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Masses, Decay Constants and Electromagnetic Form-factors with Twisted Boundary Conditions

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Abstract: Using Chiral Perturbation Theory at one-loop we analyze the consequences of twisted boundary conditions. We point out that due to the broken Lorentz and reflection symmetry a number of unexpected terms show up in the expressions. We explicitly discuss the pseudo-scalar octet masses, axial-vector and pseudo-scalar decay constants and electromagnetic form-factors. We show how the Ward identities are satisfied using the momentum dependent masses and the non-zero vacuum-expectation-values values for the electromagnetic (vector) currents. Explicit expressions at one-loop are provided and an appendix discusses the needed one-loop twisted finite volume integrals.

Keywords: Chiral Lagrangians, Lattice QCD

Contents

1 Introduction

Lattice QCD calculations of hadronic quantities of necessity happen in a finite physical volume. In a box with periodic boundary conditions this leads to spatial momentum components $p^i = (2\pi/L)n_i$ which even for a large 4 fm lattice gives a minimum spatial momentum of about 300 MeV. In order to access smaller spatial momenta it has been suggested to use twisted boundary conditions [1–3]. This allows for more momenta to be sampled. Some early numerical tests were performed in [4].

It is well known that in a finite box Lorentz invariance is broken by the boundary conditions. In particular, the spatial part of the symmetry group becomes the cubic group in case of periodic boundary conditions. Imposing twisted boundary conditions on a field ϕ in some spatial directions i via

$$\phi(x^i + L) = e^{i\theta_i L}\phi(x^i) \tag{1.1}$$

breaks the cubic symmetry down even further. In particular, reflection symmetry, $x^i \to -x^i$ in the i-direction is broken by (1.1).

In this paper we analyze the consequences of this for a number of quantities in Chiral Perturbation Theory (ChPT). In [2] ChPT for twisted boundary conditions was developed and they showed that finite volume corrections remain exponentially suppressed for large volumes. We use their method for masses, pseudo-scalar and axial-vector decay constants, the vector two-point function and electromagnetic form-factors. We have different expressions than those given in [2], the precise relation is discussed in more detail in Sect. 8.

In general, form-factors and correlators can also have a much more general structure and this has consequences for the Ward identities. We discuss three examples of this. Another result is that vector currents get a vacuum-expectation-value (VEV), which leads to non-transverse vector two-point functions. The main goal of our paper is to study all this at one-loop order in ChPT.

Sect. 2 gives the lowest order Lagrangian in ChPT and defines a few other pieces of notation. We introduce twisted boundary conditions in Sect. 3. The more technical derivation of the needed one-loop integrals is given in App. A. As a first application we calculate the vacuum expectation value of vector currents and the two-point functions. We show how they do satisfy the Ward identities at finite volume. We find, in agreement with [5], that the two-point function is not transverse. The next two sections contain the results for the meson masses and the axial-vector and pseudo-scalar decay constants. Here again we see the occurrence of extra terms. The axial-vector matrix elements is not just described by the decay constant but there are other terms. The pseudo-scalar decay constants at infinite volume were not published earlier so we have included those expressions as well. We have explicitly checked that the Ward identities relating the axial-vector and pseudo-scalar matrix elements are satisfied. The extra terms in the axial-vector matrix element are needed to achieve this. We also add the mixed matrix elements due to the fact that the twisted boundary conditions break isospin. Numerical results are presented for all masses and the charged meson axial-vector decay constants.

Sect. 7 discusses the pion electromagnetic form-factor and related quantities. We show once more how finite volume and twisting allow for extra form-factors and have checked that with the inclusion of these the Ward identities are satisfied. We study in detail the finite volume corrections from the isospin current matrix element $\left\langle \pi^0(p')|\bar{d}\gamma_\mu u|\pi^+(p)\right\rangle$ which is used in lattice QCD to obtain information on the pion radius. We find that the corrections due to twisting can be sizable. Our main conclusions are summarized in Sect. 9.

After finishing this work we became aware of the work in [6] where a number of the issues we discuss here were raised as well. The discussion there is in two-flavour theory but also includes partial twisting. We discuss the relation with our work in Sect. 8.

2 Chiral Perturbation Theory

ChPT is the effective field theory describing low energy QCD as an expansion in masses and momenta [7–9]. Finite volume ChPT was introduced in [10]. In this paper we work in the isospin limit for quark masses, i.e. $m_u = m_d = \hat{m}$, with three quark flavours. Results for two-quark flavours are obtained by simply dropping the integrals involving kaons and eta and replacing F_0, B_0 by F, B. We perform the calculations to next-to-leading order (NLO), or $\mathcal{O}(p^4)$. The Lagrangian to NLO is

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4, \tag{2.1}$$

where $\mathcal{L}_{2n}$ is the $\mathcal{O}(p^{2n})$ Lagrangian. For the mesonic fields we use the exponential representation

$$U = e^{i\sqrt{2}M/F_0} \quad \text{with} \quad M = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}. \tag{2.2}$$

We use the external field method [8, 9] to incorporate electromagnetism, quark masses as well as couplings to other quark-antiquark operators. To do this we introduce the field χ and the covariant derivative

$$\chi = 2B_0(s + ip), \quad D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu. \tag{2.3}$$

r_μ, l_μ, s and p are the external fields. Electromagnetism is included by setting

$$l_\mu = eA_\mu Q,\ r_\mu = eA_\mu Q, \tag{2.4}$$

where e is the electron charge, A_μ is the photon field and $Q = \text{diag}(2/3, -1/3, -1/3)$. Masses are included by setting $s = \mathcal{M} = \text{diag}(\hat{m}, \hat{m}, m_s)$ where $\hat{m} = (m_u + m_d)/2$.

With these definitions the lowest order Lagrangian $\mathcal{L}_2$ is

$$\mathcal{L}_2 = \frac{F_0^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U\chi^\dagger\rangle \tag{2.5}$$

where the angular brackets denotes trace over flavour indices. The expression for $\mathcal{L}_4$ can be found in for example [8].

One problem at finite volume is the definition of asymptotic states, which we need to define the wave function renormalization and matrix elements. We assume the temporal direction to be infinite in extent and use the LSZ theorem to obtain the needed wave function renormalization by keeping the spatial momentum constant and taking the limit in $(p^0)^2$ to $p^2 = m^2$. We stick here to states with at most one incoming and outgoing particle so this is sufficient. Note that since Lorentz symmetry is broken the masses are different for the same particle with different spatial momenta.

We will not present the infinite volume expressions but only the corrections at finite volume using the quantity

$$\Delta^V X = X(V) - X(\infty), \tag{2.6}$$

where X is the object under discussion.

3 Finite volume with a twist

Periodic boundary conditions on a finite volume implies that momenta become quantized. Adding a phase factor at the boundary shifts these discrete momenta. To see this, we impose for a field in one dimension at a fixed time

$$\psi(x+L) = e^{i\theta}\psi(x), \tag{3.1}$$

where L is the length of the dimension and θ is the twist angle. Developing both sides in a Fourier series we get

$$\sum_k \hat{\psi}_k e^{ik(x+L)} = \sum_k \hat{\psi} e^{ikx} e^{i\theta} \Rightarrow k = \frac{2\pi}{L}n + \frac{\theta}{L},\ n \in \mathbb{Z}. \tag{3.2}$$

The effect on anti-particles follows from the complex conjugate of (3.1); momenta are shifted in the opposite direction. It is possible to have different twists for different flavours and also different twists in different directions.

We impose now a condition like (3.1) on each quark field q in each spatial direction i

$$q(x^i+L) = e^{i\theta_q^i} q(x^i), \tag{3.3}$$

and collect the angles θ_q^i in a three vector $\vec{\theta}_q$ and a four-vector $\theta_q = (0, \vec{\theta}_q)$. The twist-angle vector for the anti-quark is minus the one for the quarks. For a meson field of flavour structure $\bar{q}'q$ this leads to a twisted boundary condition in direction i

$$\phi_{\bar{q}'q}(x^i+L) = e^{i(\theta_q^i-\theta_{q'}^i)}\phi_{\bar{q}'q}(x^i)\,. \tag{3.4}$$

We introduce the meson twist angle vector θ_ϕ in the same way as above and we will use the conventional $\pi^\pm, \ldots$ for labeling them.. Note that flavour diagonal mesons are unaffected by twisted boundary conditions. A consequence of the boundary conditions (3.4) is that charge conjugation is broken since $\phi_{\bar{q}q'}$ and $\phi_{\bar{q}'q}$ have opposite twist. A particle with spatial momentum $\vec{p}$ corresponds to an anti-particle with momentum $-\vec{p}$.

In terms of loop integrals over the momentum of a meson M this means that we have to replace the infinite volume integral by a sum over the three spatial momenta and an integral over the remaining dimensions

$$\int \frac{d^d k_M}{(2\pi)^2} \to \int_V \frac{d^d k}{(2\pi)^d} \equiv \int \frac{d^{d-3}k}{(2\pi)^{d-3}} \frac{1}{L^3} \sum_{\substack{\vec{n}\in\mathbb{Z}^3 \\ \vec{k}=(2\pi\vec{n}+\vec{\theta}_M)/L}} . \tag{3.5}$$

It is explained in [2] how this ends up with the correct allowed momenta for each propagator in a loop. The allowed momenta $\vec{k} = (2\pi\vec{n}+\vec{\theta}_M)/L$ are not symmetric around zero and thus reflection symmetry is broken. An immediate consequence is that

$$\int_V \frac{d^d k}{(2\pi)^2} \frac{k^\mu}{k^2-m^2} \neq 0\,. \tag{3.6}$$

Note also that a meson and its anti-meson carry different momenta and it is therefore important to keep track of which one is in a loop, as well as to be careful with using charge conjugation. The twist angles also bring in another source of explicit flavour symmetry breaking.

The one-loop integrals needed are worked out using the methods of [11, 12] and presented in detail in App. A. The notation we use indicates the mass of the particle but implies also the corresponding twist vector in the expressions.

4 Vector vacuum-expectation-value and two-point function

Because of (3.6) the vacuum-expectation-value of a vector-current is non-zero and we obtain

$$\begin{aligned}
\langle \bar{u}\gamma_\mu u\rangle &= -2A^V_\mu(m^2_{\pi^+}) - 2A^V_\mu(m^2_{K^+})\\
\langle \bar{d}\gamma_\mu d\rangle &= 2A^V_\mu(m^2_{\pi^+}) - 2A^V_\mu(m^2_{K^0})\\
\langle \bar{s}\gamma_\mu s\rangle &= 2A^V_\mu(m^2_{K^+}) + 2A^V_\mu(m^2_{K^0})\\
\langle j^{em}_\mu\rangle &= -2A^V_\mu(m^2_{\pi^+}) - 2A^V_\mu(m^2_{K^+})\,.
\end{aligned} \tag{4.1}$$

We used here that $\theta_{\pi^-} = -\theta_{\pi^+}$, $\theta_{K^+} = -\theta_{K^-}$, $\theta_{K^0} = -\theta_{\overline{K}^0}$ and $\theta_{\pi^0} = \theta_\eta = 0$. This non-zero result can be understood better if we look at the alternative way of including twisting in ChPT [2]. The twisted boundary conditions can be removed by a field redefinition. However, then we get a non-zero external vector field which can be seen as a constant background field. Charged particle-anti-particle vacuum fluctuations are affected by this background field thus giving rise to a non-zero current even in the vacuum.

The two-point function of a current j^μ is defined as

$$\Pi^a_{\mu\nu}(q) \equiv i\int d^4x e^{iq\cdot x}\langle T(j^a_\mu(x)j^{a\dagger}_\nu(0))\rangle\,. \tag{4.2}$$

The current $j^{\pi^+}_\mu = \bar{d}\gamma_\mu u$ satisfies the Ward identity.

$$\partial^\mu\langle T(j^{\pi^+}_\mu(x)j^{\pi^-}_\nu(0))\rangle = \delta^{(4)}(x)\langle \bar{d}\gamma_\nu d - \bar{u}\gamma_\nu u\rangle\,. \tag{4.3}$$

We used here that $m_u = m_d$ with the usual techniques to derive Ward identities. A consequence is that with twisted boundary conditions the vector two-point function is no longer transverse. However, flavour diagonal currents like the electromagnetic one remain transverse. This does not mean that they are proportional to $q_\mu q_\nu - q^2 g_{\mu\nu}$ since Lorentz symmetry is broken. A more thorough discussion at the quark level and estimates using lattice calculations can be found in [5].

The infinite volume expressions we obtain agree with those of [13]. The finite-volume corrections for the $\bar{d}\gamma_\mu u$ and electromagnetic current are

$$\begin{aligned}
\Delta^V\Pi^{\pi^+}_{\mu\nu}(q) &= 2\widetilde{\Pi}_{\mu\nu}(m^2_{\pi^+}, m^2_{\pi^0}, q) + \widetilde{\Pi}_{\mu\nu}(m^2_{K^+}, m^2_{\overline{K}^0}, q)\,,\\
\Delta^V\Pi^{em}_{\mu\nu}(q) &= \widetilde{\Pi}_{\mu\nu}(m^2_{\pi^+}, m^2_{\pi^-}, q) + \widetilde{\Pi}_{\mu\nu}(m^2_{K^+}, m^2_{K^-}, q)\,,\\
\widetilde{\Pi}_{\mu\nu}(m^2_1, m^2_2, q) &= g_{\mu\nu}\left(4B^V_{22}(m^2_1, m^2_2, q) - A^V(m^2_1) - A^V(m^2_2)\right)\\
&\quad + q_\mu q_\nu\left(4B^V_{21}(m^2_1, m^2_2, q^2) - 4B^V_1(m^2_1, m^2_2, q^2) + B^V(m^2_1, m^2_2, q^2)\right)\\
&\quad + (q_\mu g^\alpha_\nu + q_\nu g^\alpha_\mu)(-2)B^V_{2\alpha}(m^2_1, m^2_2, q) + 4B^V_{23\mu\nu}(m^2_1, m^2_2, q)\,.
\end{aligned} \tag{4.4}$$

Using the relations (A.16) it can be checked that the consequences of (4.3), namely $q^\mu \Pi^{\pi^+}_{\mu\nu} = \langle \bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\rangle$ and $q^\mu \Pi^{em}_{\mu\nu} = 0$ are satisfied.

We do not present numerical results here, the values of the vacuum expectation value are small compared to $\langle \overline{u}u\rangle$.

5 Meson masses

We define the mass here as the pole of the full propagator at fixed spatial momentum $\vec{p}$. $\vec{p}$ should be such that it satisfies the twisted boundary condition for the field under consideration. Lorentz and charge conjugation invariance are broken by the twisted boundary conditions. This leads to a mass that depends on all components of the spatial momentum $\vec{p}$. An anti-particle with spatial momentum $-\vec{p}$ has the same mass as the corresponding particle with spatial momentum $\vec{p}$.

The analytical results for the mass correction in terms of the integrals defined in App. A are

$$\begin{aligned}
\Delta^V m^2_{\pi^\pm} &= \frac{\pm p^\mu}{F_0^2}[-2A^V_\mu(m^2_{\pi^+}) - A^V_\mu(m^2_{K^+}) + A^V_\mu(m^2_{K^0})] \\
&\quad + \frac{m^2_\pi}{F_0^2}\left(-\frac{1}{2}A^V(m^2_{\pi^0}) + \frac{1}{6}A^V(m^2_\eta)\right)\,, \\
\Delta^V m^2_{\pi^0} &= \frac{m^2_\pi}{F_0^2}\left(-A^V(m^2_{\pi^+}) + \frac{1}{2}A^V(m^2_{\pi^0}) + \frac{1}{6}A^V(m^2_\eta)\right)\,, \\
\Delta^V m^2_{K^\pm} &= \pm\frac{p^\mu}{F_0^2}[-A^V_\mu(m^2_{\pi^+}) - 2A^V_\mu(m^2_{K^+}) - A^V_\mu(m^2_{K^0})] - \frac{m^2_K}{F_0^2}\frac{1}{3}A^V(m^2_\eta)\,, \\
\Delta^V m^2_{K^0(\overline{K}^0)} &= +(-)\frac{p^\mu}{F_0^2}[A^V_\mu(m^2_{\pi^+}) - A^V_\mu(m^2_{K^+}) - 2A^V_\mu(m^2_{K^0})] - \frac{m^2_K}{F_0^2}\frac{1}{3}A^V(m^2_\eta)\,, \\
\Delta^V m^2_\eta &= -\frac{m^2_K}{F_0^2}\frac{2}{3}(A^V(m^2_{K^+}) + A^V(m^2_{K^0})) + \frac{m^2_\eta}{F_0^2}\frac{2}{3}A^V(m^2_\eta)\,, \\
&\quad + \frac{m^2_\pi}{F_0^2}\frac{1}{6}(2A^V(m^2_{\pi^+}) + A^V(m^2_{\pi^0}) - A^V(m^2_\eta))\,.
\end{aligned} \tag{5.1}$$

The notation $K^0(\overline{K}^0)$ and $+(-)$ means $+$ for K^0 and $-$ for $\overline{K}^0$. We agree with the infinite volume expressions of [9] and the known untwisted finite-volume corrections [10, 11]. The relation to the results in [2, 6] is discussed in Sect. 8.

In (5.1) the masses m^2_π, m^2_K and m^2_η can be replaced by the physical masses with or without finite volume correction, or lowest order masses. The differences are higher order. The same comment applies to F_0 in (5.1). The masses in the loop functions A^V are written as the physical masses. The notation $A^V(m^2_M)$ with M the meson includes includes the dependence on θ_M. We keep for example π^+ and π^0 as notation even if they have the same infinite volume and lowest order mass, since θ_{π^+} and θ_{π^0} are different.

Note that in the case where $\vec{p} = \vec{\theta}/L$ the different signs for A^V_μ between particle and anti-particle will be canceled by the sign difference in $\vec{p}$ originating from opposite twist angles. The same cancellation occurs for the higher momentum states if the change

$2\pi\vec{n}/L \to -2\pi\vec{n}/L$ is taken. This is consistent with the fact that charge conjugation should be defined with a change of sign in momentum, as discussed above.

The twisted boundary conditions do break isospin and thus induce π^0-η mixing. This only affects the masses at next-to-next-to-leading-order (NNLO), i.e. higher order than NLO. The derivation follows the arguments as given in Sect. 2.1 in [14] .

We now show the volume and twist angle dependence for the case with

$$m_\pi = 139.5 \text{ MeV}\,, \quad m_K = 495 \text{ MeV}\,, \quad m_\eta^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2, \quad F_\pi = 92.2 \text{ MeV}\,. \tag{5.2}$$

We have used these masses in the one-loop expressions as well as the value of F_π for F_0 in the expressions. We show results for several values of the twist angle θ with

$$\vec{\theta}_u = (\theta, 0, 0)\,, \quad \vec{\theta}_d = \vec{\theta}_s = 0\,. \tag{5.3}$$

Note that this implies that for π^+ and K^+ there is a non-zero spatial momentum $\vec{p} = \vec{\theta}_u/L$, while $\vec{p}$ vanishes for π^0, K^0 and η. As can be seen in Fig. 1, the finite volume correction has a sizable dependence on the twist-angle. The correction for the K^0 does not depend on the twist angle here, since for the choice of angles in (5.3) there is only the η-loop contribution due to $\vec{p}_{K^0} = 0$. The relative correction to the kaon and eta masses remains small while for π^+ and π^0 it can become in the few % range.

6 Decay constants

We define the meson (axial-vector) decay constant in finite volume as

$$\langle 0|A_\mu^M|M(p)\rangle = i\sqrt{2}F_M p_\mu + i\sqrt{2}F^V_{M\mu}\,, \tag{6.1}$$

where $M(p)$ is a meson and $A_\mu = \bar{q}\gamma_\mu\gamma_5(\lambda^M/\sqrt{2})q$ is the axial current. The extra term is needed since the matrix element in finite volume is no longer proportional to p_μ. The first term in (6.1) can be identified by looking at the time component of the current. The second term has non-zero components only in the spatial directions and vanishes in infinite volume.

For the flavour charged mesons, the charge in the axial current and the meson is necessarily the same. In the isospin limit the same is true for the π^0 and the η. However the twisted boundary conditions do break isospin and thus the π^0 also couples to the octet current and the η to the triplet current. At NLO this coupling comes from two effects, the mixing between the isospin triplet π and the octet η as well as the direct transition to the other current. A derivation can be found in Sect. 2.2 of [14].

We also consider decay through a pseudo-scalar current. We define this decay constant as

$$\langle 0|P^M|M(p)\rangle = \frac{G_M}{\sqrt{2}} \tag{6.2}$$

where $P = \bar{q}i\gamma_5(\lambda^M/\sqrt{2})q$ is the pseudo-scalar current corresponding to the meson M. A similar comment to above about π^0 and η applies.

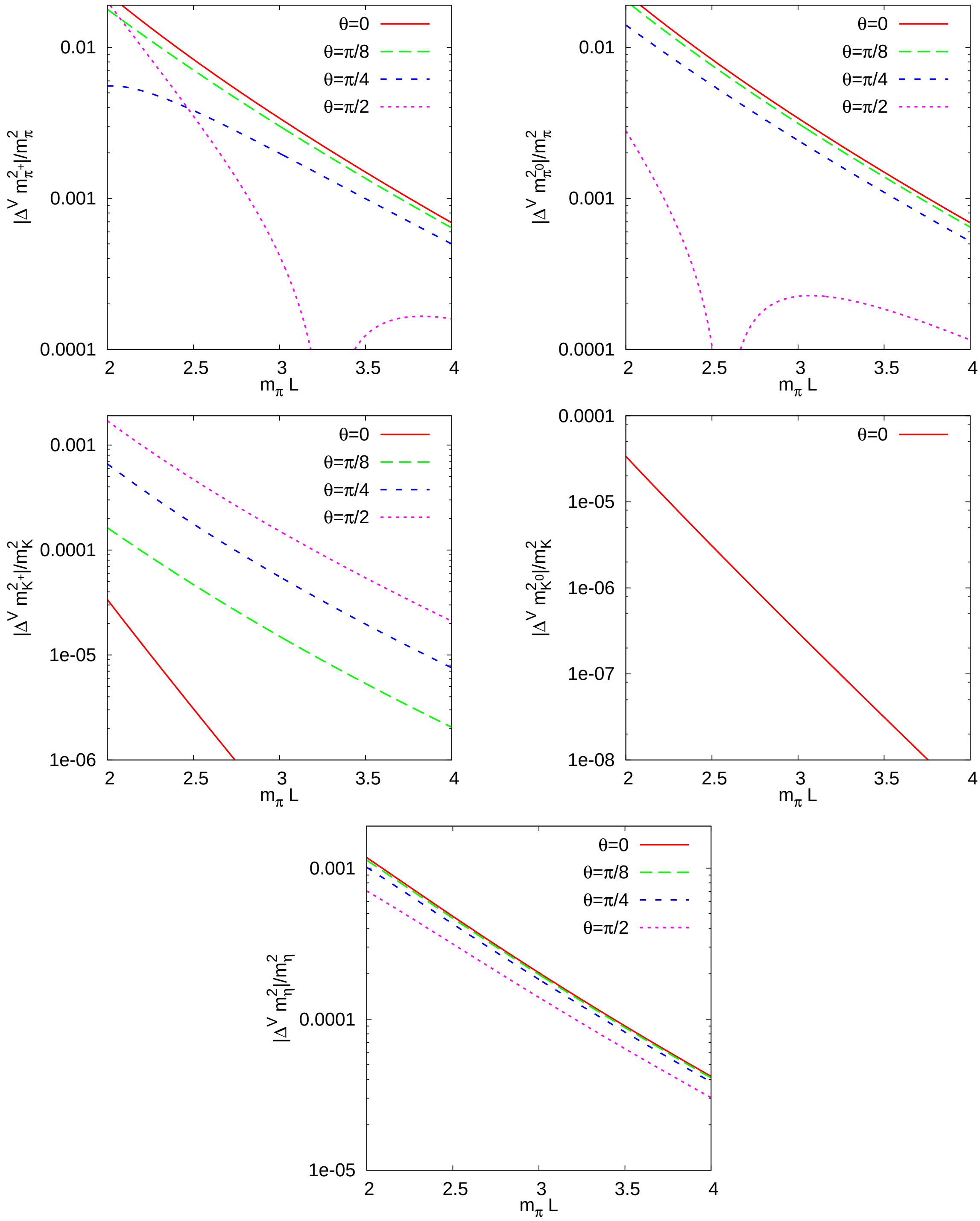

Figure 1. Absolute value of the relative finite volume correction to the masses of the light pseudoscalar mesons as a function of the box size for various twist angles. The twist is for all cases on the up quark. The input values are specified in (5.2) and (5.3). The dip in the top two plots is where the correction goes through zero

These two matrix elements satisfy the Ward identity

$$\partial^\mu \left\langle 0|A_\mu^M|M(p)\right\rangle = (m_q + m_{q'}) \left\langle 0|P^M|M(p)\right\rangle \,, \tag{6.3}$$

valid for flavour charged mesons of composition $\bar{q}q'$. This leads to

$$p^2 F_M + p^\mu F^V_{M\mu} = \frac{1}{2}(m_q + m_{q'}) G_M \,. \tag{6.4}$$

We have checked that our expressions for the charged mesons agree with this. An important part in this agreement is the use of the correct momentum-dependent mass of the meson. For the neutral mesons a somewhat more complicated relation is needed since they are sums of terms with different quark masses.

The analytical results for the finite volume effects on the axial-vector decay constants are given below in terms of the integrals defined in App. A. For the π^0 and η we listed the matrix-elements with A^3_μ and A^8_μ separately, indicating which decay is which with an extra subscript. The isospin breaking decay vanishes if the up and down quarks have the same twist angles.

Again we agree with the infinite volume results of [9]. The finite volume corrections for the axial current decay constants for the flavour charged mesons are

$$\begin{aligned}
\Delta^V F_{\pi^\pm} &= \frac{1}{F_0}\left(\frac{1}{2}A^V(m^2_{\pi^+}) + \frac{1}{2}A^V(m^2_{\pi^0}) + \frac{1}{4}A^V(m^2_{K^+}) + \frac{1}{4}A^V(m^2_{K^0})\right)\,,\\
F^V_{\pi^\pm\mu} &= \pm\frac{1}{F_0}\left[2A^V_\mu(m^2_{\pi^+}) + A^V_\mu(m^2_{K^+}) - A^V_\mu(m^2_{K^0})\right]\,,\\
\Delta^V F_{K^\pm} &= \frac{1}{F_0}\left(\frac{1}{4}A^V(m^2_{\pi^+}) + \frac{1}{8}A^V(m^2_{\pi^0}) + \frac{1}{2}A^V(m^2_{K^+}) + \frac{1}{4}A^V(m^2_{K^0}) + \frac{3}{8}A^V(m^2_\eta)\right)\,,\\
F^V_{K^\pm\mu} &= \pm\frac{1}{F_0}\left[A^V_\mu(m^2_{\pi^+}) + 2A^V_\mu(m^2_{K^+}) + A^V_\mu(m^2_{K^0})\right]\,,\\
\Delta^V F_{K^0(\bar{K}^0)} &= \frac{1}{F_0}\left(\frac{1}{4}A^V(m^2_{\pi^+}) + \frac{1}{8}A^V(m^2_{\pi^0}) + \frac{1}{4}A^V(m^2_{K^+}) + \frac{1}{2}A^V(m^2_{K^0}) + \frac{3}{8}A^V(m^2_\eta)\right)\,,\\
F^V_{K^0(\overline{K}^0)\mu} &= +(-)\frac{1}{F_0}\left[-A^V_\mu(m^2_{\pi^+}) + A^V_\mu(m^2_{K^+}) + 2A^V_\mu(m^2_{K^0})\right]\,.
\end{aligned} \tag{6.5}$$

They agree with the untwisted finite volume results of [11]. The relation to the results given in [2] is discussed in Sect. 8. The flavour neutral expressions include the effects of mixing.

$$\begin{aligned}
F^V_{\pi^0 3\mu} &= F^V_{\pi^0 8\mu} = F^V_{\eta 3\mu} = F^V_{\eta 8\mu} = 0\,,\\
\Delta^V F_{\pi^0 3} &= \frac{1}{F_0}(A^V(m^2_{\pi^+}) + \frac{1}{4}A^V(m^2_{K^+}) + \frac{1}{4}A^V(m^2_{K^0}))\,,\\
\Delta^V F_{\pi^0 8} &= \frac{3m^2_\eta - m^2_\pi}{2\sqrt{3}F_0(m^2_\eta - m^2_\pi)}(A^V(m^2_{K^+}) - A^V(m^2_{K^0}))\,,\\
\Delta^V F_{\eta 8} &= \frac{3}{4F_0}(A^V(m^2_{K^+}) + A^V(m^2_{K^0}))\,,\\
\Delta^V F_{\eta 3} &= \frac{-m^2_\pi}{\sqrt{3}F_0(m^2_\eta - m^2_\pi)}(A^V(m^2_{K^+}) - A^V(m^2_{K^0})).
\end{aligned} \tag{6.6}$$

to simplify the expressions.

The masses and F_0 in these expressions can be chosen in different ways as discussed earlier for the masses.

The lowest order value for the pseudo-scalar decay constants is $G_0 = 2F_0B_0$. We are not aware of published results for the NLO corrections at infinite volume, we thus quote those for completeness and add a superscript (4) to indicate the NLO infinite volume correction. Note that isospin is valid at infinite volume such that the mixed ones vanish and there is only an expression for the π, K and η_8 case.

$$\begin{aligned}
G_\pi^{(4)} &= \frac{G_0}{F_0^2}\left(4K_{46} + 4m_\pi^2(4L_8^r - L_5^r) + \frac{1}{2}\overline{A}(m_\pi^2) + \frac{1}{2}\overline{A}(m_K^2) + \frac{1}{6}\overline{A}(m_\eta^2)\right)\,,\\
G_K^{(4)} &= \frac{G_0}{F_0^2}\left(4K_{46} + 4m_K^2(4L_8^r - L_5^r) + \frac{3}{8}\overline{A}(m_\pi^2) + \frac{3}{4}\overline{A}(m_K^2) + \frac{1}{24}\overline{A}(m_\eta^2)\right)\,,\\
G_{\eta 8}^{(4)} &= \frac{G_0}{F_0^2}\left(4K_{46} + 4m_\eta^2(4L_8^r - L_5^r) + \frac{1}{2}\overline{A}(m_\pi^2) + \frac{1}{6}\overline{A}(m_K^2) + \frac{1}{2}\overline{A}(m_\eta^2)\right)\,,\\
K_{46} &= (2m_K^2 + m_\pi^2)(4L_6^r - L_4^r)\,.
\end{aligned}\tag{6.7}$$

The integral is

$$\overline{A}(m^2) = -\frac{m^2}{16\pi^2}\log\frac{m^2}{\mu^2}\,.\tag{6.8}$$

The finite volume effects for the pseudo-scalar decay constants for the flavour charged mesons are

$$\begin{aligned}
\Delta^V G^V_{\pi^\pm} &= \frac{G_0}{F_0^2}\left(\frac{1}{2}A^V(m_{\pi^+}^2) + \frac{1}{4}A^V(m_{K^+}^2) + \frac{1}{4}A^V(m_{K^0}^2) + \frac{1}{6}A^V(m_\eta^2)\right)\,,\\
\Delta^V G_{K^\pm} &= \frac{G_0}{F_0^2}\left(\frac{1}{4}A^V(m_{\pi^+}^2) + \frac{1}{8}A^V(m_{\pi^0}^2) + \frac{1}{2}A^V(m_{K^+}^2) + \frac{1}{4}A^V(m_{K^0}^2) + \frac{1}{24}A^V(m_\eta^2)\right)\,,\\
\Delta^V G_{K^0(\overline{K}^0)} &= \frac{G_0}{F_0^2}\left(\frac{1}{4}A^V(m_{\pi^+}^2) + \frac{1}{8}A^V(m_{\pi^0}^2) + \frac{1}{4}A^V(m_{K^+}^2) + \frac{1}{2}A^V(m_{K^0}^2) + \frac{1}{24}A^V(m_\eta^2)\right)\,.
\end{aligned}\tag{6.9}$$

For the flavour neutral cases we need to take into account mixing and obtain

$$\begin{aligned}
\Delta^V G_{\pi^0 3} &= \frac{G_0}{F_0^2}\left(\frac{1}{2}A^V(m_{\pi^0}^2) + \frac{1}{4}A^V(m_{K^+}^2) + \frac{1}{4}A^V(m_{K^0}^2) + \frac{1}{6}A^V(m_\eta^2)\right)\,,\\
\Delta^V G_{\pi^0 8} &= \frac{G_0}{F_0^2}\frac{m_\eta^2 + m_\pi^2}{2\sqrt{3}(m_\eta^2 - m_\pi^2)}\left(A^V(m_{K^+}^2) - A^V(m_{K^0}^2)\right)\,,\\
\Delta^V G_{\eta 8} &= \frac{G_0}{F_0^2}\left(\frac{1}{3}A^V(m_{\pi^+}^2) + \frac{1}{6}A^V(m_{\pi^0}^2) + \frac{1}{12}A^V(m_{K^+}^2) + \frac{1}{12}A^V(m_{K^0}^2) + \frac{1}{2}A^V(m_\eta^2)\right)\,,\\
\Delta^V G_{\eta 3} &= \frac{G_0}{F_0^2}\frac{-m_\eta^2}{\sqrt{3}(m_\eta^2 - m_\pi^2)}\left(A^V(m_{K^+}^2) - A^V(m_{K^0}^2)\right)\,.
\end{aligned}\tag{6.10}$$

At this order $G_{\pi^0 8}$ and $G_{\eta 3}$ only arise from π^0-η mixing.

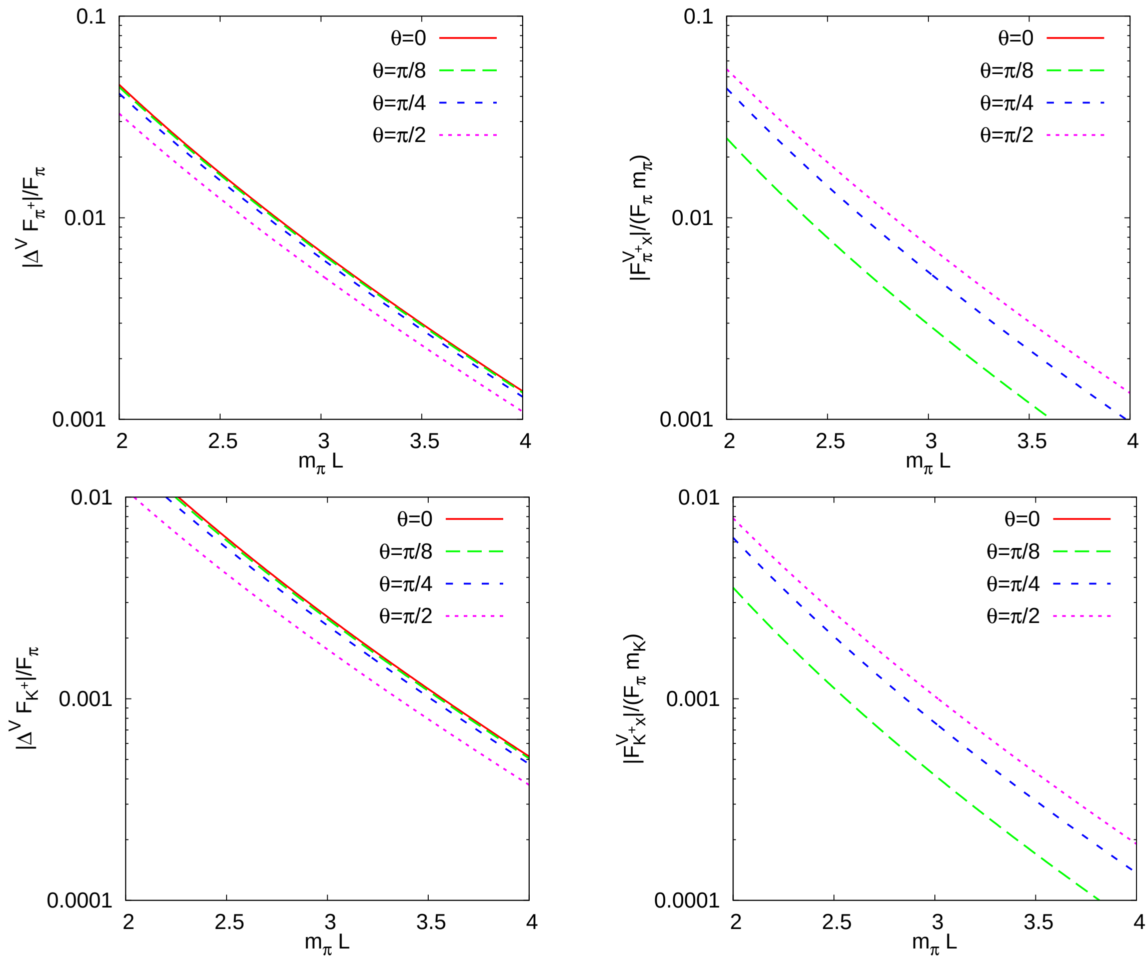

Figure 2. Relative finite volume correction for the two terms in the decay constant matrix element (6.1). On the left hand side we have plotted $\Delta^V F_M/F_\pi$ and on the right hand side $F^V_{Mx}/(F_\pi m_M)$, i.e. the x-component compared to the size of the zero-component. For the input chosen the x-component is the only non-zero one for the second term in (6.1). The top row is $M = \pi^+$ and the bottom row for $M = K^+$. Input values as in (5.2) and (5.3).

We present now some numerics for the same inputs as used for the masses given in (5.2) and (5.3).

In Fig. 2 we show the size of the finite volume corrections to the charged meson decay constants with both terms in (6.1) shown separately. We use the same input parameters as for the masses of (5.2) and (5.3). The first term in (6.1) is shown in the left plots normalized to F_π for the charged pion and kaon. The right plots shows the x-component of the second term in (6.1), which is the only non-zero component for our choice of input. It vanishes identically for $\theta = 0$. We have normalized here to the value of $F_\pi m_K$ which is roughly the value of the t-component in infinite volume. Note that the finite volume corrections can be sizable and the second term is not always negligible.

7 Electromagnetic form-factor

The electromagnetic form-factor in infinite volume is defined as

$$\langle p'|j_\mu^{em}|p\rangle = F(q^2)(p+p')_\mu \tag{7.1}$$

where $q = p - p'$ and j^μ is the electromagnetic current for the light quark flavours

$$j_\mu^{em} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}(\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu s). \tag{7.2}$$

The electromagnetic form-factor in twisted lattice QCD is not the same as in infinite volume or finite volume with periodic conditions. Instead it has the more general form

$$\begin{aligned}\langle M'(p')|j_\mu^I|M(p)\rangle &= f_{IMM'\mu} \\ &= f_{IMM'+}(p_\mu + p'_\mu) + f_{IMM'-}q_\mu + h_{IMM'\mu}\,.\end{aligned} \tag{7.3}$$

In addition to the electromagnetic current we will use

$$j_\mu^q = \bar{q}\gamma_\mu q, \qquad\qquad j_\mu^{\pi^+} = \bar{d}\gamma_\mu u\,. \tag{7.4}$$

We will also suppress the M' in the subscripts when initial and final meson are the same and sometimes the IMM'. In the infinite volume limit the functions f_- and h must go to zero and f_+ must go to $F(q^2)$ so that Eq. (7.1) is recovered. We only work with currents where the quark and anti-quark have the same mass. The result in infinite volume can be found in [15]. Results at finite volume with periodic boundary conditions are in [16, 17].

The main reason for using twisted boundary conditions is to extract physical quantities for small momenta. In the case of the electromagnetic form-factor the twist does not help when applied to correlators such as

$$\langle \pi^+(p')|j_\mu^q|\pi^+(p)\rangle \tag{7.5}$$

since the same twist is applied to the incoming and outgoing particles we get $p_i - p'_i = 2\pi n_i/L$. However, as was pointed out in [4], it is possible to extract information using isospin symmetry. To analyze this more carefully requires calculations in partially quenched ChPT and this will be the topic of forthcoming work. Here we are satisfied with noting that in the isospin limit with $m_u = m_d$ and $\theta_u = \theta_d$ we have the relation (in our sign conventions)

$$\langle \pi^+(p')|\bar{u}\gamma_\mu u|\pi^+(p)\rangle = -\langle \pi^+(p')|\bar{d}\gamma_\mu d|\pi^+(p)\rangle = -\frac{1}{\sqrt{2}}\langle \pi^0(p')|\bar{d}\gamma_\mu u|\pi^+(p)\rangle\,. \tag{7.6}$$

The relation (7.6) can in principle be used to evaluate the main part, excluding $\bar{s}\gamma_\mu s$, of the electromagnetic form-factor of the pion for arbitrary momenta. The currents $\bar{d}\gamma_\mu u$ is referred to as $\bar{d}u$ in the equations below. In practice π^0 gives rise to difficulties on the lattice, and the twisted boundary conditions explicitly break isospin. The corrections due to the latter are one of the goals of this work.

7.1 Analytic expressions

The split in f_+, f_- and h in (7.3) is not unique. The functions can depend on all components of the momenta and twist-vectors. However, we stick to the splitting among f_+, f_- and h which naturally emerges from the one-loop calculation. The integrals appearing are defined in App. A.

The results for f_+^V are most easily given in terms of the finite volume generalization of the function $\mathcal{H}$ in [15, 18].

$$H^V(m_1^2, m_2^2, q) = \frac{1}{4}A^V(m_1^2) + \frac{1}{4}A^V(m_2^2) - B_{22}^V(m_1^2, m_2^2, q) \tag{7.7}$$

The effects of π^0-η mixing appear earliest at NNLO for the form-factors listed here. The form-factors f_+ we consider are:

$$\begin{aligned}
\Delta^V f_{em\pi^\pm +} &= \frac{\pm 1}{F_0^2}\left(2H^V(m_{\pi^+}^2, m_{\pi^-}^2, q) + H^V(m_{K^+}^2, m_{K^-}^2, q)\right)\,,\\
\Delta^V f_{emK^\pm +} &= \frac{\pm 1}{F_0^2}\left(H^V(m_{\pi^+}^2, m_{\pi^-}^2, q) + 2H^V(m_{K^+}^2, m_{K^-}^2, q)\right)\,,\\
\Delta^V f_{emK^0(\overline{K}^0)+} &= \frac{\pm 1}{F_0^2}\left(-H^V(m_{\pi^+}^2, m_{\pi^-}^2, q) + H^V(m_{K^+}^2, m_{K^-}^2, q)\right)\,,\\
\Delta^V f_{em\pi^0 +} &= 0\,,\\
\Delta^V f_{\bar{d}u\pi^+\pi^0 +} &= \frac{-\sqrt{2}}{F_0^2}\Big(2H^V(m_{\pi^+}^2, m_{\pi^0}^2, q) + H^V(m_{K^+}^2, m_{\overline{K}^0}^2, q)\Big)\,.
\end{aligned} \tag{7.8}$$

The f_- form-factors for the same cases are:

$$\begin{aligned}
\Delta^V f_{em\pi^+(\pi^-)-} &= \frac{p'^\nu(-p^\nu)}{F_0^2}\left(2B_{2\nu}^V(m_{\pi^+}^2, m_{\pi^-}^2, q) + B_{2\nu}^V(m_{K^+}^2, m_{K^-}^2, q)\right)\,,\\
\Delta^V f_{emK^+(K^-)-} &= \frac{p'^\nu(-p^\nu)}{F_0^2}\left(B_{2\nu}^V(m_{\pi^+}^2, m_{\pi^-}^2, q) + 2B_{2\nu}^V(m_{K^+}^2, m_{K^-}^2, q)\right)\,,\\
\Delta^V f_{emK^0(\overline{K}^0)-} &= \frac{1}{F_0^2}\left(-(p^\nu(-p'^\nu))B_{2\nu}^V(m_{\pi^+}^2, m_{\pi^-}^2, q) + p'^\nu(-p^\nu)B_{2\nu}^V(m_{K^+}^2, m_{K^-}^2, q)\right)\,,\\
\Delta^V f_{em\pi^0 -} &= \frac{1}{F_0^2}\Bigg(m_\pi^2\left(B^V(m_{\pi^+}^2, m_{\pi^-}^2, q) - 2B_1^V(m_{\pi^+}^2, m_{\pi^-}^2, q)\right)\\
&\quad - q^\nu\left(2B_{2\nu}^V(m_{\pi^+}^2, m_{\pi^-}^2, q) + \frac{1}{2}B_{2\nu}^V(m_{K^+}^2, m_{K^-}^2, q)\right)\Bigg)\,,\\
\Delta^V f_{\bar{d}u\pi^+\pi^0 -} &= \frac{\sqrt{2}}{F_0^2}\Bigg(m_\pi^2\left(B^V(m_{\pi^+}^2, m_{\pi^0}^2, q) - 2B_1^V(m_{\pi^+}^2, m_{\pi^0}^2, q)\right)\\
&\quad - \left(2p^\nu B_{2\nu}^V(m_{\pi^+}^2, m_{\pi^-}^2, q) + \frac{1}{2}(p+p')^\nu B_{2\nu}^V(m_{K^+}^2, m_{\overline{K}^0}^2, q)\right)\Bigg)\,,
\end{aligned} \tag{7.9}$$

Finally, the h_μ at finite volume are

$$
\begin{aligned}
\Delta^V h_{em\pi^\pm\mu} &= \frac{1}{F_0^2}\Bigg(2A^V_\mu(m^2_{\pi^+}) + A^V_\mu(m^2_{K^+}) - A^V_\mu(m^2_{K^0}) \\
&\quad + q^2 B^V_{2\mu}(m^2_{\pi^+}, m^2_{\pi^-}, q) + \frac{q^2}{2} B^V_{2\mu}(m^2_{K^+}, m^2_{K^-}, q) \\
&\quad \mp (p+p')^\nu \left(2B^V_{23\mu\nu}(m^2_{\pi^+}, m^2_{\pi^-}, q) + B^V_{23\mu\nu}(m^2_{K^+}, m^2_{K^-}, q)\right)\Bigg)\,, \\
\Delta^V h_{emK^\pm\mu} &= \frac{1}{F_0^2}\Bigg(A^V_\mu(m^2_{\pi^+}) + 2A^V_\mu(m^2_{K^+}) + A^V_\mu(m^2_{K^0}) \\
&\quad + \frac{q^2}{2} B^V_{2\mu}(m^2_{\pi^+}, m^2_{\pi^-}, q) + q^2 B^V_{2\mu}(m^2_{K^+}, m^2_{K^-}, q) \\
&\quad \mp (p+p')^\nu \left(B^V_{23\mu\nu}(m^2_{\pi^+}, m^2_{\pi^-}, q) + 2B^V_{23\mu\nu}(m^2_{K^+}, m^2_{K^-}, q)\right)\Bigg)\,, \\
\Delta^V h_{emK^0(\overline{K}^0)\mu} &= \frac{1}{F_0^2}\Bigg(\frac{q^2}{2} B^V_{2\mu}(m^2_{\pi^+}, m^2_{\pi^-}, q) + \frac{q^2}{2} B^V_{2\mu}(m^2_{K^+}, m^2_{K^-}, q) \\
&\quad + (-)(p+p')^\nu \left(B^V_{23\mu\nu}(m^2_{\pi^+}, m^2_{\pi^-}, q) - B^V_{23\mu\nu}(m^2_{K^+}, m^2_{K^-}, q)\right)\Bigg)\,, \\
\Delta^V h_{em\pi^0\mu} &= \frac{1}{F_0^2}\Bigg(2(q^2 - m^2_\pi) B^V_{2\mu}(m^2_{\pi^+}, m^2_{\pi^-}, q) + \frac{q^2}{2} B^V_{2\mu}(m^2_{K^+}, m^2_{K^-}, q)\Bigg)\,, \\
\Delta^V h_{\bar{d}u\pi^+\pi^0\mu} &= \frac{\sqrt{2}}{F_0^2}\Bigg(-A^V_\mu(m^2_{\pi^+}) - \frac{1}{2}A^V_\mu(m^2_{K^+}) + \frac{1}{2}A^V_\mu(m^2_{K^0}) \\
&\quad + (q^2 - 2m^2_\pi) B^V_{2\mu}(m^2_{\pi^+}, m^2_{\pi^0}, q)\Bigg)\,, \\
&\quad + (p+p')^\nu \left(2B^V_{23\mu\nu}(m^2_{\pi^+}, m^2_{\pi^0}, q) + B^V_{23\mu\nu}(m^2_{K^+}, m^2_{\overline{K}^0}, q)\right)\Bigg)\,.
\end{aligned} \tag{7.10}
$$

We used in these formulas that the π^0 and η have no twist and that particle and anti-particle have opposite twists. Both f_- and h vanish in infinite volume.

7.2 Ward identities

All the form-factors we discuss have the same mass for the quark and anti-quark in the vector current. As a consequence they obey, even at finite volume, the Ward identity

$$
q^\mu f_{IMM'\mu} = (p^2 - p'^2) f_{IMM'+} + q^2 f_{IMM'-} + q^\mu h_{IMM'\mu} = 0\,. \tag{7.11}
$$

We have used this as a check on our results. This standard check requires a bit of caution when using twisted boundary conditions. The issue is that masses are momentum dependent when twist is applied, see Sect. 5. When performing a one loop calculation part of the mass correction is different for ingoing and outgoing meson, this means that $p^2 - p'^2 \neq 0$ even when the incoming and outgoing particle are the same. Comparing equations for the mass corrections, we see that these cancel the parts coming from A^V_μ in $h_{IMM'\mu}$. The remainder cancels between $q^2 f_{IMM'-}$ and $q^\mu h_{IMM'\mu}$ when using the identities in App. A.4.

7.3 Numerical results

Let us first remind here why twisting is useful for form-factors with the example of the pion form-factor and a lattice size of $m_\pi L = 2$. The smallest spatial momentum that can be produced is $2\pi/L = \pi m_\pi$ and the corresponding q^2 is $q^2_{min} = -0.089\ \text{GeV}^2 = -(0.3\ \text{GeV})^2$. Twisting allows for q^2 continuously varying from zero.

In this section we concentrate on the quantity

$$f_\mu = \left(1 + f_+^\infty + \Delta^V f_+\right)(p+p')_\mu + \Delta^V f_- q_\mu + \Delta^V h_\mu = -\frac{1}{\sqrt{2}} f_{\bar{d}u\pi^+\pi^0\mu}\,. \tag{7.12}$$

This is the form-factor corresponding to the right hand side of (7.6) normalized to 1 at $q^2 = 0$ in infinite volume. The finite volume parts are what is needed to obtain the pion electromagnetic form-factor, neglecting the s-quark contribution, at infinite volume. We have separated the lowest order value of 1, the infinite volume and finite volume correction to f_+ as well as the f_- and h_μ parts defined earlier.

Again we look at the case with $\vec{\theta}_u = (\theta, 0, 0)$. This means that the incoming π^+ four-momentum p, the outgoing π^0 momentum p' and q^2 are

$$\begin{aligned}
p &= \left(\sqrt{m_{\pi^+}^{V2} + (\theta/L)^2}, \theta/L, 0, 0\right)\,,\\
p' &= \left(m_{\pi^0}^{V2}, 0, 0, 0\right)\,,\\
q^2 &= m_{\pi^+}^{V2} + m_{\pi^0}^{V2} - 2m_{\pi^0}^{V}\sqrt{m_{\pi^+}^{V2} + (\theta/L)^2}\,.
\end{aligned} \tag{7.13}$$

Note that the masses at finite volume that come in here, not the infinite volume ones. We have indicated this with the superscript V in the masses. To plot the corrections we use $m_M^{V2} = m_M^2 + \Delta^V m_M^2$ in the numerics with $\Delta^V m_M^2$ given in (5.1). The size of this effect is shown in the left plot of Fig. 3. We plot the value of q^2 at finite and infinite volume and the deviation of the ratio from 1 as a function of θ/L. The endpoint of the curve is for $\theta = 2\pi$. The right plot in Fig. 3 shows the effect on the form-factor of this change in q^2. We plotted there the one-loop contribution at infinite volume to the pion electromagnetic form-factor, $f_+^\infty(q^2)$, as a function of the two different q^2 discussed here. The extra input values used are $L_9^r = 0$ and $\mu = 0.77$ GeV. The total effect of this correction is rather small.

In the remainder we will use the q^2 as calculated with the finite volume masses. In Fig. 4 we plot the different parts of the form-factor as defined in (7.12). Plotted are the infinite volume one-loop part of f_+^∞, the finite volume corrections $\Delta^V f_+$, $\Delta^V f_-$ and the two non-zero components of $\Delta^V h^\mu$. As one can see, the finite volume corrections are not small and the parts due to the extra form-factors can definitely not be neglected. The units are GeV for the two components of $\Delta^V h^\mu$.

The more relevant quantities for comparison are the components with $\mu = 0$ and $\mu = 1$. We have plotted the form-factor as defined with upper index μ. The left plot in Fig. 5 shows $\mu = 0$ and the right plot $\mu = 1$. Units are in GeV. The finite volume correction is of a size similar to the infinite volume pure one-loop contribution and the correction due to the extra terms at finite volume and twist are not negligible.

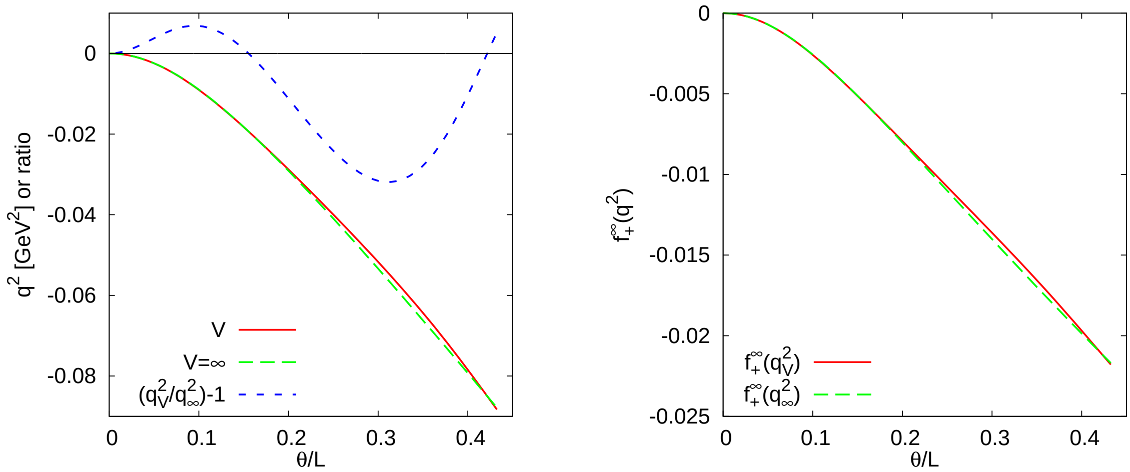

Figure 3. Left: The dependence of q^2 at a fixed $\vec{q} = (\theta/L, 0, 0)$ for the finite volume with $m_\pi L = 2$ and infinite volume as well as the difference ratio from one. The curves end at $\theta = 2\pi$. Right: The effect of this change in q^2 on the infinite volume corrections of $f_+^V(q^2)$ with $L_9^r = 0$.

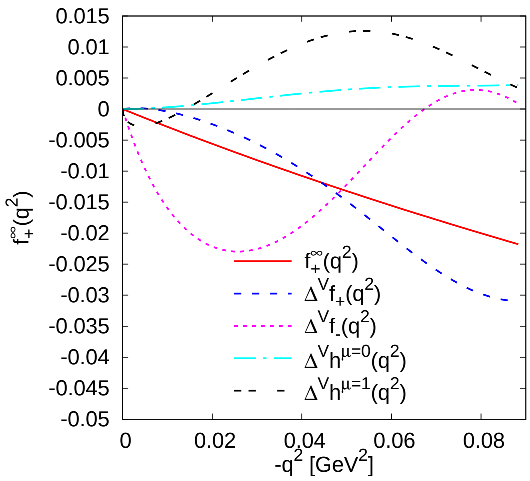

Figure 4. The various parts of the form-factor defined in (7.12). See text for a more detailed explanation.

8 Comparison with earlier work

The one and two-point Green functions of vector currents are discussed in Sect. 4. These issues were discussed in a more lattice oriented way in [5]. Here we have provided the ChPT expressions for them.

For the masses the comparison with earlier work is more subtle. In this work, we have consistently used the formulation with non-zero twist angle and no induced background field. This implies that the allowed meson momenta are of the form $\vec{p}_{BR} = (2\pi\vec{n}+\vec{\theta})/L$, with $\vec{n}$ a three-vector with integer components and $\vec{\theta}$ the twist vector for the field corresponding

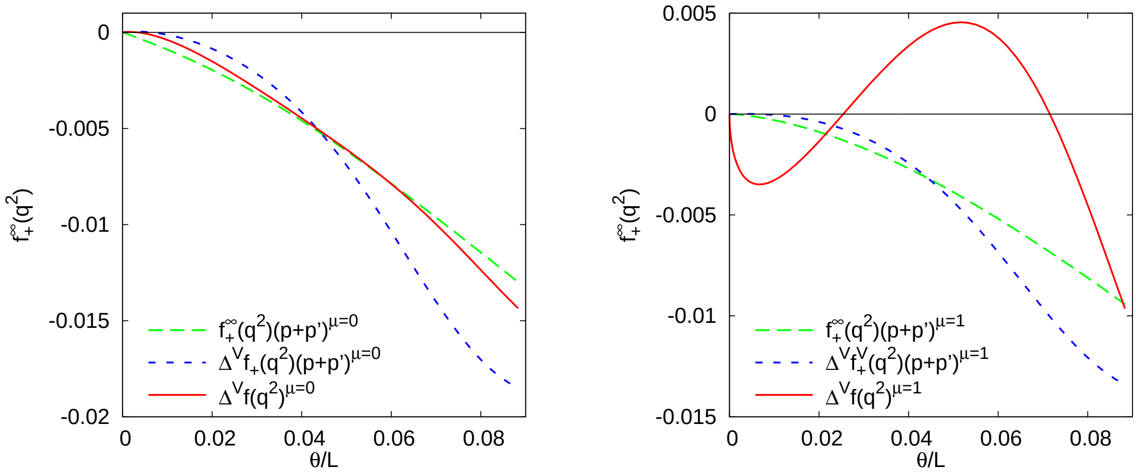

Figure 5. Left: $\mu = 0$ Right: $\mu = 1$. Plotted are those due to the one-loop infinite volume correction, $f_+^\infty(q^2)$, the finite volume correction to f_+, $\Delta^V f_+$, and the full finite volume correction, $\Delta^V f^\mu = \Delta^V f_+(p+p')^\mu + \Delta^V f_- q^\mu + \Delta^V h^\mu$.

to the meson. As mentioned in Sect. 2 we define asymptotic states as those where there is at fixed $\vec{p}$ a pole at a value, E_0, of the energy. The LSZ theorem can then be used for these single particle states to obtain matrix elements by taking the limit $E \to E_0$ allowing for the usual method with wave function renormalization and possibly mixing of external states to take into account external leg corrections. Our definition of the mass used is

$$m_{BR}^2 = E_0^2 - \vec{p}_{BR}^2 \,. \tag{8.1}$$

The mass can depend on all components of $\vec{p}$ since there is no rotation invariance and even cubic invariance[1] is no longer present. We have used the expression "momentum-dependent mass" in the text to indicate this dependence. The relation between E and $\vec{p}$ for states is called dispersion relation in some other references, see e.g. [6].

[6] discussed the pion mass, both neutral and charged, in two-flavour ChPT on the lattice. They work in the version of ChPT where the fields satisfy periodic boundary conditions but there are background fields $\vec{B} = \vec{\theta}/L$. They have periodic momenta $\vec{p}_p = (2\pi\vec{n})/L$ and define kinematical momenta $\vec{p}_k = \vec{p}_p + \vec{B}$ which coincide with our definition $\vec{p}_{BR}$. However when they define the mass they write the result in the form[2]

$$m_{JT}^2 = E_0^2 - \left(\vec{p}_p + \vec{B} + \vec{K}\right)^2 = E_0^2 - \left(\vec{p}_p + \vec{B}\right)^2 - 2\left(\vec{p}_p + \vec{B}\right) \cdot \vec{K} + \text{NNLO}. \tag{8.2}$$

$\vec{K}$ is NLO, thus we can neglect $\vec{K}^2$ as indicated. Comparing (8.1) and (8.2), the parts containing the integral A_μ^V in (5.1) can be written in the form $-2(\vec{p}_p + \vec{B}) \cdot \vec{K}$. [6] expresses this that the meson field (spatial) momentum is renormalized. When comparing the expressions, keep in mind we have also a twist on the sea quarks while [6] does not.

[1] We assume here that the t direction is infinite.

[2] We have changed their notation and conventions to make the comparison more clearly.

Comparing with the results of [2] is not obvious. The masses are not defined there. The discussion of loop diagrams in the main text indicates that they used momenta of the form $\vec{p}_p + \vec{B}$ everywhere and if one assumes that their mass is defined as

$$m_{SV1}^2 = E_0^2 - \left(\vec{p}_p + \vec{B}\right)^2 , \tag{8.3}$$

then they missed the terms with A_μ^V. If instead a definition of the mass similar to (8.2) is assumed we are in agreement. The expression corresponding to $\vec{K}$ is not present in [2].

For the decay constants a similar issue arises. They are not fully defined in [2]. If one defines the decay constant from the time component of the axial current then only the parts $\Delta^V F_M$ are relevant and we are in full agreement, if, as is natural, the neutral pion and eta decay constants in [2] are defined with the isospin and octet axial currents. It turns out that to NLO the decay constants can be defined with a shift in momentum $\vec{K}'$ similar to what was done for the masses, i.e. the full matrix element has the form

$$\left\langle 0|A_\mu^M|M(p)\right\rangle = i\sqrt{2}F_M\left(p_\mu + K'_\mu\right) + \text{NNLO}\,. \tag{8.4}$$

However, the needed shift vector is different in the two cases,

$$\vec{K} \neq \vec{K}'\,. \tag{8.5}$$

The pion form-factors as discussed in Sect. 7 were treated in the two-flavour case in [6]. They discussed the time component only but added partial twisting and quenching. The extra terms in the matrix element (7.3) are seen in (19) of [6] as well. The terms in (19) in [6] containing $G_{FV}, G_{FV}^{\text{iso}}, \mathbf{G}_{FV}^{\text{iso}}$ correspond to our $\Delta^V f_+, \Delta^V f_-, \Delta^V h_\mu$ of (7.8), (7.9) and (7.10). We have included the spatial components as well and checked that the expected Ward identity following from current conservation is satisfied when all effects of the boundary condition are taken into account. It should be noted that here the matrix element cannot be rewritten in terms of one form-factor f_+ and momenta rescaled with a shift $\vec{K}''$.

9 Conclusions

In this paper we discussed the one-loop tadpole and bubble integrals in finite volume and at non-zero twist.

We have worked out the expressions in one-loop ChPT for masses, axial-vector and pseudo-scalar decay constants as well as the vacuum expectation value and the two-point function for the electromagnetic current. We also discussed how the vector form-factors behave at finite twist angle. In particular we showed how one needs more form-factors than in the infinite volume limit and obtained expressions for those at one-loop order. We discussed how the extra terms are needed in order for the Ward identities to be satisfied.

Explicit formulas are provided for a large number of cases. We have given numerical results for all masses and the axial-vector decay constant of the charged mesons. We found that for the vector form-factor there are nontrivial finite volume effects due to the extra form-factors and have discussed the size of these effects on the form-factors. In particular, we have taken care to precisely define what all quantities are.

Work is in progress for including the effects due to partial quenching and twisting as well as the effects from staggered fermions [19].

Acknowledgments

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A Finite volume integrals with twist

The basic method to do finite volume integrals with twist can be found in [2]. The discussion below follows [12] closely.

A.1 Miscellaneous formulae

The first ingredient is the Poisson summation formula which is in one dimension

$$\frac{1}{L}\sum_{\substack{k=2\pi n/L+\theta/L\\ n\in\mathbb{Z}}} f(k) = \sum_{m\in\mathbb{Z}}\int\frac{dk}{2\pi}f(k)e^{iLmk}e^{-im\theta}. \tag{A.1}$$

The $\sum_{m\in\mathbb{Z}} e^{ima}$ projects on $a = 2\pi n$. $k-\theta/L$ is of this form, hence the sign in $e^{-im\theta}$ in (A.1).

The results for loop integrals with twist are expressed with the third Jacobi theta function and its derivatives w.r.t. to u. The definitions are

$$\begin{aligned}\Theta_3(u,q) &= \sum_{n=-\infty}^{\infty} q^{n^2}e^{2\pi i u n}, \quad \Theta_3'(u,q) = \sum_{n=-\infty}^{\infty} q^{n^2}2\pi i n e^{2\pi i u n},\\ \Theta_3''(u,q) &= -\sum_{n=-\infty}^{\infty} q^{n^2}4\pi^2 n^2 e^{2\pi i u n}.\end{aligned} \tag{A.2}$$

Some useful properties can be found in [12].

A.2 Tadpole integral

We define the tadpole integral in finite volume with twist as

$$A^{\{\ ,\mu,\mu\nu\}}(m_M^2,n) = \frac{1}{i}\int_V\frac{d^dk}{(2\pi)^d}\frac{\{1,k^\mu,k^\mu k^\nu\}}{(k^2-m_M^2)^n}\,. \tag{A.3}$$

The blank in the superscript indicates no superscript. $\int_V d^dk/(2\pi)^d$ is defined in (3.5). The momentum $\vec{k}$ which is summed over must be such that the boundary condition for the propagating meson M is satisfied,

$$\vec{k} = \frac{2\pi}{L}\vec{n} + \frac{\vec{\theta}_M}{L}, \quad \vec{\theta}_M = (\theta_M^x,\theta_M^y,\theta_M^z)\,. \tag{A.4}$$

We also introduce a fourvector $\theta_M = (0, \vec{\theta})$. Note that this implies that the tadpole integral is not invariant under $\vec{k} \to -\vec{k}$ since $-\vec{k}$ does not satisfy the boundary conditions for non-zero twist. The direction of propagation is important. We drop the subscript M below for clarity.

To describe the evaluation of these integrals, we restrict to the case $\{1\}$ and then quote the results for the other cases. We Wick rotate to Euclidean space and apply Poisson's summation formula from Eq. (A.1), giving

$$A(m^2, n) = (-1)^n \sum_{\vec{l} \in \mathbb{Z}^3} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + m^2)^n} e^{iL\vec{l}\cdot\vec{k} - i\vec{l}\cdot\vec{\theta}}. \tag{A.5}$$

The term with $\vec{l} = 0$ gives the infinite volume result. We focus on the finite volume part and use a prime on the sum to indicate that we sum over $\vec{l} \neq 0$. Using $1/a^n = (1/\Gamma(n)) \int_0^\infty d\lambda \lambda^{n-1} e^{-a\lambda}$, we get

$$A^V(m^2, n) = (-1)^n {\sum_{\vec{l} \in \mathbb{Z}^3}}' \int \frac{d^d k_E}{(2\pi)^d} \int \frac{d\lambda}{\Gamma(n)} \lambda^{n-1} e^{-\lambda(k^2+m^2)} e^{iL\vec{l}\cdot\vec{k} - i\vec{l}\cdot\vec{\theta}}. \tag{A.6}$$

The shift of integration variable via $k = \bar{k} + iLl/(2\lambda)$, with $l = (0, \vec{l})$, completes the square:

$$A^V(m^2, n) = (-1)^n {\sum_{\vec{l} \in \mathbb{Z}^3}}' \int \frac{d^d \bar{k}_E}{(2\pi)^d} \int \frac{d\lambda}{\Gamma(n)} \lambda^{n-1} e^{-\lambda(\bar{k}^2+m^2)} e^{-L^2\vec{l}^2/(4\lambda) - i\vec{l}\cdot\vec{\theta}}. \tag{A.7}$$

We can now perform the Gaussian integral and we end up with

$$A^V(m^2, n) = (-1)^n {\sum_{\vec{l} \in \mathbb{Z}^3}}' \int \frac{d\lambda}{\Gamma(n)} \frac{\lambda^{n-1-d/2}}{(4\pi)^{d/2}} e^{-\lambda m^2} e^{-L^2\vec{l}^2/(4\lambda) - i\vec{l}\cdot\vec{\theta}}. \tag{A.8}$$

Changing variables $\lambda \to \lambda L^2/4$ and using the Jacobi theta function of (A.2), we arrive at

$$A^V(m^2, n) = (-1)^n \left(\frac{L^2}{4}\right)^{n-2} \int \frac{d\lambda}{\Gamma(n)} \frac{\lambda^{n-3}}{(4\pi)^2} e^{-\lambda m^2 L^2/4} \left(\prod_{j=x,y,z} \Theta_3\left(\frac{-\theta^j}{2\pi}, e^{-1/\lambda}\right) - 1 \right). \tag{A.9}$$

The -1 removes the case with $\vec{l} = 0$ and the triple product comes from the triple sum and we set $d = 4$.

Performing the same operations using the other elements in X gives for the finite volume corrections

$$\begin{aligned} A^{V\mu}(m^2, n) = (-1)^n \frac{1}{\pi L} \left(\frac{L^2}{4}\right)^{n-2} & \int \frac{d\lambda}{\Gamma(n)} \frac{\lambda^{n-4}}{(4\pi)^2} e^{-\lambda m^2 L^2/4} \\ \times \Theta_3'\left(\frac{-\theta^\mu}{2\pi}, e^{-1/\lambda}\right) & \prod_{\substack{j=x,y,z \\ j \neq \mu}} \Theta_3\left(\frac{-\theta^j}{2\pi}, e^{-1/\lambda}\right). \end{aligned} \tag{A.10}$$

Note that the component $\mu = 0$ vanishes.

$$A^{V\mu\nu}(m^2, n) = g^{\mu\nu} A^V_{22}(m^2, n) + A^{V\mu\nu}_{23}(m^2, n)\,,$$

$$A^V_{22}(m^2, n) = \frac{(-1)^{n-1}}{2}\left(\frac{L^2}{4}\right)^{n-3}\int \frac{d\lambda}{\Gamma(n)}\frac{\lambda^{n-4}}{(4\pi)^2}e^{-\lambda m^2 L^2/4}\left(\prod_{j=x,y,z}\Theta_3\left(\frac{-\theta^j}{2\pi}, e^{-1/\lambda}\right) - 1\right),$$

$$\begin{aligned} A^{V\mu\nu}_{23}(m^2, n) = {}& \frac{(-1)^n}{4\pi^2}\left(\frac{L^2}{4}\right)^{n-3}\int \frac{d\lambda}{\Gamma(n)}\frac{\lambda^{n-5}}{(4\pi)^2}e^{-\lambda m^2 L^2/4} \\ & ((a)\mu = 0 \text{ or } \nu = 0) \quad \times 0 \\ & ((b) 0 \neq \mu \neq \nu \neq 0) \quad \times \Theta_3'\left(\frac{-\theta^\mu}{2\pi}, e^{-1/\lambda}\right)\Theta_3'\left(\frac{-\theta^\nu}{2\pi}, e^{-1/\lambda}\right)\prod_{\substack{j=x,y,z\\ j\neq\mu,\nu}}\Theta_3\left(\frac{-\theta^j}{2\pi}, e^{-1/\lambda}\right) \\ & ((c)\mu = \nu \neq 0) \qquad \times \Theta_3''\left(\frac{-\theta^\mu}{2\pi}, e^{-1/\lambda}\right)\prod_{\substack{j=x,y,z\\ j\neq\mu}}\Theta_3\left(\frac{-\theta^j}{2\pi}, e^{-1/\lambda}\right) \end{aligned} \tag{A.11}$$

$A^{V\mu\nu}_{23}$ vanishes for $\mu = 0$ or $\nu = 0$, case (a). For $\mu \neq \nu$ one uses the line (b), otherwise (c). $A^{V\mu\nu}_{23}$ is from the $l^\mu l^\nu$ part after the shift of k to $\bar{k}$. The sign conventions are Minkowski with upper indices as indicated. In the main text we have dropped the argument n, we only need $n = 1$.

A.3 Two propagator integrals

We define two propagator integrals as

$$B^{\{\ ,\mu,\mu\nu\}}(m_1^2, m_2^2, n_1, n_2) = \frac{1}{i}\int_V \frac{d^d k}{(2\pi)^d}\frac{\{1, k^\mu, k^\mu k^\nu\}}{(k^2 - m_1^2)^{n_1}((q-k)^2 - m_2^2)^{n_2}}. \tag{A.12}$$

As in the tadpole case, the direction of the propagators is important. We use the convention that the particles propagate in the direction of the momentum indicated in the propagator. We thus writing k and $q - k$ in the propagators to indicate this, even if the sign in the denominator at first sight is not relevant.

We have in principle a twist angle vector for each of the two particles in the denominators. However, it is sufficient to specify only the twist vector for the first propagator, with m_1^2, and the external momentum q. The latter must be such that $q - k$ automatically produces the correct boundary conditions for the particle corresponding to m_2^2. This is discussed in detail in [2].

We first do the Poisson summation trick to get full integrals over k. We combine the two propagators in (A.12) using a Feynman parameter x and shift integration variable by $k = \tilde{k} + xq$. We then have expressions of the form of the previous subsection but with $\tilde{k}$ as integration variable and $\tilde{m}^2 = (1-x)m_1^2 + xm_2^2 - x(1-x)q^2$ instead of m^2, as well as $\vec{\tilde{\theta}} = \vec{\theta}_1 - x\vec{q}$.

The final result is

$$
\begin{aligned}
B^V(m_1^2, m_2^2, n_1, n_2, q) &= \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx (1-x)^{n_1-1} x^{n_2-1} A^V(\tilde{m}^2, n_1+n_2)\,,\\
B^{V\mu}(m_1^2, m_2^2, n_1, n_2, q) &= \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx (1-x)^{n_1-1} x^{n_2-1}\\
&\quad \times \left(A^{V\mu}(\tilde{m}^2, n_1+n_2) + x q^\mu A^V(\tilde{m}^2, n_1+n_2)\right),\\
B^{V\mu\nu}(m_1^2, m_2^2, n_1, n_2) &= \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx (1-x)^{n_1-1} x^{n_2-1} \left(A^{V\mu\nu}(\tilde{m}^2, n_1+n_2)\right.\\
&\quad \left. + x(q^\mu g^\nu_\alpha + q^\nu g^\mu_\alpha) A^{V\alpha}(\tilde{m}^2, n_1+n_2) + x^2 q^\mu q^\nu A^V(\tilde{m}^2, n_1+n_2)\right). \qquad \text{(A.13)}
\end{aligned}
$$

The signs are for upper indices in Minkowski space as indicated. For the numerical evaluation it is useful to treat the integral over x and λ together. In the main text we have dropped the indices n_1 and n_2 and used the components as defined below in (A.15).

A.4 Integral relations

It is possible to derive relations between integrals using the relation

$$
2k\cdot q = (k^2 - m_1^2) - ((q-k)^2 - m_2^2) + m_1^2 - m_2^2 + q^2\,. \qquad \text{(A.14)}
$$

These were done in infinite volume in [20] and in [13] in the same conventions as ours. The trick remains valid at finite volume. Care has to be taken in the shift of integration momentum for some of the tadpole integrals (from k to $q-k$) but that is consistent with the boundary conditions.

We define components

$$
\begin{aligned}
B^{V\mu}(m_1^2, m_2^2) &= q^\mu B_1^V(m_1^2, m_2^2, q) + B_2^{V\mu}(m_1^2, m_2^2, q)\\
B^{V\mu\nu}(m_1^2, m_2^2, q) &= q^\mu q^\nu B_{21}^V(m_1^2, m_2^2, q) + g^{\mu\nu} B_{22}^V(m_1^2, m_2^2, q) + B_{23}^{V\mu\nu}(m_1^2, m_2^2, q)\,. \qquad \text{(A.15)}
\end{aligned}
$$

The relations we get from using (A.14) are, suppressing the arguments (m_1^2, m_2^2, q),

$$
\begin{aligned}
2q^2 B_1^V &= -A^V(m_1^2) + A^V(m_2^2) + (q^2 + m_1^2 - m_2^2) B^V - 2B_2^{V\mu} q_\mu\,,\\
q_\mu B_{23}^{V\mu\nu} &= -q^2 q^\nu B_{21}^V - q^\nu B_{22}^V\\
&\quad + \frac{1}{2}\left(-A^{V\nu}(m_2^2) - A^{V\nu}(m_1^2) + q^\nu A(m_2^2) + (q^2 + m_1^2 - m_2^2) B^{V\nu}\right). \qquad \text{(A.16)}
\end{aligned}
$$

These are valid for $n_1 = n_2 = 1$ and $n = 1$ in the tadpole integrals. They are needed to prove the Ward identities in the main text. We have also used them to simplify the expressions.

References

[1] P. F. Bedaque, *Aharonov-Bohm effect and nucleon-nucleon phase shifts on the lattice*, Phys. Lett. B **593** (2004) 82 [arXiv:nucl-th/0402051]

[2] C. T. Sachrajda and G. Villadoro, *Twisted boundary conditions in lattice simulations*, Phys. Lett. B **609** (2005) 73 [arXiv:hep-lat/0411033].

[3] G.M. de Divitiis, R. Petronzio, N. Tantalo, *On the discretization of physical momenta in lattice QCD*, Phys. Lett. B, **595** (2004) 408 [arXiv:hep-lat/0405002].

[4] J. M. Flynn *et al.* [UKQCD Collaboration], *A Numerical study of partially twisted boundary conditions*, Phys. Lett. B **632** (2006) 313 [arXiv:hep-lat/0506016].

[5] C. Aubin, T. Blum, M. Golterman and S. Peris, *The hadronic vacuum polarization with twisted boundary conditions*, Phys. Rev. D **88** (2013) 074505 [arXiv:1307.4701 [hep-lat]].

[6] F. -J. Jiang and B. C. Tiburzi, Phys. Lett. B **645** (2007) 314 [hep-lat/0610103].

[7] S. Weinberg, *Phenomenological Lagrangians*, Physica A **96** (1979) 327.

[8] J. Gasser and H. Leutwyler, *Chiral Perturbation Theory To One Loop*, Annals Phys. **158** (1984) 142;

[9] J. Gasser and H. Leutwyler, *Chiral Perturbation Theory: Expansions In The Mass Of The Strange Quark*, Nucl. Phys. B **250** (1985) 465.

[10] J. Gasser and H. Leutwyler, *Spontaneously Broken Symmetries: Effective Lagrangians At Finite Volume*, Nucl. Phys. B **307** (1988) 763.

[11] D. Becirevic and G. Villadoro, *Impact of the finite volume effects on the chiral behavior of f_K and B_K*, Phys. Rev. D **69** (2004) 054010 [arXiv:hep-lat/0311028].

[12] J. Bijnens, E. Boström and T. A. Lähde, *Two-loop Sunset Integrals at Finite Volume*, JHEP **1401** (2014) 019 [arXiv:1311.3531 [hep-lat]].

[13] G. Amorós, J. Bijnens and P. Talavera *Two point functions at two loops in three flavor chiral perturbation theory*, Nucl. Phys. B **568** (2000) 319 [arXiv:hep-ph/9907264].

[14] G. Amorós, J. Bijnens and P. Talavera, *QCD isospin breaking in meson masses, decay constants and quark mass ratios*, Nucl. Phys. B **602** (2001) 87 [arXiv:hep-ph/0101127].

[15] J. Gasser and H. Leutwyler, *Low-Energy Expansion of Meson Form-Factors*, Nucl. Phys. B **250** (1985) 517.

[16] T. B. Bunton, F. -J. Jiang and B. C. Tiburzi, Phys. Rev. D **74** (2006) 034514 [Erratum-ibid. D **74** (2006) 099902] [hep-lat/0607001].

[17] K. Ghorbani, *Chiral and Volume Extrapolation of Pion and Kaon Electromagnetic form Factor within SU(3) ChPT*, arXiv:1112.0729 [hep-ph].

[18] J. Bijnens and P. Talavera, *Pion and kaon electromagnetic form factors*, JHEP **0203** (2002) 046 [arXiv:hep-ph/0203049].

[19] C. Bernard, J. Bijnens, E. Gámiz and J. Relefors, work in progress.

[20] G. Passarino and M. J. G. Veltman, *One Loop Corrections For e^+e^- Annihilation Into $\mu^+\mu^-$ In The Weinberg Model*, Nucl. Phys. B **160** (1979) 151.