplitude also depends on k_1^2 and k_2^2. Since both vectors carry the same flavour index, the amplitude must satisfy

$$k_1^\mu V_{\mu\nu} = k_2^\nu V_{\mu\nu} = 0\,. \tag{43}$$

This follows from the Ward identity. If the vectors had different flavour there could have been an equal time part in the Ward identity but it vanishes here. As a result the amplitude can be be decomposed into gauge invariant quantities as

$$V^{\mu\nu}(s,t,u) = A(s,t,u)T_1^{\mu\nu} + B(s,t,u)T_2^{\mu\nu} + C(s,t,u)T_3^{\mu\nu} + D(s,t,u)T_4^{\mu\nu} + E(s,t,u)T_5^{\mu\nu}, \tag{44}$$

where

$$
\begin{aligned}
T_{1\mu\nu} &= k_1 \cdot k_2 g_{\mu\nu} - k_{2\mu}k_{1\nu}\,, \\
T_{2\mu\nu} &= k_1 \cdot k_2 \Delta_\mu \Delta_\nu + \frac{1}{2}(t-u)k_{2\mu}\Delta_\nu - \frac{1}{2}(t-u)\Delta_\mu k_{1\nu} - \frac{1}{4}(t-u)^2\, g_{\mu\nu}\,, \\
T_{3\mu\nu} &= k_1 \cdot k_2 k_{1\mu}k_{2\nu} - k_1^2 k_{2\mu}k_{2\nu} - k_2^2 k_{1\mu}k_{1\nu} + k_1^2 k_2^2 g_{\mu\nu}\,, \\
T_{4\mu\nu} &= k_1 \cdot k_2 k_{1\mu}\Delta_\nu - k_1 \cdot k_2 \Delta_\mu k_{2\nu} - k_1^2 k_{2\mu}\Delta_\nu + k_2^2 \Delta_\mu k_{1\nu} \\
&\quad + \frac{1}{2}(t-u)(k_1^2+k_2^2)g_{\mu\nu} - \frac{1}{2}(t-u)k_{1\mu}k_{1\nu} - \frac{1}{2}(t-u)k_{2\mu}k_{2\nu}\,, \\
T_{5\mu\nu} &= k_1^2 k_2^2 \Delta_\mu \Delta_\nu - \frac{1}{2}(t-u)k_1^2\Delta_\mu k_{2\nu} + \frac{1}{2}(t-u)k_2^2 k_{1\mu}\Delta_\nu - \frac{1}{4}(t-u)^2 k_{1\mu}k_{2\nu}\,, \tag{45}
\end{aligned}
$$

with $\Delta = p_1 - p_2$. For $k_1^2 = k_2^2 = 0$, $T_{5\mu\nu}$ is equivalent to $T_{3\mu\nu}$, such that only four independent quantities remain, which are identical to those given in [25] up to normalization factors. For on-shell photons, one has in addition $\epsilon_1 \cdot k_1 = \epsilon_2 \cdot k_2 = 0$, which only leaves $T_{1\mu\nu}$ and $T_{2\mu\nu}$. The $T_{i\mu\nu}$ satisfy the identities

$$k_1^\mu T_{i\mu\nu} = k_2^\nu T_{i\mu\nu} = 0\,, \tag{46}$$

which can be readily checked using

$$-\Delta \cdot k_1 = \Delta \cdot k_2 = \frac{1}{2}(t-u)\,. \tag{47}$$

The flavour structure of the amplitude consists of all possible traces of T^a, T^b and T^c. But the total amplitude is symmetric under $(p_1, a) \leftrightarrow (p_2, b)$ and under $(k_1, \epsilon_1) \leftrightarrow (k_2, \epsilon_2)$. Using charge conjugation[1] one can prove that the amplitude must be separately invariant under $(a \leftrightarrow b)$.

In contrast to all the earlier quantities discussed in this paper, we must here consider several classes of diagrams in order to calculate the physical $\gamma\gamma \to \pi\pi$ amplitude. There are one-particle-reducible contributions here other than those taken care of by wave function renormalization. The three needed types of diagrams are depicted in Figure 8. On the one hand, there is the direct contribution involving the $\gamma\gamma\pi\pi$ leading-order vertex, on the other hand, there are two types of diagrams involving the $\gamma\pi\pi$ vertex twice. Loop contributions to the latter vertex have been calculated already for the vector form factor and the results

---

[1]This is valid at least for the real part of the amplitude.
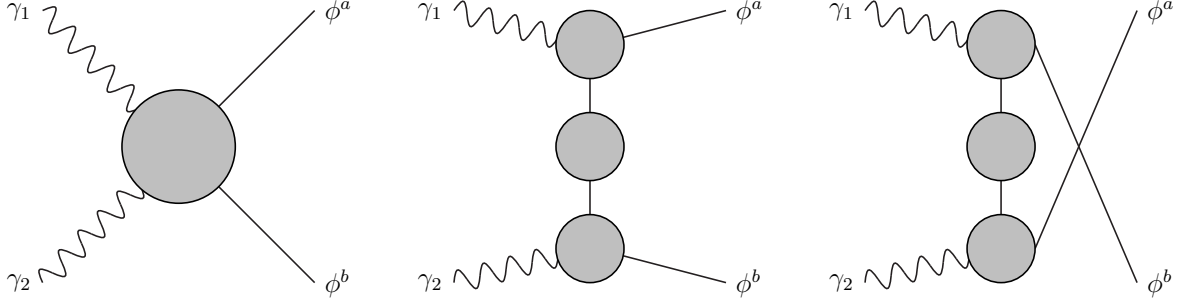
Figure 8: The three types of diagrams that contribute to $\gamma\gamma \to \pi\pi$. The gray circles stand for the sum over all one-particle-irreducible diagrams. Note that the intermediate meson propagator in the two rightmost diagrams is off-shell.

from there can be reused. The corrections to the two-point function are incorporated by using the physical propagator for off-shell momenta, which is given by

$$\frac{1}{1+\overline{\Sigma}(p^2)}\frac{i}{p^2-M_\pi^2} \quad \text{with} \quad \overline{\Sigma}(p^2) = \frac{\Sigma(p^2)-\Sigma(M_\pi^2)}{p^2-M_\pi^2}, \tag{48}$$

where $\Sigma(p^2)$ denotes the self-energy of the meson. The contributions from the diagrams involving a propagator can be written in a compact form because these terms depend explicitly on the vector form factor. In particular, because of the LSZ theorem, the residue of the amplitude when the intermediate propagator is on-shell must contain the vector form factor twice.

We introduce a notation for the flavour traces that is a little shorter to write:

$$
\begin{aligned}
t_1 &= \langle T^a T^b T^c T^c \rangle + \langle T^b T^a T^c T^c \rangle\,, & t_2 &= \langle T^a T^c T^b T^c \rangle\,, \\
t_3 &= \langle T^a T^b \rangle \langle T^c T^c \rangle\,, & t_4 &= \langle T^a T^c \rangle \langle T^b T^c \rangle\,.
\end{aligned}
\tag{49}
$$

That the traces in t_1 only appear together is due to the symmetry under $a \leftrightarrow b$.

A large part of the amplitude is contained in the generalized Born amplitude because of the argument given above, with

$$
\begin{aligned}
\epsilon_1^\mu \epsilon_2^\mu V_{\mu\nu}^{Born} = (t_1 - 2t_2)\, F_V(k_1^2) F_V(k_2^2) \Bigg[&2\,\epsilon_1 \cdot \epsilon_2 \\
&+ \frac{-\Delta \cdot \epsilon_1\, \Delta \cdot \epsilon_2 + \Delta \cdot \epsilon_1 k_1 \cdot \epsilon_2 - k_2 \cdot \epsilon_1\, \Delta \cdot \epsilon_2 + k_2 \cdot \epsilon_1\, k_1 \cdot \epsilon_2}{t - M_\pi^2} \\
&+ \frac{-\Delta \cdot \epsilon_1\, \Delta \cdot \epsilon_2 - \Delta \cdot \epsilon_1 k_1 \cdot \epsilon_2 + k_2 \cdot \epsilon_1\, \Delta \cdot \epsilon_2 + k_2 \cdot \epsilon_1\, k_1 \cdot \epsilon_2}{u - M_\pi^2} \Bigg].
\end{aligned}
\tag{50}
$$

24

The remainder of the amplitude now has no poles in the t or u channel. The pole amplitude also has the commutator structure expected from tree-level couplings to external vectors, visible in T^{abc}. The expression for $F_V(t)$ in terms of the LLs can be found in (16). The generalized Born amplitude can be decomposed in the functions defined in (44), but is somewhat simpler in the form given in (50).

We write the full amplitude now as

$$V_{\mu\nu} = V_{\mu\nu}^{Born} + \overline{A}/M_\pi^2 \, T_{1\mu\nu} + \overline{B}/M_\pi^4 \, T_{2\mu\nu} + \overline{C}/M_\pi^4 \, T_{3\mu\nu} + \overline{D}/M_\pi^4 \, T_{4\mu\nu} + \overline{E}/M_\pi^6 \, T_{5\mu\nu} \,. \quad (51)$$

The factors of M_π^2 are introduced to make the functions dimensionless. The partial amplitudes we write as functions of

$$\tilde{k}_1 = k_1^2/M_\pi^2 \,, \quad \tilde{k}_2 = k_2^2/M_\pi^2 \,, \quad \tilde{k}_{12} = k_1 \cdot k_2/M_\pi^2 \,, \quad \tilde{\Delta}_{tu} = (t-u)/M_\pi^2 \,. \quad (52)$$

Each of the amplitudes we then write as

$$\overline{A} = A^{(2)} L_\mathcal{M}^2 + A^{(3)} L_\mathcal{M}^3 + A^{(4)} L_\mathcal{M}^4 + \cdots \,. \quad (53)$$

The leading logarithms at one-loop order are already fully contained in (50). The two-loop leading logarithms are quite simple:

$$B^{(2)} = (1/72\,N^2 + 1/12)t_1 - 1/6\,t_2 + 1/9\,Nt_3 - 1/36\,Nt_4 \,,$$
$$A^{(2)} = (\tilde{k}_{12} - 2)B^{(2)} \,, \quad C^{(2)} = -B^{(2)} \,, \quad D^{(2)} = E^{(2)} = 0 \,. \quad (54)$$

The third-order expressions are still reasonable in full generality:

$$
\begin{aligned}
A^{(3)} = {}& t_4[1/216\,(30\tilde{k}_{12}^2 + 9\tilde{k}_2\tilde{k}_{12} - 5\tilde{k}_2^2 + 9\tilde{k}_1\tilde{k}_{12} - 2\tilde{k}_1\tilde{k}_2 - 5\tilde{k}_1^2 + 64\tilde{k}_{12} + 54\tilde{k}_2 + 54\tilde{k}_1 - 104) \\
& + 1/1296(30\tilde{k}_{12}^2 + 14\tilde{k}_2\tilde{k}_{12} - \tilde{k}_2^2 + 14\tilde{k}_1\tilde{k}_{12} + 4\tilde{k}_1\tilde{k}_2 - \tilde{k}_1^2 - 32\tilde{k}_{12} + 50\tilde{k}_2 + 50\tilde{k}_1 + 88)N^2] \\
& + t_3[1/432(30\tilde{k}_{12}^2 + 9\tilde{k}_2\tilde{k}_{12} - 5\tilde{k}_2^2 + 9\tilde{k}_1\tilde{k}_{12} - 2\tilde{k}_1\tilde{k}_2 - 5\tilde{k}_1^2 + 64\tilde{k}_{12} + 54\tilde{k}_2 + 54\tilde{k}_1 - 104) \\
& + 1/2592(30\tilde{k}_{12}^2 - 31\tilde{k}_2\tilde{k}_{12} - 37\tilde{k}_2^2 - 31\tilde{k}_1\tilde{k}_{12} - 50\tilde{k}_1\tilde{k}_2 - 37\tilde{k}_1^2 + 548\tilde{k}_{12} - 1072)N^2] \\
& + t_2[1/1296(210\tilde{k}_{12}^2 + 21\tilde{k}_2\tilde{k}_{12} - 299\tilde{k}_2^2 + 21\tilde{k}_1\tilde{k}_{12} + 58\tilde{k}_1\tilde{k}_2 - 299\tilde{k}_1^2 + 608\tilde{k}_{12} \\
& + 2042\tilde{k}_2 + 2042\tilde{k}_1 + 1832)N - 1/1296(2\tilde{k}_2\tilde{k}_{12} + 8\tilde{k}_2^2 + 2\tilde{k}_1\tilde{k}_{12} - \tilde{k}_1\tilde{k}_2 + 8\tilde{k}_1^2 - 26\tilde{k}_{12} \\
& - 52\tilde{k}_2 - 52\tilde{k}_1 + 12)N^3] \\
& + t_1[-1/1296(60\tilde{k}_{12}^2 - 3\tilde{k}_2\tilde{k}_{12} - 142\tilde{k}_2^2 - 3\tilde{k}_1\tilde{k}_{12} + 32\tilde{k}_1\tilde{k}_2 - 142\tilde{k}_1^2 + 208\tilde{k}_{12} + 940\tilde{k}_2 \\
& + 940\tilde{k}_1 + 1072)N - 1/5184(30\tilde{k}_{12}^2 + 15\tilde{k}_2\tilde{k}_{12} - 13\tilde{k}_2^2 + 15\tilde{k}_1\tilde{k}_{12} + 12\tilde{k}_1\tilde{k}_2 - 13\tilde{k}_1^2 - 40\tilde{k}_{12} \\
& + 160\tilde{k}_2 + 160\tilde{k}_1 + 184)N^3] \,, \\
B^{(3)} = {}& t_4[1/216(30\tilde{k}_{12} + 29\tilde{k}_2 + 29\tilde{k}_1 - 20) + 1/648(15\tilde{k}_{12} + 13\tilde{k}_2 + 13\tilde{k}_1 - 58)N^2] \\
& + t_3[1/432(30\tilde{k}_{12} + 29\tilde{k}_2 + 29\tilde{k}_1 - 20) + 1/2592(30\tilde{k}_{12} + 53\tilde{k}_2 + 53\tilde{k}_1 + 464)N^2] \\
& + t_2[1/1296(210\tilde{k}_{12} + 293\tilde{k}_2 + 293\tilde{k}_1 - 1996)N + 1/432(\tilde{k}_2 + \tilde{k}_1 - 10)N^3] \\
& + t_1[-1/1296(60\tilde{k}_{12} + 103\tilde{k}_2 + 103\tilde{k}_1 - 968)N - 1/5184(30\tilde{k}_{12} + 29\tilde{k}_2 + 29\tilde{k}_1 - 236)N^3]
\end{aligned}
$$

$$C^{(3)} = t_4[-1/216(34\tilde{k}_{12} + 17\tilde{k}_2 + 17\tilde{k}_1 - 36) - 1/324(3\tilde{k}_{12} + \tilde{k}_2 + \tilde{k}_1 - 27)N^2]$$
$$+ t_3[-1/432(34\tilde{k}_{12} + 17\tilde{k}_2 + 17\tilde{k}_1 - 44) - 1/2592(210\tilde{k}_{12} + 121\tilde{k}_2 + 121\tilde{k}_1 + 452)N^2]$$
$$+ t_2[-1/1296(106\tilde{k}_{12} + 77\tilde{k}_2 + 77\tilde{k}_1 - 1484)N - 1/1296(\tilde{k}_2 + \tilde{k}_1 - 18)N^3]$$
$$+ t_1[1/1296(2\tilde{k}_{12} + 13\tilde{k}_2 + 13\tilde{k}_1 - 676)N - 1/5184(10\tilde{k}_{12} + 7\tilde{k}_2 + 7\tilde{k}_1 + 200)N^3]$$
$$D^{(3)} = t_4\tilde{\Delta}_{tu}(5/216 + 5/1296\,N^2) + t_3\tilde{\Delta}_{tu}(5/432 + 5/2592\,N^2) + t_2\tilde{\Delta}_{tu}(53/1296\,N + 1/2592\,N^3)$$
$$+ t_1\tilde{\Delta}_{tu}(-19/1296\,N - 1/864\,N^3)$$
$$E^{(3)} = t_4(7/54 + 13/648\,N^2) + t_3(7/108 + 11/648\,N^2) + t_2(23/162\,N)$$
$$+ t_1(-25/648\,N - 1/216\,N^3)\,. \tag{55}$$

The fourth-order expression is very long. We therefore only quote the on-shell case with $k_1^2 = k_2^2 = \epsilon_1 \cdot k_1 = \epsilon_2 \cdot k_2 = 0$, where the amplitude is reduced to the contributions from $\overline{A}$ and $\overline{B}$:

$$A^{(4)} = t_4[-85/324(\tilde{k}_{12} - 2)N^{-1} + 1/777600(129\tilde{k}_{12}\tilde{\Delta}_{tu}^2 - 40908\tilde{k}_{12}^3 + 12292\tilde{\Delta}_{tu}^2 - 573518\tilde{k}_{12}^2$$
$$+ 3289918\tilde{k}_{12} - 2678500)N - 1/1555200(174\tilde{k}_{12}\tilde{\Delta}_{tu}^2 + 14452\tilde{k}_{12}^3 + 1027\tilde{\Delta}_{tu}^2$$
$$- 79808\tilde{k}_{12}^2 - 67142\tilde{k}_{12} - 75300)N^3]$$
$$+ t_3[43/324(\tilde{k}_{12} - 2)N^{-1} + 1/1555200(5772\tilde{k}_{12}\tilde{\Delta}_{tu}^2 + 346056\tilde{k}_{12}^3 + 14281\tilde{\Delta}_{tu}^2 - 187274\tilde{k}_{12}^2$$
$$+ 1943624\tilde{k}_{12} - 2697000)N + 1/3110400(453\tilde{k}_{12}\tilde{\Delta}_{tu}^2 + 37544\tilde{k}_{12}^3 - 806\tilde{\Delta}_{tu}^2 + 6624\tilde{k}_{12}^2$$
$$+ 1042176\tilde{k}_{12} - 1985600)N^3]$$
$$+ t_2[-19/54(\tilde{k}_{12} - 2)N^{-2} - 1/388800(1881\tilde{k}_{12}\tilde{\Delta}_{tu}^2 + 128988\tilde{k}_{12}^3 + 663\tilde{\Delta}_{tu}^2 + 128748\tilde{k}_{12}^2$$
$$- 444098\tilde{k}_{12} - 862700) - 1/1555200(4824\tilde{k}_{12}\tilde{\Delta}_{tu}^2 + 326352\tilde{k}_{12}^3 + 50627\tilde{\Delta}_{tu}^2 - 1077658\tilde{k}_{12}^2$$
$$- 5276492\tilde{k}_{12} - 3115200)N^2 - 1/62208(39\tilde{\Delta}_{tu}^2 + 280\tilde{k}_{12}^2 - 5810\tilde{k}_{12} + 6228)N^4]$$
$$+ t_1[19/108(\tilde{k}_{12} - 2)N^{-2} + 1/777600(1881\tilde{k}_{12}\tilde{\Delta}_{tu}^2 + 128988\tilde{k}_{12}^3 + 663\tilde{\Delta}_{tu}^2 + 128748\tilde{k}_{12}^2$$
$$- 460898\tilde{k}_{12} - 829100) + 1/518400(1139\tilde{k}_{12}\tilde{\Delta}_{tu}^2 + 69072\tilde{k}_{12}^3 + 10597\tilde{\Delta}_{tu}^2 - 253738\tilde{k}_{12}^2$$
$$- 478512\tilde{k}_{12} - 925400)N^2 + 1/6220800(219\tilde{k}_{12}\tilde{\Delta}_{tu}^2 + 12512\tilde{k}_{12}^3 + 5012\tilde{\Delta}_{tu}^2 - 174348\tilde{k}_{12}^2$$
$$+ 48248\tilde{k}_{12} - 20800)N^4]$$
$$B^{(4)} = t_4[-85/324\,N^{-1} + 1/777600(129\tilde{\Delta}_{tu}^2 - 40908\tilde{k}_{12}^2 + 564866\tilde{k}_{12} - 362950)N$$
$$- 1/777600(87\tilde{\Delta}_{tu}^2 + 7226\tilde{k}_{12}^2 - 83402\tilde{k}_{12} + 170475)N^3]$$
$$+ t_3[43/324\,N^{-1} + 1/777600(2886\tilde{\Delta}_{tu}^2 + 173028\tilde{k}_{12}^2 - 323231\tilde{k}_{12} + 216900)N$$
$$+ 1/3110400(453\tilde{\Delta}_{tu}^2 + 37544\tilde{k}_{12}^2 - 65888\tilde{k}_{12} + 812200)N^3]$$
$$+ t_2[-19/54\,N^{-2} - 1/388800(1881\tilde{\Delta}_{tu}^2 + 128988\tilde{k}_{12}^2 - 403776\tilde{k}_{12} + 68650)$$
$$- 1/777600(2412\tilde{\Delta}_{tu}^2 + 163176\tilde{k}_{12}^2 - 1391027\tilde{k}_{12} + 4833750)N^2 + 29/10368(2\tilde{k}_{12} - 29)N^4]$$
$$+ t_1[19/108\,N^{-2} + 1/777600(1881\tilde{\Delta}_{tu}^2 + 128988\tilde{k}_{12}^2 - 403776\tilde{k}_{12} + 51850)$$
$$+ 1/1555200(3417\tilde{\Delta}_{tu}^2 + 207216\tilde{k}_{12}^2 - 1310482\tilde{k}_{12} + 4768800)N^2$$
$$+ 1/6220800(219\tilde{\Delta}_{tu}^2 + 12512\tilde{k}_{12}^2 - 162724\tilde{k}_{12} + 691400)N^4] \tag{56}$$

The two-loop leading logarithms[2] for $N = 2$ agree with those of [28, 29] for $\overline{A}$ and $\overline{B}$. Note that the $\bar{l}_i$ in the formulas given there also contain a logarithm denoted by ℓ.

We can now use these results to find the polarizabilities. These are defined from the helicity amplitudes,

$$H_{++} = \frac{\overline{A}}{M_\pi^2} + \frac{4M_\pi^2 - s}{2M_\pi^4}\overline{B}, \quad H_{+-} = \frac{2(M_\pi^4 - tu)}{M_\pi^4 s}\overline{B}, \tag{57}$$

at fixed $t = M_\pi^2$:

$$\frac{\alpha}{M_\pi}H_{+\mp}(s, t = M_\pi^2) = (\alpha_1 \pm \beta_1) + \frac{s}{12}(\alpha_2 \pm \beta_2) + \mathcal{O}(s^2). \tag{58}$$

The leading terms are the dipole, the next-to-leading terms the quadrupole polarizabilities. They can then be expanded as

$$\alpha_i \pm \beta_i = \frac{\alpha}{16\pi^2 F_\pi^2 M_\pi}\left(c_{i\pm} + \frac{M_\pi^2 d_{i\pm}}{16\pi^2 F_\pi^2} + \mathcal{O}(M_\pi^4)\right). \tag{59}$$

At one-loop order the leading logarithms vanish. This was shown for $N = 2, 3$ in the earlier works [23, 24, 30]. The two-loop LLs for $N = 2$ can be extracted most easily from the expressions in [28, 29]. The conclusion is that the only terms containing LLs are the d_{1+} and from [28, 29] we get

$$d_{1+}(\pi^0) = \frac{2}{9}\log^2(\mu^2/M_\pi^2), \quad d_{1+}(\pi^+) = \frac{4}{9}\log^2(\mu^2/M_\pi^2). \tag{60}$$

Alternatively we can write the expression for the polarizabilities using our notation as

$$\alpha_1 \pm \beta_1 = \frac{\alpha}{M_\pi^3}\sum_i c_i L_{\text{phys}}^i,$$

$$\alpha_2 \pm \beta_2 = \frac{\alpha}{M_\pi^5}\sum_i c_i L_{\text{phys}}^i. \tag{61}$$

The resulting coefficients are given in Table 14. Note that they should not be confused with the coefficients $c_{i\pm}$ in (59). We only quote the results for $N = 2$. The general N results depend on how one extends the charge matrix to more flavours.

In Table 15 we have given the numerical LL contributions together with the full two-loop results in units of 10^{-4} fm^3 for the dipole and 10^{-4} fm^3 for the quadrupole polarizabilities. The results are somewhat mixed. For the phenomenologically most relevant case, $\alpha_i - \beta_i$, the LL are always small and well below the uncertainty quoted in [28, 29]. Also $\alpha_1 + \beta_1$ for the neutral pion is not much affected but the other $\alpha_i + \beta_i$ obtain significant corrections to their actual value. The estimate in [28] of the p^8 contributions from omega exchange to $\alpha_2 + \beta_2$ neutral is -0.25 in the same units, again much larger than the LL. The charged $\alpha_i + \beta_i$ obtain larger relative corrections since the two-loop estimate of [29] is very small. Note that the chiral logarithm contributions in [28, 29] also include nonleading logarithms which is why those numbers differ from the ones in Table 15.

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[2]Our calculation is with a charge matrix with vanishing trace. However, the singlet part does not appear in the lowest-order Lagrangian, hence we get the correct result for $N = 2$ using $Q = \text{diag}(1/2, -1/2)$ rather than $Q = \text{diag}(2/3, -1/3)$.

	neutral				charged			
i	$\alpha_1+\beta_1$	$\alpha_1-\beta_1$	$\alpha_2+\beta_2$	$\alpha_2-\beta_2$	$\alpha_1+\beta_1$	$\alpha_1-\beta_1$	$\alpha_2+\beta_2$	$\alpha_2-\beta_2$
1	0	0	0	0	0	0	0	0
2	2/9	0	0	0	4/9	0	0	0
3	4/9	−8/9	20/9	16/3	337/81	14/9	−10/9	−352/27
4	$\frac{203}{243}$	$-\frac{155}{18}$	$\frac{4631}{675}$	$\frac{4694}{81}$	$\frac{29239}{972}$	$\frac{1045}{54}$	$-\frac{114829}{2025}$	$-\frac{4529}{27}$

Table 14: The coefficients c_i of the leading logarithms for the pion polarizabilities as defined in (61) for the physical case $N = 2$.

	neutral				charged			
i	$\alpha_1+\beta_1$	$\alpha_1-\beta_1$	$\alpha_2+\beta_2$	$\alpha_2-\beta_2$	$\alpha_1+\beta_1$	$\alpha_1-\beta_1$	$\alpha_2+\beta_2$	$\alpha_2-\beta_2$
1	0	0	0	0	0	0	0	0
2	0.11	0	0	0	0.23	0	0	0
3	0.011	−0.022	0.11	0.26	0.10	0.039	−0.056	−0.65
4	0.001	−0.011	0.017	0.14	0.037	0.024	−0.14	−0.42
[28, 29]	1.1	−1.9	0.04	37.6	0.16	5.7	−0.001	16.2

Table 15: The numerical contribution of the leading logarithms at each order to the pion polarizabilities with the physical charged pion mass $M_\pi = 139.57$ MeV, $F_\pi = 92.2$ MeV and $\mu = 0.77$ GeV. The units are 10^{-4} fm^3 for the dipole and 10^{-4} fm^5 for the quadrupole polarizabilities.

11 Conclusions

In this work we extended the earlier work on leading logarithms in effective theories and especially massive nonlinear sigma models to the case of $SU(N) \times SU(N)/SU(N)$ with equal meson masses. We presented results for the leading logarithms for up to 7 loops for the mass, decay constant, vacuum expectation value, vector form factor, and scalar form factor, as well as for a number of quantities connected with meson-meson scattering and $\gamma\gamma \to \pi\pi$. When applicable we have provided results for both physically relevant cases $N = 2$ and $N = 3$ and for general N. In all cases we find reasonable convergence for $N = 3$, while it is even better for $N = 2$.

All the results presented here have been checked through the use of several parameterizations. Furthermore, we have compared to existing one- and two-loop results. The higher orders could be checked for $N = 2$ by comparison with our previous work on the $O(3)$ model.

We have not done a general study of how well the large N limit works. But looking at the coefficients in the various tables, one notes that the coefficients of the subleading terms are often larger than the leading coefficients even though usually not by much. Some cases with substantial corrections to the large N limit can however be found. In conclusion, the

large N limit is not typically a good approximation to the full result but is significantly better than was found for the $O(n)$ case, both due to the size of the coefficients and the fact that the suppression is now in powers of N^2.

One of the motivations behind this work was the hope that knowing many of the leading coefficients in N would allow for an educated guess at the all-order series. We did not succeed in this. Our work can serve as a starting point for future studies in this direction.

Acknowledgments

A   Powers of N in the results

The formulas and tables in the main text show a clear step of 2 in the powers of the number of flavours N that show up. In this appendix we show that this must be the case to all orders. The same is true for the colour factors in QCD with N_c-colours due to gluonic contributions but we are not aware of a simple published proof even though it might be well known in the QCD perturbation theory community.

If we had a theory with the symmetry breaking pattern $U(N) \times U(N)/U(N)$ we would have N^2 Goldstone bosons and the proof could be done using 't Hooft's double line notation with the only difference that the lines indicate flavour of the quarks rather than colour. The argument in [31] in the purely gluonic theory with surfaces with handles goes through in the same way and one concludes that only positive powers of N should appear and that adding a handle changes the power by two.

Since in our case we do not have N^2 particles but rather $N^2 - 1$, we cannot simply take over the arguments from [31]. Those still determine the highest power of N at each loop order, though.

In the main text we determined the leading logarithms from one-loop diagrams only. But the leading logarithms are in principle also determined from the diagrams with largest number of loops at any order. We use the fact that in these diagrams, all vertices come from the single trace lowest-order Lagrangian. The proof immediately generalizes if external fields are included. We give the proof for one-particle-irreducible diagrams only but the generalization should be fairly clear.

For a given one-particle-irreducible diagram, the number of loops N_L, vertices N_V, and propagators N_P satisfy the relation

$$N_L \;=\; N_P - N_V + 1 \,. \tag{62}$$

For the parametrization U_1 in (4), exactly one flavour trace appears for each vertex, such that the total number of traces in the diagram is N_V. For each of the N_P propagators there is a sum over a flavour index, which can be evaluated using the relations

$$\langle T^a A \rangle \langle T^a B \rangle = \langle AB \rangle - \frac{1}{N}\langle A \rangle \langle B \rangle, \qquad \langle T^a A T^a B \rangle = \langle A \rangle \langle B \rangle - \frac{1}{N}\langle AB \rangle \,. \tag{63}$$

We denote the total number of traces in a term by N_{tr} and the power of N by N_N. The sum of the two is abbreviated by $N_{trN} = N_{tr} + N_N$. Each time the first one of the relations (63) is used, N_{trN} is reduced by one. The first term of the second relation, on the other hand, adds one to N_{trN}, while the second term again subtracts one. Positive powers of N can be generated by $\langle \mathbf{1} \rangle = N$.

After all propagator flavour traces have been removed, a diagram with N_V vertices can contain terms with $N_{trN} = N_V - N_P, N_V - N_P + 2, \ldots$. The minimal value is achieved if N_{trN} has been decreased by one N_P times. If the first term of the second relation in (63) has entered exactly once, one gets instead $N_{trN} = N_V - (N_P - 1) + 1$ and so on. From (62) then follows that N_{trN} is odd (even) for even (odd) N_L. The tree-level one-particle-irreducible diagrams clearly satisfy this since they contain no powers of N and one flavour trace.

For a given number of loops, the lowest number that can occur is $N_{trN} = 1 - N_L$ using (62). The highest requires a little more work because not all contractions can increase N_{trN}. In a one-particle-irreducible loop diagram with N_V vertices, we need to use the first relation in (63), which only lowers N_{trN}, at least $N_V - 1$ times since at least that many operations have the T^a in different traces. So the maximum is

$$N_{trN} = N_V - (N_V - 1) + (N_P - (N_V - 1)) = N_P - N_V + 2 = 1 + N_L \,. \tag{64}$$

This coincides with the maximum power derived with the double line method of [31].

References

[1] S. Weinberg, *Phenomenological Lagrangians*, *Physica* **A96** (1979) 327. Festschrift honoring Julian Schwinger on his 60th birthday.

[2] J. Gasser and H. Leutwyler, *Chiral perturbation theory to one loop*, *Annals Phys.* **158** (1984) 142.

[3] J. Gasser and H. Leutwyler, *Chiral perturbation theory: Expansions in the mass of the strange quark*, *Nucl.Phys.* **B250** (1985) 465.

[4] J. Bijnens, G. Colangelo and G. Ecker, *Double chiral logs*, *Phys.Lett.* **B441** (1998) 437–446, [hep-ph/9808421].

[5] M. Büchler and G. Colangelo, *Renormalization group equations for effective field theories*, *Eur.Phys.J.* **C32** (2003) 427–442, [`hep-ph/0309049`].

[6] J. Bijnens and L. Carloni, *Leading logarithms in the massive O(N) nonlinear sigma model*, *Nucl.Phys.* **B827** (2010) 237–255, [`arXiv:0909.5086`].

[7] N. Kivel, M. Polyakov and A. Vladimirov, *Chiral logarithms in the massless limit tamed*, *Phys.Rev.Lett.* **101** (2008) 262001, [`arXiv:0809.3236`].

[8] N. Kivel, M. Polyakov and A. Vladimirov, *Leading chiral logarithms for pion form factors to arbitrary number of loops*, *JETP Lett.* **89** (2009) 529–534, [`arXiv:0904.3008`].

[9] J. Koschinski, M. V. Polyakov and A. A. Vladimirov, *Leading infrared logarithms from unitarity, analyticity and crossing*, *Phys.Rev.* **D82** (2010) 014014, [`arXiv:1004.2197`].

[10] M. Polyakov and A. Vladimirov, *Leading infrared logarithms for σ-model with fields on arbitrary Riemann manifold*, *Theor.Math.Phys.* **169** (2011) 1499–1506, [`arXiv:1012.4205`].

[11] J. Bijnens and L. Carloni, *The massive O(N) non-linear sigma model at high orders*, *Nucl.Phys.* **B843** (2011) 55–83, [`arXiv:1008.3499`].

[12] J. Bijnens, K. Kampf and S. Lanz, *Leading logarithms in the anomalous sector of two-flavour QCD*, *Nucl.Phys.* **B860** (2012) 245–266, [`arXiv:1201.2608`].

[13] J. Bijnens, *Chiral perturbation theory beyond one loop*, *Prog.Part.Nucl.Phys.* **58** (2007) 521–586, [`hep-ph/0604043`].

[14] J. Bijnens and J. Lu, *Technicolor and other QCD-like theories at next-to-next-to-leading order*, *JHEP* **0911** (2009) 116, [`arXiv:0910.5424`].

[15] J. Vermaseren, *New features of FORM*, `math-ph/0010025`.

[16] J. Gasser and H. Leutwyler, *Light quarks at low temperatures*, *Phys.Lett.* **B184** (1987) 83.

[17] J. Gasser and H. Leutwyler, *Low-energy expansion of meson form-factors*, *Nucl.Phys.* **B250** (1985) 517–538.

[18] J. Bijnens, G. Colangelo and P. Talavera, *The vector and scalar form-factors of the pion to two loops*, *JHEP* **9805** (1998) 014, [`hep-ph/9805389`].

[19] J. Bijnens and P. Talavera, *Pion and kaon electromagnetic form-factors*, *JHEP* **0203** (2002) 046, [`hep-ph/0203049`].

[20] J. Bijnens and P. Dhonte, *Scalar form-factors in $SU(3)$ chiral perturbation theory*, *JHEP* **0310** (2003) 061, [`hep-ph/0307044`].

[21] R. Chivukula, M. J. Dugan and M. Golden, *Analyticity, crossing symmetry and the limits of chiral perturbation theory*, *Phys.Rev.* **D47** (1993) 2930–2939, [`hep-ph/9206222`].

[22] J. Bijnens and J. Lu, *Meson-meson scattering in QCD-like theories*, *JHEP* **1103** (2011) 028, [`arXiv:1102.0172`].

[23] J. Bijnens and F. Cornet, *Two pion production in photon-photon collisions*, *Nucl.Phys.* **B296** (1988) 557.

[24] J. F. Donoghue, B. R. Holstein and Y. Lin, *The reaction $\gamma\gamma \to \pi^0\pi^0$ and chiral loops*, *Phys.Rev.* **D37** (1988) 2423.

[25] S. Bellucci, J. Gasser and M. Sainio, *Low-energy photon-photon collisions to two loop order*, *Nucl.Phys.* **B423** (1994) 80–122, [`hep-ph/9401206`].

[26] U. Bürgi, *Charged pion polarizabilities to two loops*, *Phys.Lett.* **B377** (1996) 147–152, [`hep-ph/9602421`].

[27] U. Bürgi, *Charged pion pair production and pion polarizabilities to two loops*, *Nucl.Phys.* **B479** (1996) 392–426, [`hep-ph/9602429`].

[28] J. Gasser, M. A. Ivanov and M. E. Sainio, *Low-energy photon-photon collisions to two loops revisited*, *Nucl.Phys.* **B728** (2005) 31–54, [`hep-ph/0506265`].

[29] J. Gasser, M. A. Ivanov and M. E. Sainio, *Revisiting $\gamma\gamma \to \pi^+\pi^-$ at low energies*, *Nucl.Phys.* **B745** (2006) 84–108, [`hep-ph/0602234`].

[30] F. Guerrero and J. Prades, *Kaon polarizabilities in chiral perturbation theory*, *Phys.Lett.* **B405** (1997) 341–346, [`hep-ph/9702303`].

[31] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl.Phys.* **B72** (1974) 461.