LU TP 13-25

PROCEEDINGS OF SCIENCE

Tools for calculations in color space

Malin SJODAHL*
Dept. of Astronomy and Theoretical Physics, Lund University, Sölvegatan 14A, 223 62 Lund, Sweden, E-mail: malin.sjodahl@thep.lu.se

Stefan KEPPELER
Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany, E-mail: stefan.keppeler@uni-tuebingen.de

Both the higher energy and the initial state colored partons contribute to making exact calculations in QCD color space more important at the LHC than at its predecessors. This is applicable whether the method of assessing QCD is fixed order calculation, resummation, or parton showers. In this talk we discuss tools for tackling the problem of performing exact color summed calculations. We start with theoretical tools in the form of the (standard) trace bases and the orthogonal multiplet bases (for which a general method of construction was recently presented). Following this, we focus on two new packages for performing color structure calculations: one easy to use Mathematica package, ColorMath, and one C++ package, ColorFull, which is suitable for more demanding calculations, and for interfacing with event generators.

XXI International Workshop on Deep-Inelastic Scattering and Related Subjects
22-26 April, 2013
Marseilles, France

*Speaker.

1. Introduction

The high energy available at the LHC and the initial state colored partons both contribute to increase the number of colored partons involved in collisions, and indeed, QCD is a major background for most interesting processes. An accurate treatment of QCD and the color space associated with it is therefore of increased importance. This is applicable independent of the method of accessing QCD, fixed order calculations, resummation, or – as lately for the speaker – parton showers [1].

We here discuss several tools for dealing with exact calculations in QCD color space. After a few words of introduction in Section 2, we briefly review the standard "trace bases" method in Section 3.

In Section 4, we present recent results on how to instead deal with color space using orthogonal group theory based multiplet bases [2]. Finally, in Section 5, we introduce two computer algebra codes for dealing with calculations in color space, the Mathematica package ColorMath [3], which is designed do deal with color summed calculations of moderate complexity in a user friendly way, and the C++ package ColorFull [4] which aims at dealing with trace bases in an event generator context.

2. Color space

Due to confinement we never observe the color of individual partons in QCD. This distinguishes color from other internal degrees of freedom like spin, insofar as for the latter we actually care about in which state a physical object is, rather than just under what representation it transforms.

This property also opens up for the possibility to study summed (averaged) quantities *only*. It is not hard to argue that for given external partons, the color summed (averaged) space is a finite dimensional vector space equipped with a scalar product

$$\langle A, B\rangle = \sum_{a,b,c,\ldots} (A_{a,b,c,\ldots})^* B_{a,b,c,\ldots} \ , \tag{2.1}$$

where the sum runs over all quarks, anti-quarks and gluons. For example, if

$$A = \sum_g (t^g)^a{}_b (t^g)^c{}_d = \sum_g$$  ,

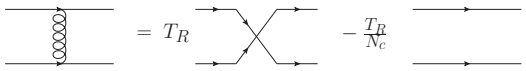
then

$$\langle A|A\rangle = \sum_{a,b,c,d,g,h} (t^g)^b{}_a (t^g)^d{}_c (t^h)^a{}_b (t^h)^c{}_d \ . \tag{2.2}$$

One way of dealing with the color space is to square the amplitudes one by one as they are encountered. Alternatively – and this is likely the preferred method for processes with many colored partons – one may want to use a basis or a spanning set.

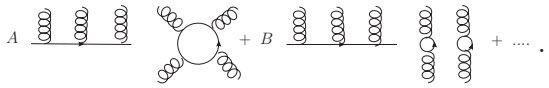
3. Trace bases

A standard way of dealing exactly with QCD color space is to note that every four-gluon vertex can be rewritten in terms of three-gluon vertices. The three gluon vertices in turn can be replaced using $if_{abc} = (1/T_R)[\mathrm{tr}(t^a t^b t^c - t^b t^a t^c)]$, where T_R is defined by $\mathrm{tr}[t^a t^b] = T_R \delta^{ab}$, and all internal gluon propagators can be removed using the Fierz or completeness relation,



. (3.1)

This can be applied to any QCD amplitude, tree level or beyond, and the result is in general a linear combination of products of traces over gluon indices and traces that have been cut open, i.e. color structures of the form



. (3.2)

For obvious reasons this type of basis is here referred to as a trace basis.

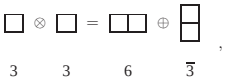
These bases [5 – 12] have several advantages. It is easy to see that a basis vector of this type, results in at most two new basis vectors (in a larger vector space) once a gluon is emitted. Furthermore, starting with any basis vector and exchanging a gluon between two partons results in a linear combination of at most four basis vectors [12]. On top of this, powerful recursion relations exist for the amplitudes multiplying the various color structures.

Trace bases, however, also come with significant drawbacks. Most importantly, they are not orthogonal, and for more than N_c gluons plus $q\overline{q}$ -pairs the "bases" are also overcomplete. Furthermore, as the number of spanning vectors in these bases grows roughly as a factorial in $N_g + N_{q\overline{q}}$ this rapidly becomes an issue [2].

4. Multiplet bases

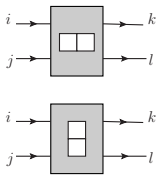
It is therefore desirable to use minimal orthogonal bases. As QCD is based on SU(3), one way to construct orthogonal bases is to use bases corresponding to irreducible representations in color space. Basis vectors where at least one subset of partons transforms under a different representation will then automatically be orthogonal. One way to enforce this is to sub-group the partons in order to make sure that parton 1 and 2 are in a manifest multiplet M^{12}, at the same time as partons 1,2, and 3 are in a manifest multiplet M^{123} etc [13, 2].

As the decomposition of color space into irreducible representations can be enumerated using Young tableau multiplication, the expectations on multiplet type bases are clear: In general (to arbitrary order in perturbation theory), we expect to encounter any state where the incoming partons are in multiplet M, and the outgoing partons in the same multiplet M. For example, for $qq \to qq$ we have the Young tableau decomposition



,

and the corresponding orthogonal basis vectors are


$$= \frac{1}{2}\left(\delta^i{}_k\delta^j{}_l + \delta^i{}_l\delta^j{}_l\right) \qquad (4.1)$$

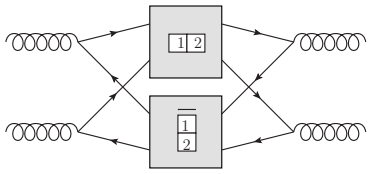
$$= \frac{1}{2}\left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_l\right) \ .$$

For processes with only quarks and anti-quarks (an incoming anti-quark can always be traded for an outgoing quark etc., so we may always treat the color space as if we had N_q incoming quarks and N_q outgoing quarks) orthogonal bases can be constructed similarly by using Hermitian versions of Young projection operators [2, 8, 14].

For processes with gluons, the translation from Young tableaux to basis vectors is far from obvious as the Young tableaux operate with quark units, rather than gluon units. We can enumerate basis vectors using Young tableau multiplication, the problem lays in the construction of the corresponding basis vectors.

Let us start with considering processes with gluons only. In the case of $gg \to gg$ the problem of constructing orthogonal bases corresponding to the multiplets in $8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} + \oplus 27 \oplus 0$ [1] has been solved a long time ago [15 – 19]. Several solutions involve splitting the gluons into $q\overline{q}$ -pairs and using that for each set of symmetries among the quarks and anti-quarks there is precisely one "new" multiplet, i.e. precisely one multiplet that could not occur for fewer gluons.

For example, the decuplet corresponds to symmetrizing the quarks and anti-symmetrizing the anti-quarks and can be obtained from the color structure

$\mathbf{T}^{10} \sim$  . (4.2)

Similarly the anti-decuplet corresponds to $\boxed{\frac{1}{2}} \otimes \overline{\boxed{1\,2}}$, the 27-plet corresponds to $\boxed{1\,2} \otimes \overline{\boxed{1\,2}}$ and the 0-plet to $\boxed{\frac{1}{2}} \otimes \overline{\boxed{\frac{1}{2}}}$. By projecting out parts corresponding to "old" multiplets, i.e., multiplets that can appear also for fewer gluons, projection operators and basis vectors can be constructed. For more than two gluons, the above picture is complicated in several ways:

(i) For each (anti-)quark Young diagram (Young tableau shape) there are many Young tableaux.

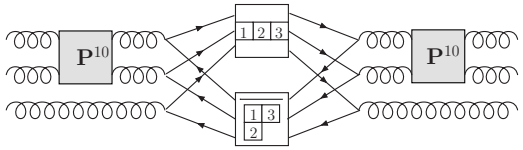
(ii) For each new overall multiplet, for example a 35-plet, there are in general several ways of obtaining it, for example there is one 35-plet in $10 \otimes 8$ and one in $27 \otimes 8$.

On top of this, when constructing the basis vectors there are issues with multiple occurrences of the same multiplet as well as with the construction of all vectors corresponding to "old" multiplets.

That this method for constructing basis tensors can be fruitful for more than two gluons therefore appears far from obvious. However, to make a long story short, we have shown that it is [2]. The proof largely depends on one key observation, namely that starting in a given multiplet,

[1] For $N_c > 3$ there is an additional multiplet which vanishes for SU(3).

corresponding to some $q\overline{q}$ -symmetries (such as 10, from $\square\square\otimes\overline{\square}$) it turns out that for each way of attaching a quark box to the quark Young-tableau ($\square\square$) and an anti-quark box to the anti-quark Young tableau ($\overline{\square}$), there is at most one new multiplet. For example, the projector $\mathbf{P}^{10,35}$ can be seen as coming from

$$\mathbf{T}^{10,35} \sim$$ 

, (4.3)

after having projected out "old" multiplets. In fact, for sufficiently large N_c, there is precisely one new multiplet for each set of $q\overline{q}$ -symmetries. What appears as a problem in *(i)* is thus in fact the resolution to the problem in *(ii)*!

In this way we can construct all projection operators for an arbitrary number of gluons, and from these we can construct orthogonal minimal bases for any number of gluons. For the three gluon case, we have explicitly constructed all 51 projectors and 265 bases vectors (for general N_c). The generalization to processes involving both quarks and gluons is straightforward, as each $q\overline{q}$ -pair either is in an octet, in which case it can be treated as a gluon or in a singlet.

5. Computer tools

In order to facilitate automatic color summed calculations the speaker has developed and cross checked two independent computer algebra packages.

5.1 ColorFull

For the purpose of treating a general QCD color structure in the trace basis, a C++ color algebra code, ColorFull [4], which creates trace bases for any number and kind of partons and to any order in α_s has been written. ColorFull also describes the effect of gluon emission and exchange, squares color amplitudes and is planned to be published separately later this year.

5.2 ColorMath

ColorMath [3] is a user friendly Mathematica package for calculations in color space of moderate complexity.

In its simplest form, the idea of ColorMath is that one should just write down the color structure, much like on paper, and then run **CSimplify** to contract color indices, for example

```
In[2]:= Amplitude = I f[g1, g2, g] t[{g}, q1, q2]
Out[2]= i t{g}q1q2 f{g1,g2,g}

In[3]:= CSimplify[Amplitude Conjugate[Amplitude /. g → h]]
Out[3]= 2 Nc (-1 + Nc^2) TR^2
```
. (5.1)

6. Conclusions

One way of dealing with exact calculations in color space is to use trace bases. This method has advantages when it comes to simplicity, recursion relations and the effect of gluon exchange and gluon emission. It is also the basis for the C++ code ColorFull [4], which is intended for enabling advanced color calculations in an event generators context. This type of basis is, however, overcomplete and not orthogonal, which becomes an issue for many partons due to the rapid growth of the number of spanning vectors.

It is therefore desirable to construct minimal orthogonal bases, and we have recently outlined a general recipe for group theory based minimal orthogonal multiplet bases for any QCD process [2]. This can potentially very significantly speed up exact calculations in the color space of SU(N_c).

We have also presented a Mathematica package ColorMath [3] for performing color summed calculations in SU(N_c).

References

[1] S. Platzer and M. Sjodahl, *Subleading N_c improved parton showers*, *JHEP* **1207** (2012) 042, [arXiv:1201.0260].

[2] S. Keppeler and M. Sjodahl, *Orthogonal multiplet bases in SU(Nc) color space*, *JHEP* **1209** (2012) 124, [arXiv:1207.0609].

[3] M. Sjodahl, *ColorMath - A package for color summed calculations in SU(Nc)*, *Eur. Phys. J.* **C73** (2013) 2310, [arXiv:1211.2099].

[4] M. Sjodahl, *ColorFull – A C++ package for color space calculations*, *work in preparation*.

[5] J. E. Paton and H.-M. Chan, *Generalized Veneziano model with isospin*, *Nucl. Phys. B* **10** (1969) 516–520.

[6] P. Dittner, *Invariant tensors in SU(3). II*, *Commun. Math. Phys.* **27** (1972) 44–52.

[7] P. Cvitanović, *Group theory for Feynman diagrams in non-Abelian gauge theories*, *Phys. Rev. D* **14** (1976) 1536–1553.

[8] P. Cvitanović, P. Lauwers, and P. Scharbach, *Gauge invariance structure of quantum chromodynamics*, *Nucl. Phys. B* **186** (1981) 165–186.

[9] M. L. Mangano, S. J. Parke, and Z. Xu, *Duality and multi-gluon scattering*, *Nucl. Phys. B* **298** (1988) 653.

[10] M. L. Mangano, *The color structure of gluon emission*, *Nucl. Phys. B* **309** (1988) 461.

[11] Z. Nagy and D. E. Soper, *Parton showers with quantum interference*, *JHEP* **09** (2007) 114, [arXiv:0706.0017].

[12] M. Sjodahl, *Color structure for soft gluon resummation – a general recipe*, *JHEP* **0909** (2009) 087, [arXiv:0906.1121].

[13] M. Sjodahl, *Color evolution of $2 \to 3$ processes*, *JHEP* **12** (2008) 083, [arXiv:0807.0555].

[14] S. Keppeler and M. Sjodahl (in preparation).

[15] A. Macfarlane, A. Sudbery, and P. Weisz, *On Gell-Mann's λ-matrices, d- and f-tensors, octets, and parametrizations of SU(3)*, *Commun. Math. Phys.* **11** (1968) 77–90.

[16] P. Butera, G. M. Cicuta, and M. Enriotti, *Group weight and vanishing graphs*, *Phys. Rev.* **D21** (1980) 972.

[17] P. Cvitanović, *Group Theory – Classics Illustrated – part I*. Nordita, Copenhagen, 1984.

[18] P. Cvitanović, *Group Theory: Birdtracks, Lie's, and Exceptional Groups*. Princeton University Press, 2008. URL: `www.birdtracks.eu`.

[19] Y. L. Dokshitzer and G. Marchesini, *Soft gluons at large angles in hadron collisions*, *JHEP* **01** (2006) 007, [`hep-ph/0509078`].