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Dual-color decompositions at one-loop level in Yang-Mills theory

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ABSTRACT: In this work, we extend the construction of dual color decomposition in Yang-Mills theory to one-loop level, i.e., we show how to write one-loop integrands in Yang-Mills theory to the dual DDM-form and the dual trace-form. In dual forms, integrands are decomposed in terms of color-ordered one-loop integrands for color scalar theory with proper dual color coefficients. In dual DDM decomposition, The dual color coefficients can be obtained directly from BCJ-form by applying Jacobi-like identities for kinematic factors. In dual trace decomposition, the dual trace factors can be obtained by imposing one-loop KK relations, reflection relation and their relation with the kinematic factors in dual DDM-form.

Contents

1. Introduction

One of the significant recent progresses in the study of scattering amplitude is the discovery of color-kinematic duality made by Bern, Carrasco and Johansson (BCJ) [1]. It is conjectured that a generic L-loop Yang-Mills amplitude can be written to the double-copy formula (we will call the form as BCJ-form)

$$\mathcal{A}_{tot}^L = i^L g^{m-2+2L} \sum_{D_i} \int \prod_{j=1}^{l} \frac{d^D l_j}{(2\pi)^D} \frac{1}{S_i} \frac{n_i(l)c_i}{\prod_k P_{ki}(l)}, \quad \text{[L-loop double copy]} \tag{1.1}$$

where the sum is over all possible cubic Feynman-like diagrams and the S_i is the symmetric factor. In the formula, the c_i is the color factor given by the group structure constants f^{abc} and $n_i(l)$ is the kinematic factor satisfying the following properties: whenever two color factors c_i, c_j are related by antisymmetry or three color factors c_i, c_j, c_k by Jacobi identity, so are the corresponding kinematic factors n.

$$\text{antisymmetry}: \quad c_i = -c_j \quad \Rightarrow n_i = -n_j$$

$$\text{Jacobi} - \text{like identity}: c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0. \tag{1.2}$$

The BCJ-form of Yang-Mills theory was proved at the tree-level from string theory in [2, 3], from twistor string theory in [4, 5, 6, 7], from field theory in [8, 9, 10] and further studied in [11, 12, 13, 14, 15, 16, 17, 18, 19] (see also a nice review [20]). The BCJ-form at the loop level is still a conjecture, but many studies have appeared[21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. The apparent equal-footing treatment on the color and kinematic factors of (1.1) introduced a very interesting perspective to the understanding of the structure of Yang-Mills amplitudes. To see its implication more clearly, let us review some results at tree-level.

The tree-level case

Based on results [32, 33] as well as [1], at tree-level we can write Yang-Mills amplitudes in the following three color decomposition forms:

$$\text{double} - \text{copy} \,(\text{or BCJ}-)\,\text{form}: \qquad \mathcal{A}_{tot} = \sum_i \frac{c_i n_i}{D_i} \qquad \text{[BCJ-form]} \tag{1.3}$$

$$\text{Trace form}: \qquad \mathcal{A}_{tot} = \sum_{\sigma \in S_{n-1}} \text{Tr}(T^{\sigma_1}...T^{\sigma_n}) A(\sigma) \qquad \text{[Trace-form]} \tag{1.4}$$

$$\text{DDM form}: \qquad \mathcal{A}_{tot} = \sum_{\sigma \in S_{n-2}} c_{1|\sigma(2,..,n-1)|n} A(1, \sigma, n) \qquad \text{[DDM-form]} \tag{1.5}$$

Here Roman As represent color ordered amplitudes, T^a are generators of $U(N)$ in fundamental representation and $c_i, c_{1|\sigma(2,..,n-1)|n}$ represent strings of structure constants f^{abc}

$$c_{1|\sigma(2,..,n-1)|n} = f^{1\sigma_2 x_1} f^{x_1 \sigma_3 x_2}...f^{x_{n-3}\sigma_{n-1}n} \,. \qquad \text{[DDM-c]} \tag{1.6}$$

Among these three forms, the relation between trace-form and DDM-form has been well understood using the following two properties of the Lie algebra of $U(N)$ gauge group. (See Ref. [33])

$$\text{Property One}: \qquad (f^a)_{ij} = f^{aij} = \text{Tr}(T^a[T^i, T^j]), \qquad \text{[group-1]} \tag{1.7}$$

$$\text{Property Two}: \qquad \sum_a \text{Tr}(XT^a)\text{Tr}(T^a Y) = \text{Tr}(XY) \qquad \text{[group-2]} \tag{1.8}$$

On the other hand, the existence of BCJ-form (1.3) is very nontrivial and recently many works have been devoted to its understanding as reviewed in the previous paragraph. For special helicity configurations, it was shown that the kinematic numerators correspond to area-preserving diffeomorphism algebra[16, 17]. Using this idea, an explicit construction of the BCJ-numerators $n_i(l)$ was given in [18], thereby providing a support to an algebra-manifest formulation. Also that Mafra, Schlotterer and Stieberger have given an explicit construction in [15] using Berkovits' pure spinor formalism. Finally, using the twistor string theory, Cachazo, He and Yuan gave an algorithm for $n_i(l)$ using solutions from scattering equations [7].

Although it is very hard[1] to derive the BCJ-form from the trace-form or the DDM-form, it is not hard to establish the trace-form and the DDM-form from the BCJ-form. Explicitly, using Jacobi relations, one

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[1]In fact, we are not clear how to do so.

can construct a basis for all color factors c_i, which are nothing, but the factors given in (1.6). Knowing the basis, we can expand arbitrary color factor $c_i = \sum_\sigma \alpha_\sigma c_{1|\sigma|n}$ and putting them back to (1.3). After collecting terms according to factors $c_{1|\sigma|n}$, (1.3) becomes the following form

$$\mathcal{A}_{tot} = \sum_{\sigma \in S_{n-2}} c_{1|\sigma(2,..,n-1)|n} \widehat{A}(1, \sigma, n) \qquad \text{[DDM-form-tilde]} \tag{1.9}$$

Then we need to ask whether the $\widehat{A}(1, \sigma, n)$ defined here is the same as $A(1, \sigma, n)$ given in (1.5). This identification can been done either from $c_{1|\sigma(2,..,n-1)|n}$ as the basis of color factor and both $\widehat{A}$ and A are minimum gauge invariant objects, or using the KLT relation [34] in [35].

However, since in the double-copy formulation (BCJ-form) n_i acquire the same status as c_i, it is natural to exchange the roles between c_i and n_i and consider the following two dual formulations

$$\text{Dual Trace} - \text{form}: \quad \mathcal{A}_{tot} = \sum_{\sigma \in S_{n-1}} \tau_{\sigma_1 ... \sigma_n} \widetilde{A}(\sigma) \qquad \text{[Dual-Trace-form]} \tag{1.10}$$

$$\text{Dual DDM} - \text{form}: \quad \mathcal{A}_{tot} = \sum_{\sigma \in S_{n-2}} n_{1|\sigma(2,..,n-1)|n} \widetilde{A}(1, \sigma, n) \qquad \text{[dual-DDM-form]} \tag{1.11}$$

where $\widetilde{A}$ is the color ordered scalar amplitude with f^{abc} as its cubic coupling constant (See Ref. [36, 35]) and τ is required to be cyclic invariant.

The idea of dual DDM-form first appeared in the literature in [37]. Using Jacobi identity we can find the basis $n_{1|\sigma|n}$ and expand any other $n_i = \sum_\sigma n_{1|\sigma|n}$ as we did for color factor c_i. Putting them back to BCJ-form (1.3) and collecting terms leads to the form given in (1.11). However, it is not clear whether the $\widetilde{A}$ obtained by this way is the same color-ordered scalar amplitude with f^{abc} coupling as claimed in (1.11). To establish this fact, one idea is to use the KLT relation as was done in [35] to establish the existence of dual DDM-form (however, now we need to use the off-shell BCJ relation for gauge theory amplitudes presented in [18]). Then by the independence of basis $n_{1|\sigma|n}$, the identification of $\widetilde{A}$ is done.

Based on the established existence of dual DDM-form, the dual trace-form was conjectured in [38] with explicit constructions given for the first few lower-point amplitudes. In addition two constructions for the dual trace-form was discussed in [40] and [41].

One-loop level

Having reviewed the tree-level case, now we move to the one-loop case, which will be our focus in this paper. At one-loop level we have the following three color decompositions:

$$\text{BCJ} - \text{form}: \quad \mathcal{A}_{tot}^{1-loop} = ig^n \sum_{diagrams \ \Gamma^i} \int \prod \frac{d^D l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i(l)c_i}{\prod_k P_{ki}(l)} \qquad \text{[1loop-BCJ-form]} \tag{1.12}$$

$$\text{Trace} - \text{form}: \quad \mathcal{A}_{tot}^{1-loop} = N_c \sum_{S_n/Z_n} \text{Tr}(T^{\sigma_1}...T^{\sigma_n}) A_{n;0}(\sigma_1, ..., \sigma_n)$$

$$+\sum_{m=1}^{\lfloor n/2\rfloor}\sum_{\sigma\in S_n/S_{n;m}}\mathrm{Tr}(T^{\sigma_1}...T^{\sigma_{n-m}})\mathrm{Tr}(T^{\sigma_{n-m+1}}...T^{\sigma_n})A_{n-m;m}(\sigma_1,...,\sigma_{n-m};\sigma_{n-m+1},...,\sigma_n)$$

<div align="right">[1loop-Trace-form]   (1.13)</div>

$$\mathrm{DDM-form}:\ \mathcal{A}_{tot}^{1-loop}=\sum_{\sigma\in S_{n-1}/R}f^{x_n\sigma_1 x_1}f^{x_1\sigma_2 x_2}....f^{x_{n-1}\sigma_n x_n}A_{n,0}(\sigma_1,...,\sigma_n)\quad {}^{\text{[1loop-DDM-form]}}\qquad(1.14)$$

In (1.13), Z_n denote cyclic symmetry and $S_{n;m}$ is the subsets of S_n that leaves the double-trace structure invariant, $\lfloor n/2\rfloor$ is the greatest integer less than or equal to $n/2$. In (1.13), R denotes reflection. Among these three forms, the last two, i.e., (1.13) and (1.14), are well established while the first one (1.12) is still a conjecture.

The trace-form (1.13) was given by Bern, Dixon, Dunbar and Kosower in [42]. In the formula there are single and double trace parts, where the partial amplitude $A_{n-m;m}(\sigma_1,...,\sigma_{n-m};\sigma_{n-m+1},...,\sigma_n)$ associated to the double trace part can be obtained by linear combination of those to the single trace part $A_{n;0}(\sigma_1,...,\sigma_n)$. In other words, we do not need to calculate $A_{n-m;m}(\sigma_1,...,\sigma_{n-m};\sigma_{n-m+1},...,\sigma_n)$ for one-loop amplitudes. The DDM-form, on the other hand, was given in [33], where the sum is over noncylic permutation up to reflections $R:(12...n)=(n...21)$. Here the $A_{n,0}(\sigma_1,...,\sigma_n)$ in (1.14) is nothing but the single trace partial amplitude appeared in (1.13). In fact, starting from DDM-form, it is easy to derive the trace-form as demonstrated in [33]. As a byproduct, the relation between single and double trace partial amplitudes will appear automatically.

Assuming the existence of (1.12), to go from BCJ-form to trace-form (1.13) and DDM-form (1.14) is easy. As shown by [33], for one-loop color factors, annuli of structure constants of the form

$$c^{1-loop}(\sigma_1...\sigma_n)\equiv f^{x_n\sigma_1 x_1}f^{x_1\sigma_2 x_2}....f^{x_{n-1}\sigma_n x_n}\quad {}^{\text{[1loop-color-basis]}}\qquad(1.15)$$

serve as a basis. Using it we can expand any c_i in BCJ-form and collect terms with factor $c^{1-loop}(\sigma_1...\sigma_n)$. These terms as a whole can be denoted by $\widehat{A}_{n,0}(\sigma_1,...,\sigma_n)$. Again the problem is whether it is equal to the one $A_{n,0}(\sigma_1,...,\sigma_n)$ defined in (1.14)? The identification is again easy by using the following facts: the color basis $c^{1-loop}(\sigma_1...\sigma_n)$ is independent to each other, and $\widehat{A}$ and A are gauge invariant objects.

The above discussions are parallel to the one given for tree-level case. Considering the duality between n_i and c_i in (1.12), it is natural to investigate the dual form [38], where the interest of this paper lies[2]. Unlike color numerator, the n_i depends on the loop momentum in general, so the dual formulations at one-loop should be given by[3]

$$\mathcal{A}_{1-loop}=ig^n\int\frac{d^D l}{(2\pi)^D}\sum_{\sigma\in S_{n-1}}n_{1,\sigma}^{DDM}(l)\widetilde{I}(1,\sigma).\quad {}^{\text{[1loop-Dual-DDM]}}\qquad(1.16)$$

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[2]In supergravity theory, a DDM-form of decomposition at one-loop level of supergravity has already been suggested in [39] and the dual DDM-form in Yang-Mills theory has the similar form. Nevertheless, in this work, we would like to give a general discussion on dual DDM-form at one-loop level in Yang-Mills theory in the introduction and some explicit examples in section 2, because the dual DDM-form is crucial for the construction of dual trace-form.

[3]One may notice that in one-loop DDM form (1.14), reflection has been modded out, in the dual DDM-form, we just leave the reflection symmetry and only consider it when we discuss on the dual trace-form.

for the dual DDM-form, and

$$\mathcal{A}_{1-loop} = ig^n \int \frac{d^D l}{(2\pi)^D} \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{\sigma \in S_n/S_{n;m}} \tau_{\{\sigma_1,\ldots,\sigma_{n-m}\},\{\sigma_{n-m+1},\ldots,\sigma_n\}}(l) \widetilde{I}(\{\sigma_1,\ldots,\sigma_{n-m}\}; \{\sigma_{n-m+1},\ldots,\sigma_n\}),$$
<div style="text-align:right">[Dual-trace]</div>

$$(1.17)$$

for the dual trace-form[4]. In other words, it is the *integrand* taking the dual form. Using the same idea, to get dual DDM-form from BCJ-form, first we need to find a suitable basis for n_i, then put it back to BCJ-form and collect terms to get the dual DDM-form. After this step, again the key step is to identify what integrand these collected terms correspond to. An intuition might be provided by making use of the existing tree-level results. Naively if we start with an $(n+2)$-point Feynman diagrams, a one-loop diagram can be constructed by connecting the $(n+1)$-th and $(n+2)$-th external lines. Since we know at tree-level these collected terms correspond to color-ordered scalar amplitudes, it is very natural to do the identification at the one-loop level.

Now we make above observation more accurately. Starting from BCJ-form, note that in order to get DDM-form, one decomposes color numerators $c_i = \sum_\alpha \kappa_{i\alpha} c_\alpha$, where c_α is a basis of the color part constructed only by antisymmetry and the Jacobi identity. After that we get

$$\sum_\alpha c_\alpha \left(\sum_{i:diagrams} \kappa_{i\alpha} \frac{n_i}{D_i} \right) \qquad \text{[test-1]} \tag{1.18}$$

where the factors inside the bracket together constitute the color-ordered gauge theory amplitude. Likewise, to get the dual DDM-form one decomposes kinematic numerators according to $n_i = \sum_\alpha \kappa_{i\alpha} n_\alpha$ where n_α is another basis of the kinematic part constructed, again, only by antisymmetry and the Jacobi identity. It is crucial the construction of basis used only the topology of cubic diagrams, thus we can take bases c_α and n_α sharing the same diagram topology. In other words, we should have the same expansion coefficients $\kappa_{i\alpha}$ for both constructions of DDM-form and dual DDM-form

$$\sum_\alpha n_\alpha \left(\sum_{i:diagrams} \kappa_{i\alpha} \frac{c_i}{D_i} \right) \qquad \text{[test-2]} \tag{1.19}$$

Comparing (1.18) and (1.19), we see that the difference is just the exchange of $c_i \leftrightarrow n_i$, whereas c_i and n_i correspond to the same cubic diagram. Because the c_i has local construction, i.e., each cubic vertex is decorated with the coupling constant f^{abc}, we conclude that the integrand of one-loop dual DDM-form is indeed the one of color-ordered scalar theory. In section 2, we will use explicit calculations to demonstrate above arguments.

Having obtained the dual DDM-form, the next step is to construct the dual trace-form. Going from dual DDM-form to dual trace-form, we need to find a way to rewrite basis kinematic numerators n_α to

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[4]Since we only discuss on one-loop case, we will use n and τ and $\widetilde{I}$ instead of n^{1-loop}, τ^{1-loop}, $\widetilde{I}^{1-loop}$ for convenience.

a linear combination of some kind "single trace" part and "double trace" part n_{trace} as did for tree-level case in [38, 40, 41]. However note that, as in the tree-level case, the number of n_{trace} is much more than n_α, thus proper extra relations need to be manually imposed in order to solve n_{trace} by n_α. Choosing the appropriate relations is nevertheless far from trivial, in particular one needs to avoid over-constraint and maintain relabeling symmetry if possible. A 4-point example at one-loop level was provided by Bern and Dennen in [38], where cyclic and KK-relations were implemented. In section 3, we will generalize the result in [38] and give a general algorithm for the construction of dual trace-form at one-loop. Our algorithm gives the solution satisfying natural relabeling symmetry.

In section 4, we will use the relabeling symmetry to give another construction of dual trace-form. Finally in section 5 a brief conclusion is given.

2. Dual DDM-form

Having the general discussion for the dual DDM-form at one-loop, in this section, we will use explicit example to demonstrate the construction. In the discussion below we will follow the convention where the loop momentum l is defined to be the momentum carried by the propagator next to leg 1.
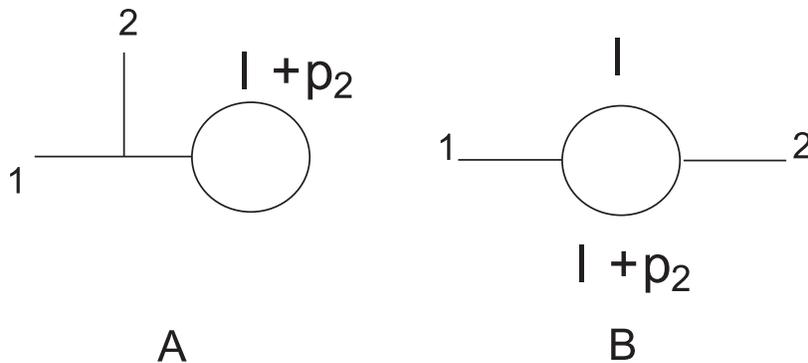
2.1 Two-point example

Figure 1: Feynman-like diagrams for two-point one-loop integrand using only cubic vertex.

For two point case, using only cubic vertex, two diagrams A and B are constructed as given in Figure 1. With our convention, the corresponding integrands are

$$I_A(l) = \frac{C_A n_A(l)}{s_{12}(l+p_1+p_2)^2}, \quad I_B(l) = \frac{C_B n_B(l)}{l^2(l+p_2)^2}.$$

(2.1)

where

$$C_A = f^{12e} f^{ee'e'}, \quad C_B = f^{e1e'} f^{e'2e}$$

(2.2)

and the Einstein summation convention has been used. Since the structure constant is antisymmetric, $f^{ee'e'} = 0$, so $C_A = 0$. The two-point one-loop integrand becomes

$$I_{2-pt}(l) = I_B(l) = n_B(l) \left[\frac{f^{e1e'} f^{e'2e}}{l^2(l+p_2)^2} \right]. \tag{2.3}$$

Comparing with (1.16), we see that the part inside the bracket in above equation is nothing, but the integrand $\widetilde{I}(1,2)$ we are looking for. It is obvious from the expression that $\widetilde{I}(1,2)$ is the one-loop integrand of color-ordered scalar theory with two external lines.
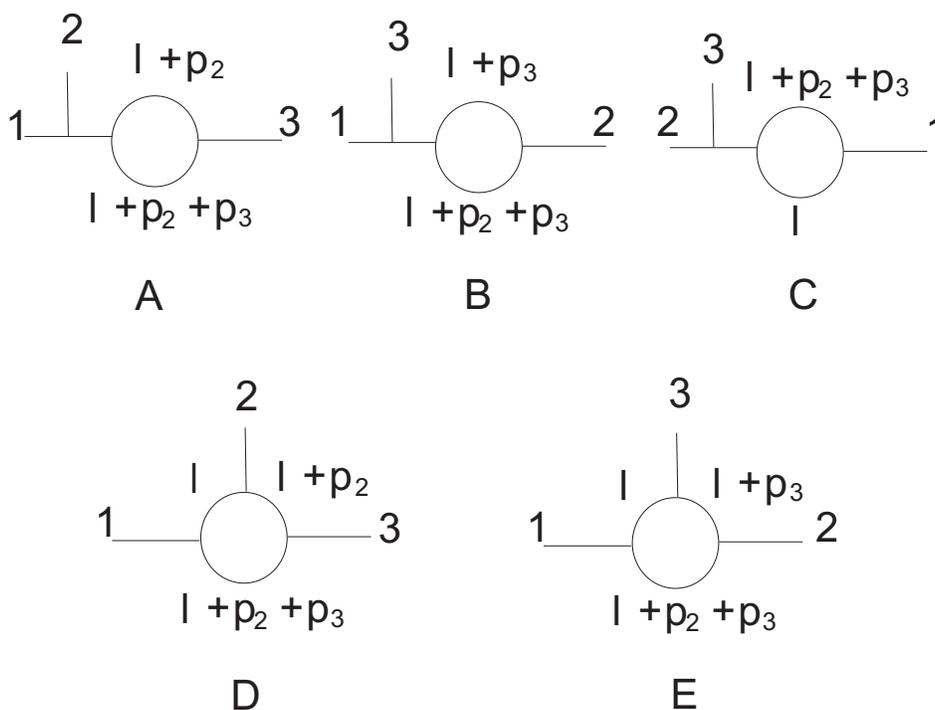
2.2 Three-point example

Figure 2: Feynman-like diagrams for three-point one-loop integrand.

For the three-point case, using only cubic vertex there are three kinds of topologies (see Figure 2). The first kind of topologies is the tadpole diagrams, i.e., there is only one line connected to the loop. Because the antisymmetric property of group structure constants, the contribution is zero, just like the diagram A of two-point case in Figure 1. The second kind of topologies has two lines connected to the loop directly, i.e., diagrams A, B, C in Fig. 2. Expressions are given by

$$I_A(l) = \frac{C_A n_A(l)}{s_{12}(l+p_2)^2(l+p_2+p_3)^2}, I_B(l) = \frac{C_A n_B(l)}{s_{13}(l+p_3)^2(l+p_3+p_2)^2}, I_C(l) = \frac{C_C n_C(l)}{s_{23}l^2(l+p_2+p_3)^2}. \tag{2.4}$$

where color factors C_A, C_B, C_C can easily be read out from corresponding diagrams. The third kind of topologies have three lines connected to the loop directly, i.e., diagrams D, E in Fig. 2 and expressions are

$$I_D(l) = \frac{C_D n_D(l)}{l^2(l+p_2)^2(l+p_2+p_3)^2}, \quad I_E(l) = \frac{C_E n_E(l)}{l^2(l+p_3)^2(l+p_3+p_2)^2}. \tag{2.5}$$

where C_D, C_E are corresponding color factors. Using Jacobi-like identity and taking n_D and n_E as basis, we find following expansions

$$n_A(l) = n_D(l) - n_E(l+p_2), \quad n_B(l) = n_E(l) - n_D(l+p_3), \quad n_C(l) = n_D(l) - n_E(l). \tag{2.6}$$

Thus I_A, I_B, I_C can be written as

$$I_A(l) = \frac{C_A[n_D(l) - n_E(l+p_2)]}{s_{12}(l+p_2)^2(l+p_2+p_3)^2} = \frac{C_A n_D(l)}{s_{12}(l+p_2)^2(l+p_2+p_3)^2} - \frac{C_A n_E(l)}{s_{12}l^2(l+p_3)^2} + T_A$$

$$I_B(l) = \frac{C_B[n_E(l) - n_D(l+p_3)]}{s_{13}(l+p_3)^2(l+p_3+p_2)^2} = \frac{C_B n_E(l)}{s_{13}(l+p_3)^2(l+p_3+p_2)^2} - \frac{C_B n_D(l)}{s_{13}(l)^2(l+p_2)^2} + T_B$$

$$I_C(l) = \frac{C_C[n_D(l) - n_E(l)]}{s_{23}l^2(l+p_2+p_3)^2}, \tag{2.7}$$

where T_A and T_B are terms integrated to zero[5]

$$T_A = \frac{C_A}{s_{12}}\left[\frac{n_E(l)}{l^2(l+p_3)^2} - \frac{n_E(l+p_2)}{(l+p_2)^2(l+p_2+p_3)^2}\right],$$

$$T_B = \frac{C_B}{s_{13}}\left[\frac{n_D(l)}{l^2(l+p_2)^2} - \frac{n_D(l+p_3)}{(l+p_3)^2(l+p_3+p_2)^2}\right]. \tag{2.8}$$

Up to terms integrated to zero, the total integrand is given as

$$I(1,2,3)(l) = n_D(l)\left[\frac{C_A}{s_{12}(l+p_2)^2(l+p_2+p_3)^2} - \frac{C_B}{s_{13}(l)^2(l+p_2)^2} + \frac{C_C}{s_{23}l^2(l+p_2+p_3)^2}\right.$$

$$\left.+ \frac{C_D}{l^2(l+p_2)^2(l+p_2+p_3)^2}\right] + n_E(l)\left[-\frac{C_A}{s_{12}l^2(l+p_3)^2} + \frac{C_B}{s_{13}(l+p_3)^2(l+p_3+p_2)^2} - \frac{C_C}{s_{23}l^2(l+p_2+p_3)^2}\right.$$

$$\left.+ \frac{C_E}{l^2(l+p_3)^2(l+p_3+p_2)^2}\right] \equiv n_D(l)\widetilde{I}(1,2,3)(l) + n_E(l)\widetilde{I}(1,3,2)(l), \tag{2.9}$$

Above form is exactly the dual DDM-form for one-loop amplitude (1.16), where n_D and n_E are just the kinematic basis. The corresponding expressions for $\widetilde{I}(1,2,3)(l)$ and $\widetilde{I}(1,3,2)(l)$ are again the three-point one-loop integrands of color-ordered scalar theory.

2.3 Four-point example

For four-point case, there are many diagrams and they can be classified as follows:

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[5]It can easily be seen by shifting the loop momentum of the first term.
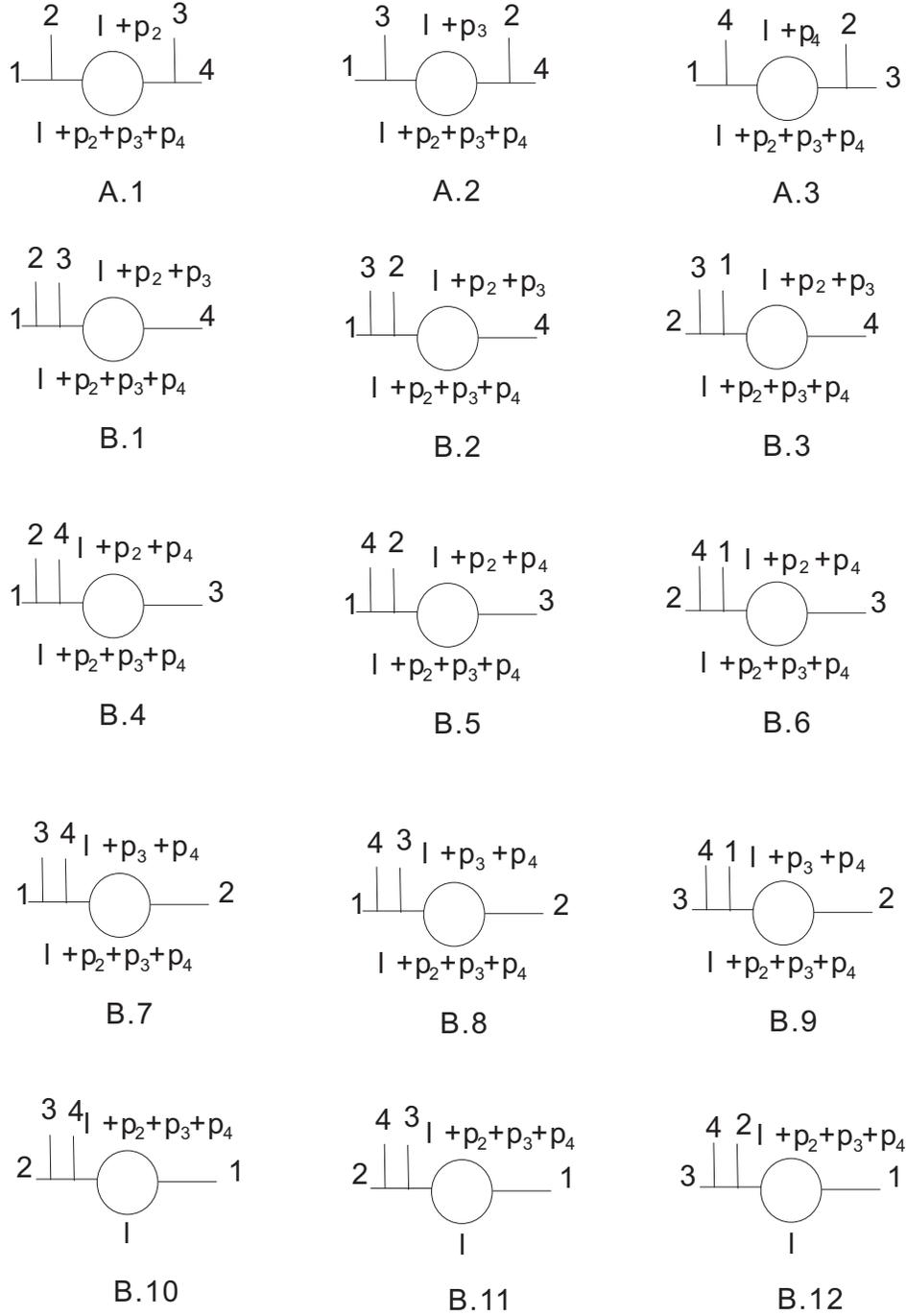
Figure 3: Feynman-like diagrams with two lines connected to the loop in four-point case.

- (1) For tadpole diagrams with only one line connected to the loop directly, their contributions are zero due to the antisymmetry of group structure constant.

- (2) For diagrams with two lines connected to the loop directly, they are listed in Fig. 3. Their
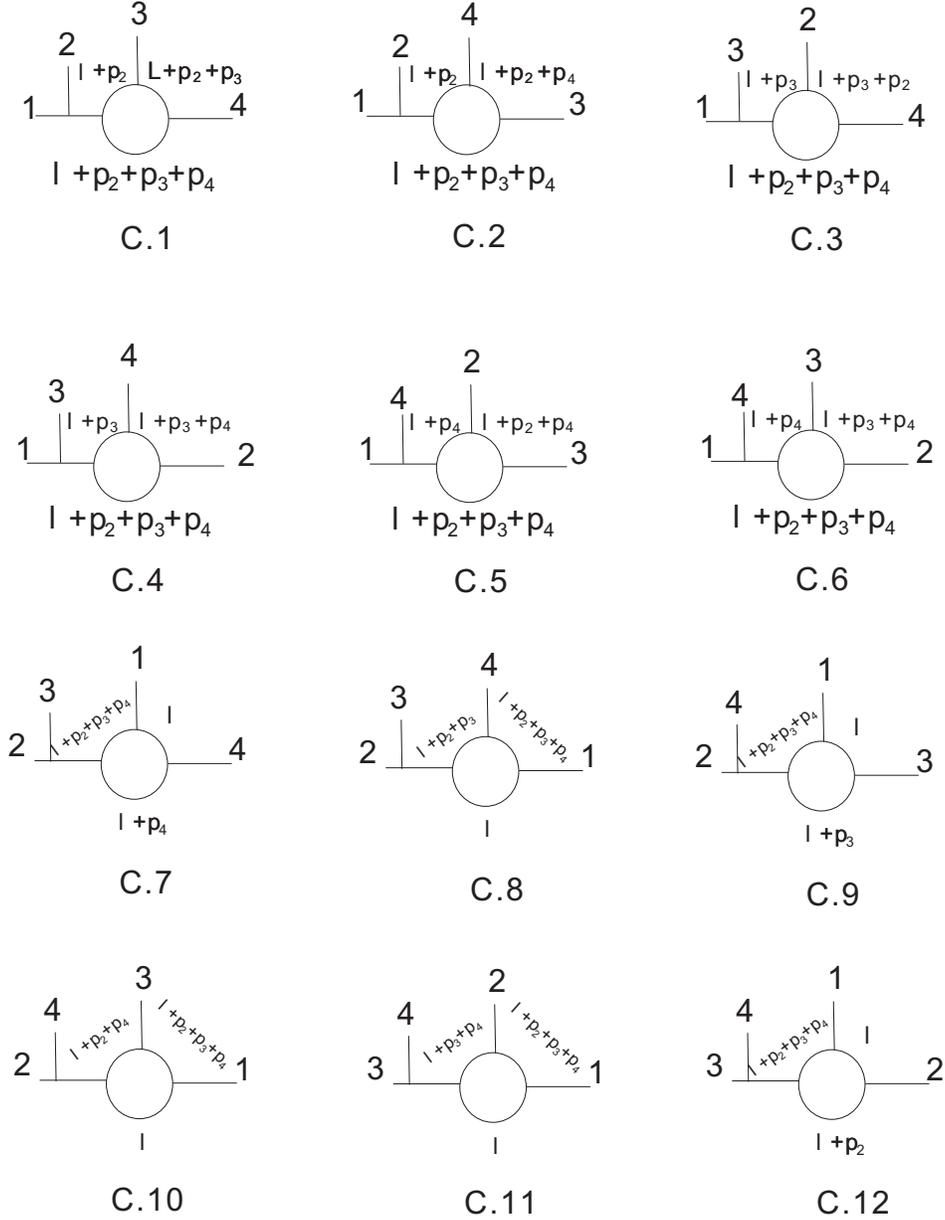
3

2

$l+p_2$ $L+p_2+p_3$

1 ——————— 4

$l+p_2+p_3+p_4$

C.1

4

2

$l+p_2$ $l+p_2+p_4$

1 ——————— 3

$l+p_2+p_3+p_4$

C.2

2

3

$l+p_3$ $l+p_3+p_2$

1 ——————— 4

$l+p_2+p_3+p_4$

C.3

4

3

$l+p_3$ $l+p_3+p_4$

1 ——————— 2

$l+p_2+p_3+p_4$

C.4

2

4

$l+p_4$ $l+p_2+p_4$

1 ——————— 3

$l+p_2+p_3+p_4$

C.5

3

4

$l+p_4$ $l+p_3+p_4$

1 ——————— 2

$l+p_2+p_3+p_4$

C.6

1

3

$l+p_2+p_3+p_4$ l

2 ——————— 4

$l+p_4$

C.7

4

3

$l+p_2+p_3$ $l+p_2+p_3+p_4$

2 ——————— 1

l

C.8

1

4

$l+p_2+p_3+p_4$ l

2 ——————— 3

$l+p_3$

C.9

3

4

$l+p_2+p_4$ $l+p_2+p_3+p_4$

2 ——————— 1

l

C.10

2

4

$l+p_3+p_4$ $l+p_2+p_3+p_4$

3 ——————— 1

l

C.11

1

4

$l+p_2+p_3+p_4$ l

3 ——————— 2

$l+p_2$

C.12

Figure 4: Feynman-like diagrams with three lines connected to the loop in four-point case.

corresponding expressions can be read out easily.

- (3) For diagrams with three lines connected to the loop directly, they are listed in Fig. 4. From these diagrams, it is easy to write down corresponding expressions.

- (4) For diagrams with four lines connected to the loop directly, they are listed in Fig. 5. From these diagrams, it is easy to write down corresponding expressions.
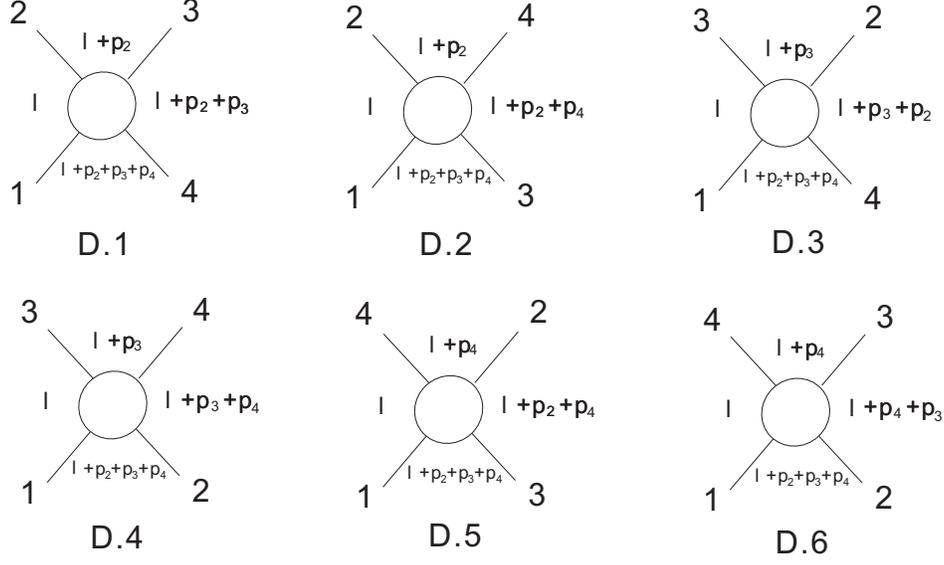
Figure 5: Feynman-like diagrams with four lines connected to the loop in four-point case.

We will choose the kinematic basis n_α as these given by Fig. 5(i.e., the D.1-D.6) in dual DDM-form, and expand other n_i given by Fig. 3 and Fig. 4 using Jacobi identities. For example, the coefficient of n_{1234}^{DDM} will get contributions from diagrams A.1, A.3, B.1, B.3, B.5, B.8, B.9, B.10, B.12, C.1, C.5, C.8, C.12, D.1, as

$$\frac{C_{A.1}}{s_{12}s_{34}(l+p_2)^2(l-p_1)^2} + \frac{C_{A.3}}{s_{14}s_{23}(l+p_4)^2(l-p_1)^2}$$

$$+ \frac{C_{B.1}}{s_{12}s_{123}(l+p_2+p_3)^2(l-p_1)^2} + \frac{C_{B.3}}{s_{23}s_{123}(l+p_2+p_3)^2(l-p_1)^2} + \frac{C_{B.5}}{s_{14}s_{124}(l+p_2+p_4)^2(l-p_1)^2}$$

$$+ \frac{C_{B.7}}{s_{13}s_{134}(l+p_3+p_4)^2(l-p_1)^2} + \frac{C_{B.9}}{s_{34}s_{134}(l+p_3+p_4)^2(l-p_1)^2} + \frac{C_{B.10}}{s_{23}s_{234}l^2(l-p_1)^2}$$

$$+ \frac{C_{B.12}}{s_{34}s_{234}l^2(l-p_1)^2} + \frac{C_{C.1}}{s_{12}(l+p_2)^2(l+p_2+p_3)^2(l-p_1)^2} + \frac{C_{C.5}}{s_{14}(l+p_4)^2(l+p_2+p_4)^2(l-p_1)^2}$$

$$+ \frac{C_{C.8}}{s_{23}l^2(l+p_2+p_3)^2(l-p_1)^2} + \frac{C_{C.12}}{s_{34}l^2(l+p_2)^2(l-p_1)^2} + \frac{C_{D.1}}{l^2(l+p_3)^2(l+p_3+p_4)^2(l-p_1)^2}, \qquad (2.10)$$

where we have neglect terms integrated to zero. The above expression is nothing but four-point one-loop integrand $\widetilde{I}(1,2,3,4)$ of color-ordered scalar theory . After similar calculations for other ordering, we do get the claimed form (1.16)

$$I(1,2,3,4)(l) = \sum_{\sigma \in permutations\ of\{2,3,4\}} n_{1,\sigma}(l)\widetilde{I}(1,\sigma). \qquad (2.11)$$

up to terms vanishing after loop integration, where each $\widetilde{I}(1,\sigma)$ is identified to the integrand of color-ordered scalar theory at one-loop. For higher points, the general procedure is same although computations will be much more complicated.

3. Dual trace-form

In the discussions above we saw that the dual DDM-form can be derived through relatively straightforward manipulations. Deriving a corresponding dual trace-form at one-loop however turns out to be less direct, especially because of the extra conditions required to define dual trace factors [38, 40].

Recall that at tree-level, the set of numerators $n_{1\sigma n}$, consisting of $(n-2)!$ elements, having legs 1 and n fixed at two ends, were translated into $(n-1)!$ dual traces $\tau_{1\tilde{\sigma}}$, which are counterparts of the single color trace factors. To uniquely determine τ we need to impose KK-relation among $\tau_{1\tilde{\sigma}}$'s, so the number of independent dual traces can be reduced to $(n-2)!$. The algorithm formally picks a fixed pair $(1, n)$ to define basis numerators. To examine if the solution satisfy relabling symmetry we need to inspect the transformation under permutaions of legs 1 and n.

At one-loop level, similar constraints are required to properly define dual traces. For the purpose of discussion let us first review the $U(N_c)$ color structure at one-loop, which also serves as input to the definition of dual traces.

3.1 general structure of the defining conditions

Generically, the color factors appear in the DDM-form at one-loop level can be translated into double trace factors,

$$
\begin{aligned}
c_{\{\sigma\}}^{1-loop} &= f^{x_1 a_1 x_2} f^{x_2 a_2 x_3} \dots f^{x^n a_n x_1} = \text{Tr}(T^{x_1}[T^{a_1}, [T^{a_2}, \dots [T_n^a, T^{x_1}]]]) \\
&= \sum_{\sigma \in OP(\{\alpha\} \bigcup \{\beta\})} (-1)^{n_\beta} \text{Tr}(T^{x_1} T^{a_{\alpha_1}} \dots T^{a_{\alpha_{n_\alpha}}} T^{x_1} T^{\beta_{n_\beta}} \dots T^{\beta_1}) \\
&= \sum_{\sigma \in OP(\{\alpha\} \bigcup \{\beta\})} (-1)^{n_\beta} \text{Tr}(T^{a_{\alpha_1}} \dots T^{a_{\alpha_{n_\alpha}}}) \text{Tr}(T^{\beta_{n_\beta}} \dots T^{\beta_1}), \quad \text{[c-Tr]}
\end{aligned}
\tag{3.1}
$$

where in the last line we used the property of $U(N_c)$,

$$
\sum_{x_1} \text{Tr}(T^{x_1} A T^{x_1} B) = \text{Tr}(A)\text{Tr}(B).
\tag{3.2}
$$

Note however, two exceptional cases call for special attention. When the repeated generators are adjacent, single trace factors are produced instead. This can happen in equation (3.1) as

$$
\sum_{x_1} \text{Tr}(T^{x_1} T^{a_1} \dots T^{a_n} T^{x_1}) = N_c \text{Tr}(T^{a_1} \dots T^{a_n})
\tag{3.3}
$$

or as

$$
(-)^n \sum_{x_1} \text{Tr}(T^{x_1} T^{x_1} T^{a_n} T^{a_{n-1}} \dots T^{a_1}) = (-)^n N_c \text{Tr}(T^{a_n} T^{a_{n-1}} \dots T^{a_1}).
\tag{3.4}
$$

Inspired by the above algebraic structure, it is natural to assume that there are kinematic correspondence of the following color trace factors

$$
\text{Tr}(T^{\alpha_1} \dots T^{\alpha_m}) \text{Tr}(T^{\beta_1} \dots T^{\beta_n}) \to \tau_{\alpha;\beta}, \quad \text{Tr}(T^{a_1} \dots T^{a_n}) \to \tau_\alpha, \quad \text{[map-1]}
\tag{3.5}
$$

where kinematic trace factors $\tau_{\alpha;\beta}$ and τ_α are cyclic invariant. Thus we can impose following relation between n_α in dual DDM-form and kinematic trace structure $\tau_{\alpha;\beta}$ in dual trace-form as

$$n_{\{\sigma\}}^{1-loop} = \sum_{\sigma \in OP(\{\alpha\} \bigcup \{\beta\})} (-1)^{n_\beta} \tau_{\alpha;\beta^T}, \quad \text{[n-tau-rel]} \tag{3.6}$$

where β^T means reversing the ordering in the subset β. In (3.6), there are again two special cases: when $\alpha = \emptyset$, $\tau_{\alpha;\beta^T} \to N_c \tau_{\beta^T}$ and when $\beta = \emptyset$, $\tau_{\alpha;\beta^T} \to N_c \tau_\alpha$. In other words, there are two single traces and we have also kept the possible freedom of "kinematic rank N_c".

From equation (3.6), one can see that the number of all τ together clearly exceeds that of n. In fact, there are only $(n-1)!/2$ independent n_α's because n_α is cyclic invariant and satisfies following reflection relation

$$n_\alpha = (-)^M n_{\alpha^T} \quad \text{[n-reverse]} \tag{3.7}$$

where M is the number of elements of the set α. To be able to solve τ by n_α, we need to impose extra equations. In viewing of the solution that works at tree-level, a natural generalization is to impose the one-loop KK-relation [38] between kinematic single and double trace parts

$$\tau_{\alpha;\beta} = (-)^{n_\beta} \sum_{C \in Z_{n_\beta}} \sum_{\sigma \in \mathcal{COP}(\alpha \bigcup C(\beta)^T)} \tau_\sigma \quad \text{[map-2]} \tag{3.8}$$

It is worth to notice that in (3.8), subsets α and β have been treated differently at the right handed side. However, for double trace part, by the correspondence to color part it is very naturally to impose

$$\tau_{\alpha;\beta} = \tau_{\beta;\alpha} \quad \text{[tau-double-sym]} \tag{3.9}$$

Thus, to be consistence between (3.8) and (3.9), we need to impose reflection relation in addition

$$\tau_\alpha = (-1)^M \tau_{\alpha^T}. \quad \text{[tau-reverse]} \tag{3.10}$$

where M is the number of elements of the set α. With these extra conditions (3.8), (3.9) and (3.10), the number of independent kinematic trace factors is reduced $(n-1)!/2$. So finally the original equations (3.6) become an $(n-1)!/2$ by $(n-1)!/2$ matrix equation,

$$n_{1\sigma} = \sum_{\sigma' \in S_{n-1}} G[\sigma|\sigma'] \tau_{1\sigma'} \quad \text{[n-tau-final]} \tag{3.11}$$

Knowing the matrix G, we can solve $\tau_{1\sigma'}$ by $n_{1\sigma}$ and finally determine all kinematic trace factors.

general algorithm

To summarize, the general algorithm of constructing kinematic trace factors is given by the following:

- Starting with any dual-DDM basis numerator $n_{1,...,n}$ we consider all possible splittings of its label $\{1,...,n\}$ into two subsets α, β, each can be empty. Generically there will be 2^n splittings. For example at four-points, denoting the one-loop dual-DDM factor as $n_{1,\sigma}^{1-loop}$, $\sigma \in perm(2,3,4)$. the relation between n and τ is given by

$$n_{1234}^{1-loop} = N_c\tau_{\{1234\}} - \tau_{\{234\},\{1\}} - \tau_{\{134\},\{2\}} - \tau_{\{124\},\{3\}} - \tau_{\{123\},\{4\}}$$
$$+\tau_{\{34\},\{21\}} + \tau_{\{24\},\{31\}} + \tau_{\{23\},\{41\}} + \tau_{\{13\},\{42\}} + \tau_{\{14\},\{32\}} + \tau_{\{12\},\{43\}}$$
$$-\tau_{\{4\},\{321\}} - \tau_{\{3\},\{421\}} - \tau_{\{2\},\{431\}} - \tau_{\{1\},\{432\}} + N_c\tau_{\{4321\}}^{[4\text{-pt}]} \tag{3.12}$$

- We then impose KK relation on τ (3.8). In the four-point case, we have

$$\tau_{\{bcd\},\{a\}} = -\tau_{\{abcd\}} - \tau_{\{bacd\}} - \tau_{\{bcad\}},$$
$$\tau_{\{cd\},\{ba\}} = \tau_{\{abcd\}} + \tau_{\{acbd\}} + \tau_{\{cabd\}} + \tau_{\{bacd\}} + \tau_{\{bcad\}} + \tau_{\{cbad\}},$$
$$\tau_{\{d\},\{cba\}} = -\tau_{\{abcd\}} - \tau_{\{abdc\}} - \tau_{\{adbc\}}, \tag{3.13}$$

Substituting these relations into (3.12) and using cyclic symmetry $\tau_{\{abcd\}} = \tau_{\{dabc\}}$, we get

$$n_{1234}^{1-loop} = (15 + N_c)\tau_{\{1234\}} + 10\tau_{\{1243\}} + 10\tau_{\{1324\}} + 10\tau_{\{1342\}} + 10\tau_{\{1423\}} + (5 + N_c)\tau_{\{1432\}}. \tag{3.14}$$

- Using reflection relation (3.10) we can reduce the obtained equations further. For example, above equation is reduced to

$$n_{1234}^{1-loop} = (20 + 2N_c)\tau_{1234} + 20\tau_{1243} + 20\tau_{1324}, \qquad [4\text{-point-rel}] \tag{3.15}$$

Repeating the same manipulations for all basis numerators, we arrive at the matrix equation (3.11), from which we can solve for all dual traces.

G-matrix:

Now we discuss the computation of G-matrix. The calculation can be divided into two steps. The first step is to calculate extended $\widetilde{G}$-matrix $\widetilde{G}[\sigma|\rho]$ where $\sigma, \rho \in S_n/Z_n$ (i.e., all permutations up to cyclic ordering). The second step is to impose the reflection relation (3.10), i.e,

$$G[\sigma|\rho] = \widetilde{G}[\sigma|\rho] + (-)^n\widetilde{G}[\sigma|\rho^T], \qquad \sigma, \rho \in (S_n/Z_n)/Z_2 \qquad [\text{G-gen}] \tag{3.16}$$

Since the second step is easy, we will focus on the first step, i.e., the extended $\widetilde{G}$-matrix. Elements of extended $\widetilde{G}$-matrix depend on N_c only for following two kinds of structures

$$\widetilde{G}[\sigma|\sigma] = a_0 + N_c, \qquad \widetilde{G}[\sigma|\sigma^T] = (-)^n(b_0 + N_c) \tag{3.17}$$

where a_0, b_0 are constants. Because this dependence, if we know the extended $\widetilde{G}$-matrix for $N_c = 1$, we will know the extended $\widetilde{G}$-matrix for general N_c.

To demonstrate the calculation of element $\widetilde{G}[\sigma|\rho]$, let us use four-point result (3.14) with $N_c = 1$ as an example. For this example, we have $\sigma = \{1,2,3,4\}$ fixed and $3! = 6$ different choices of ρ. Given the ordering of σ, there are $2^4 = 16$ different splittings to two subsets. Among them, 8 of them with 1 at the first subset are given by (remembering to keep relative ordering)

$$\{1,2,3,4\} \to (\sigma_L, \sigma_R)$$
$$= (1234, \emptyset)/(123, 4)/(124, 3)/(134, 2)/(12, 34)/(13, 24)/(14, 23)/(1, 234) \quad \text{[sigma-split]} \quad (3.18)$$

and other 8 are obtained by exchanging these two subsets. We do similar splitting to the ordering ρ, but now we will allow the cyclic shifting of one subset. For example, with $\rho = \{1,2,4,3\}$ we will have following splitting with 1 at the first position of the first subset (by cyclic symmetry, we can always fix one element)

$$\{1,2,4,3\} \to (\rho_L, \rho_R) = (1243, \emptyset)/(124, 3)/(123, 4)/(143, 2)/(12, 43)/(12, 34)/(14, 23)/(14, 32)$$
$$/(13, 24)/(13, 42)/(1, 243)/(1, 432)/(1, 324) \quad \text{[rho-split]} \quad (3.19)$$

where since we have fixed 1, we have to include the cyclic shifting of ρ_R. Comparing these two splitting (3.18) and (3.19), we see that there are five splittings to be same:

$$(123, 4)/(124, 3)/(12, 34)/(13, 24)/(14, 23) \Longrightarrow \widetilde{G}[\{1,2,3,4\}|\{1,2,4,3\}] = 2 \times 5 = 10 \quad (3.20)$$

where factor 2 comes from exchanging of two subsets. One can easily check that all other five coefficients in (3.14) can be obtained by same way. For $\rho = \{1,2,3,4\}$ there are 8×2 splitting. For $\rho = \{1,3,2,4\}$, $(124, 3)/(134, 2)/(12, 34)/(13, 24)/(14, 23)$ from (3.18) are taken, so $5 \times 2 = 10$. For $\rho = \{1,3,4,2\}$, $(134, 2)/(12, 34)/(13, 24)/(14, 23)/(1, 234)$ from (3.18) are taken, so $5 \times 2 = 10$. For $\rho = \{1,4,2,3\}$, $(123, 4)/(12, 34)/(13, 24)/(14, 23)/(1, 234)$ from (3.18) are taken, so $5 \times 2 = 10$. Finally for $\rho = \{1,4,3,2\}$, $(12, 34)/(13, 24)/(14, 23)$ from (3.18) are taken, so $3 \times 2 = 6$.

Having about general discussions, now we demonstrate our algorithm by several examples.

3.2 Four-point dual traces

Under our imposed conditions (3.8), (3.9) and (3.10) the number of independent n's and τ's is $\frac{(4-1)!}{2} = 3$. We take the liberty to choose following three orderings (1234), (1243) and (1324) as our basis. Using our algorithm for G-matrix, equation (3.11) yields

$$n_{1234}^{1-loop} = (20 + 2N_c)\tau_{1234} + 20\tau_{1243} + 20\tau_{1324},$$
$$n_{1243}^{1-loop} = 20\tau_{1234} + (20 + 2N_c)\tau_{1243} + 20\tau_{1324},$$
$$n_{1324}^{1-loop} = 20\tau_{1234} + 22\tau_{1243} + (20 + 2N_c)\tau_{1324}. \quad (3.21)$$

The determinant of G-matrix is $\det(G) = 8N_c^2(30 + N_c)$ for generic N_c, from which we derive the solution for τ_{1234},

$$\tau_{1234} = \frac{1}{2N_c(30 + N_c)} \left((20 + N_c)n_{1234} - 10n_{1243} - 10n_{1324} \right). \quad \text{[4pt-tau]} \quad (3.22)$$

Expressions of other orderings τ_ρ can be obtained by relabeling symmetry.

This result seems to differ from the result previously obtained for $N = 4$ SYM theory in [38]. To connect the two results, notice that for $N = 4$ SYM, only have box diagrams contribute and the corresponding n is given as

$$n_{abcd}^{1-loop} = s_{ab}s_{ad}A^{tree}(a, b, c, d). \quad \text{[4pt-N=4-n]} \tag{3.23}$$

Substituting (3.23) into (3.22) and using tree-level amplitude relation to write all the four-point tree amplitudes in terms of $A(1234)$, we get

$$\tau_{1234}^{1-loop} = \frac{1}{62}stA^{tree}(1, 2, 3, 4). \tag{3.24}$$

which is just the result given by Bern and Dennen when N_c is chosen to be 1.

Our result (3.22) has a free parameter N_c. It is easy to see that N_c can not be 0 or -30 because for these two values, determinant of G-matrix is zero, i.e., G-matrix is degenerated and we can not solve τ by n_α. Also, with particular choice of N_c, we may get simpler expressions. For example, when $N_c = -10$ we get

$$\tau_{1234} = \frac{-1}{40}\left(+n_{1234} - n_{1243} - n_{1324}\right) \tag{3.25}$$

while when $N_c = -20$ we get

$$\tau_{1234} = \frac{1}{40}\left(n_{1243} + n_{1324}\right) \tag{3.26}$$

3.3 Five-point case

Let us apply the same algorithm to five points. The relation between DDM basis numerator n and dual trace τ is given as

$$n_{a_1a_2a_3a_4a_5} = (30 + 2N_c)\tau_{\{a_1a_2a_3a_4a_5\}} + 12\tau_{\{a_1a_2a_3a_5a_4\}} + 12\tau_{\{a_1a_2a_4a_3a_5\}} + 12\tau_{\{a_1a_3a_2a_4a_5\}}$$

$$+6\tau_{\{a_1a_2a_4a_5a_3\}} + 6\tau_{\{a_1a_2a_5a_3a_4\}} + 6\tau_{\{a_1a_3a_4a_2a_5\}} + 6\tau_{\{a_1a_4a_2a_3a_5\}}$$

$$-12\tau_{\{a_1a_2a_5a_4a_3\}} - 12\tau_{\{a_1a_4a_3a_2a_5\}} - 6\tau_{\{a_1a_3a_2a_5a_4\}} + 0\tau_{\{a_1a_3a_5a_2a_4\}} \quad \text{[5-point-rel]} \tag{3.27}$$

The number of independent ns and τs is $(5 - 1)!/2 = 12$. We choose the our basis to be (12345), (12354), (12435), (12453), (12434), (12543), (13245), (13254), (13425), (13524), (14235), (14325) in order, which leads to following matrix G

$$G = \begin{pmatrix}
2Nc + 30 & 12 & 12 & 6 & 6 & -12 & 12 & -6 & 6 & 0 & 6 & -12 \\
12 & 2Nc + 30 & 6 & -12 & 12 & 6 & -6 & 12 & 0 & 6 & 12 & -6 \\
12 & 6 & 2Nc + 30 & 12 & -12 & 6 & 6 & 0 & -12 & 6 & 12 & 6 \\
6 & -12 & 12 & 2Nc + 30 & 6 & 12 & 12 & -6 & -6 & -12 & -6 & 0 \\
6 & 12 & -12 & 6 & 2Nc + 30 & 12 & 0 & 6 & 6 & -12 & -6 & -12 \\
-12 & 6 & 6 & 12 & 12 & 2Nc + 30 & -6 & 12 & -12 & -6 & 0 & 6 \\
12 & -6 & 6 & 12 & 0 & -6 & 2Nc + 30 & 12 & 12 & 6 & -12 & 6 \\
-6 & 12 & 0 & -6 & 6 & 12 & 12 & 2Nc + 30 & 6 & 12 & -6 & 12 \\
6 & 0 & -12 & -6 & 6 & -12 & 12 & 6 & 2Nc + 30 & -12 & 6 & 12 \\
0 & 6 & 6 & -12 & -12 & -6 & 6 & 12 & -12 & 2Nc + 30 & -12 & -6 \\
6 & 12 & 12 & -6 & -6 & 0 & -12 & -6 & 6 & -12 & 2Nc + 30 & 12 \\
-12 & -6 & 6 & 0 & -12 & 6 & 6 & 12 & 12 & -6 & 12 & 2Nc + 30
\end{pmatrix} \tag{3.28}$$

with determinant $\det(G) = 2^{12}N_c^6(N_c+30)^6$. Therefore the solution is

$$\tau_{\{12345\}} = \frac{1}{2N_c(30+N_c)}\{(15+N_c)n_{12345} - 6n_{12354} - 6n_{12435} - 3n_{12453}$$

$$-3n_{12534} + 6n_{12543} - 6n_{13245} + 3n_{13254} - 3n_{13425} - 3n_{14235} + 6n_{14325}\} \qquad (3.29)$$

Other τ's can be obtained using relabeling symmetry. For this expression, if we choose $N_c = -15$, all coefficients are $\frac{\pm 2}{150}$ and $\frac{1}{150}$. Especially the first coefficients $(15+N_c) \to 0$.

3.4 Six-point example

At six-points, the basis can be labeled by the following $(6-1)!/2 = 60$ orderings

$$\{1,2,3,4,5,6\}, \{1,2,3,4,6,5\}, \{1,2,3,5,4,6\}, \{1,2,3,5,6,4\}, \{1,2,3,6,4,5\}, \{1,2,3,6,5,4\},$$

$$\{1,2,4,3,5,6\}, \{1,2,4,3,6,5\}, \{1,2,4,5,3,6\}, \{1,2,4,5,6,3\}, \{1,2,4,6,3,5\}, \{1,2,4,6,5,3\},$$

$$\{1,2,5,3,4,6\}, \{1,2,5,3,6,4\}, \{1,2,5,4,3,6\}, \{1,2,5,4,6,3\}, \{1,2,5,6,3,4\}, \{1,2,5,6,4,3\},$$

$$\{1,2,6,3,4,5\}, \{1,2,6,3,5,4\}, \{1,2,6,4,3,5\}, \{1,2,6,4,5,3\}, \{1,2,6,5,3,4\}, \{1,2,6,5,4,3\},$$

$$\{1,3,2,4,5,6\}, \{1,3,2,4,6,5\}, \{1,3,2,5,4,6\}, \{1,3,2,5,6,4\}, \{1,3,2,6,4,5\}, \{1,3,2,6,5,4\},$$

$$\{1,3,4,2,5,6\}, \{1,3,4,2,6,5\}, \{1,3,4,5,2,6\}, \{1,3,4,6,2,5\}, \{1,3,5,2,4,6\}, \{1,3,5,2,6,4\},$$

$$\{1,3,5,4,2,6\}, \{1,3,5,6,2,4\}, \{1,3,6,2,4,5\}, \{1,3,6,2,5,4\}, \{1,3,6,4,2,5\}, \{1,3,6,5,2,4\},$$

$$\{1,4,2,3,5,6\}, \{1,4,2,3,6,5\}, \{1,4,2,5,3,6\}, \{1,4,2,6,3,5\}, \{1,4,3,2,5,6\}, \{1,4,3,2,6,5\},$$

$$\{1,4,3,5,2,6\}, \{1,4,3,6,2,5\}, \{1,4,5,2,3,6\}, \{1,4,5,3,2,6\}, \{1,4,6,2,3,5\}, \{1,4,6,3,2,5\},$$

$$\{1,5,2,3,4,6\}, \{1,5,2,4,3,6\}, \{1,5,3,2,4,6\}, \{1,5,3,4,2,6\}, \{1,5,4,2,3,6\}, \{1,5,4,3,2,6\} \qquad (3.30)$$

The expansion coefficients of n_{123456} into τs, i.e, $G[\{123456\}|\rho] = G_{1i}$ $i = 1,...,60$ respectively by the orderings listed above, is given by

$$G_{1i} = \{62+2N_c, \quad 34,34,22,22,12,34,22,22,22,$$

$$18, \quad 16,22,18,12,16,26,22,22,16,$$

$$16, \quad 22,22,34,34,22,22,16,16,22,$$

$$22, \quad 16,22,18,18,12,16,18,18,18,$$

$$12, \quad 18,22,16,18,12,12,22,16,18,$$

$$26, \quad 22,18,18,22,16,16,22,22,34\}, \qquad (3.31)$$

Other G_{ij} can be obtained by relabeling symmetry. The determinant of matrix G is

$$\det(G) = 2^{60}N_c^{24}(N_c+18)^5(N_c+21)^{16}(N_c+56)^9(N_c+60)^5(N_c+630), \qquad (3.32)$$

and the solution is given by

$$\tau_\sigma = G^{-1}[\sigma|\rho]n_\rho \tag{3.33}$$

The inverse of matrix G is very complicated, but with relabeling symmetry, it is enough to give the first row, i.e., G_{1i}^{-1} with $i = 1, ..., 60$. To have a feeling about the N_c-dependence, we list all 60 elements as following:

$$G_{11,12,13,14,15}^{-1} = \left\{ \frac{1}{120}\left(\frac{5}{N_c+18} + \frac{16}{N_c+21} + \frac{9}{N_c+56} + \frac{5}{N_c+60} + \frac{1}{N_c+630} + \frac{24}{N_c} \right), \right.$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} + \frac{99}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{176}{N_c} \right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} + \frac{99}{N_c+56} + \frac{35}{N\,c+60} + \frac{21}{N_c+630} - \frac{176}{N_c} \right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{28}{N_c+21} + \frac{27}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{76}{N_c} \right),$$

$$\left. \frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{28}{N_c+21} + \frac{27}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{76}{N_c} \right) \right\} \tag{3.34}$$

$$G_{16,17,18,19,1(10)}^{-1} = \left\{ \frac{1}{2520}\left(-\frac{105}{N_c+18} + \frac{45}{N_c+56} - \frac{105}{N_c+60} + \frac{21}{N_c+630} + \frac{144}{N_c} \right), \right.$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} + \frac{99}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{176}{N_c} \right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18} - \frac{112}{N_c+21} + \frac{27}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{64}{N_c} \right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{28}{N_c+21} + \frac{27}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{76}{N_c} \right),$$

$$\left. \frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{28}{N_c+21} + \frac{27}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{76}{N_c} \right) \right\} \tag{3.35}$$

$$G_{1(11),1(12),1(13),1(14),1(15)}^{-1} = \left\{ \frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} - \frac{81}{N_c+56} + \frac{35}{Nc+60} + \frac{21}{N_c+630} + \frac{4}{N_c} \right), \right.$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18} - \frac{56}{N_c+21} - \frac{9}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{44}{N_c} \right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{28}{N_c+21} + \frac{27}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{76}{N_c} \right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} - \frac{81}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{4}{N_c} \right),$$

$$\frac{1}{2520}\left(-\frac{105}{N_c+18}+\frac{45}{N_c+56}-\frac{105}{N_c+60}+\frac{21}{N_c+630}+\frac{144}{N_c}\right)\bigg\}\qquad(3.36)$$

$$G^{-1}_{1(16),1(17),1(18),1(19),1(20)}=\bigg\{\frac{1}{2520}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{44}{N_c}\right),$$

$$\frac{1}{840}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}+\frac{35}{N_c+60}+\frac{7}{N_c+630}-\frac{12}{N_c}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}-\frac{112}{N_c+21}+\frac{27}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{64}{N_c}\right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18}+\frac{28}{N_c+21}+\frac{27}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}-\frac{76}{N_c}\right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{44}{N_c}\right)\bigg\}(3.37)$$

$$G^{-1}_{1(21),1(22),1(23),1(24),1(25)}=\bigg\{\frac{1}{2520}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{44}{N_c}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}-\frac{112}{N_c+21}+\frac{27}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{64}{N_c}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}-\frac{112}{N_c+21}+\frac{27}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{64}{N_c}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}+\frac{56}{N_c+21}+\frac{99}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}-\frac{176}{N_c}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}+\frac{56}{N_c+21}+\frac{99}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}-\frac{176}{N_c}\right)\bigg\}(3.38)$$

$$G^{-1}_{1(26),1(27),1(28),1(29),1(30)}=\bigg\{\frac{1}{2520}\left(-\frac{35}{N_c+18}-\frac{112}{N_c+21}+\frac{27}{N_c+56}+\frac{35}{Nc+60}+\frac{21}{N_c+630}+\frac{64}{N_c}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}-\frac{112}{N_c+21}+\frac{27}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{64}{N_c}\right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{44}{N_c}\right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{44}{N_c}\right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18}+\frac{28}{N_c+21}+\frac{27}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}-\frac{76}{N_c}\right)\bigg\}(3.39)$$

$$G^{-1}_{1(31),1(32),1(33),1(34),1(35)} = \left\{ \frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{28}{N_c+21} + \frac{27}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{76}{N_c} \right), \right.$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18} - \frac{56}{N_c+21} - \frac{9}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{44}{N_c} \right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{28}{N_c+21} + \frac{27}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{76}{N_c} \right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} - \frac{81}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{4}{N_c} \right),$$

$$\left. \frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} - \frac{81}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{4}{N_c} \right) \right\} \quad (3.40)$$

$$G^{-1}_{1(36),1(37),1(38),1(39),1(40)} = \left\{ \frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{112}{N_c+21} - \frac{117}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{16}{N_c} \right), \right.$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18} - \frac{56}{N_c+21} - \frac{9}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{44}{N_c} \right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} - \frac{81}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{4}{N_c} \right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} - \frac{81}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{4}{N_c} \right),$$

$$\left. \frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} - \frac{81}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{4}{N_c} \right) \right\} \quad (3.41)$$

$$G^{-1}_{1(41),1(42),1(43),1(44),1(45)} = \left\{ \frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{112}{N_c+21} - \frac{117}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{16}{N_c} \right), \right.$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} - \frac{81}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{4}{N_c} \right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{28}{N_c+21} + \frac{27}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{76}{N_c} \right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18} - \frac{56}{N_c+21} - \frac{9}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{44}{N_c} \right),$$

$$\left. \frac{1}{2520}\left(-\frac{35}{N_c+18} + \frac{56}{N_c+21} - \frac{81}{N_c+56} + \frac{35}{N_c+60} + \frac{21}{N_c+630} + \frac{4}{N_c} \right) \right\} \quad (3.42)$$

$$G^{-1}_{1(46),1(47),1(48),1(49),1(50)} = \left\{ \frac{1}{2520}\left(\frac{35}{N_c+18} + \frac{112}{N_c+21} - \frac{117}{N_c+56} - \frac{35}{N_c+60} + \frac{21}{N_c+630} - \frac{16}{N_c} \right), \right.$$

$$\frac{1}{2520}\left(-\frac{105}{N_c+18}+\frac{45}{N_c+56}-\frac{105}{N_c+60}+\frac{21}{N_c+630}+\frac{144}{N_c}\right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18}+\frac{28}{N_c+21}+\frac{27}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}-\frac{76}{N_c}\right),$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{44}{N_c}\right),$$

$$\left.\frac{1}{2520}\left(-\frac{35}{N_c+18}+\frac{56}{N_c+21}-\frac{81}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{4}{N_c}\right)\right\} \quad (3.43)$$

$$G^{-1}_{1(51),1(52),1(53),1(54),1(55)} = \left\{\frac{1}{840}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}+\frac{35}{N_c+60}+\frac{7}{N_c+630}-\frac{12}{N_c}\right),\right.$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}-\frac{112}{Nc+21}+\frac{27}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{64}{Nc}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}+\frac{56}{N_c+21}-\frac{81}{N_c+56}+\frac{35}{textNc+60}+\frac{21}{N_c+630}+\frac{4}{N_c}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}+\frac{56}{N_c+21}-\frac{81}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{4}{N_c}\right),$$

$$\left.\frac{1}{2520}\left(\frac{35}{N_c+18}+\frac{28}{N_c+21}+\frac{27}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}-\frac{76}{N_c}\right)\right\} \quad (3.44)$$

$$G^{-1}_{1(56),1(57),1(58),1(59),1(60)} = \left\{\frac{1}{2520}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{44}{N_c}\right),\right.$$

$$\frac{1}{2520}\left(\frac{35}{N_c+18}-\frac{56}{N_c+21}-\frac{9}{N_c+56}-\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{44}{N_c}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}-\frac{112}{N_c+21}+\frac{27}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{64}{N_c}\right),$$

$$\frac{1}{2520}\left(-\frac{35}{N_c+18}-\frac{112}{N_c+21}+\frac{27}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}+\frac{64}{N_c}\right),$$

$$\left.\frac{1}{2520}\left(-\frac{35}{N_c+18}+\frac{56}{N_c+21}+\frac{99}{N_c+56}+\frac{35}{N_c+60}+\frac{21}{N_c+630}-\frac{176}{N_c}\right)\right\} \quad (3.45)$$

Remarks

Before concluding this section, let us make a few remarks on the degrees of freedom introduced by N_c. First we notice that $\det(G)$ will depend on N_c, thus there are solutions of N_c such that $\det(G) = 0$. When this happens, $G \cdot \tau = n$ will not have solution. In other words, for these specific values, N_c and the imposed loop-KK relations are not compatible to each other. At this moment, we are not clear what is the physical

meaning of these degenerated values of N_c. However from explicit examples discussed above, it seems that N_c that lead to degenerating matrix G are always negative integer. For positive N_c there is no problem for it. It is perhaps possible to choose special values of N_c such that the final expression dramatically simplifies or manifest patten can be observed.

4. An alternative approach

In previous section, we solve τ by n using the G-matrix directly. Since all conditions we imposed, such as (3.8), (3.9), (3.10) and (3.11), are relabeling symmetric, the solutions τ_σ for different ordering σ's are also related by relabeling symmetry. This property can be used to solve τ without using the G-matrix, which will be the purpose of this section. In fact, similar method has been used in tree-level case in [40]. For simplicity, in this section we assume $N_c = 1$.

Four-point example:

In the four-point case we assume that τ can be expanded by n, i.e.,

$$\tau_{1234} = an_{1234} + bn_{1243} + cn_{1324}. \tag{4.1}$$

Under the relabeling $1 \leftrightarrow 2$, we get

$$\tau_{2134} = an_{2134} + bn_{2143} + cn_{2314}, \tag{4.2}$$

which can be recast into the original basis using reflection and cyclic symmetry of τ and n

$$\tau_{1342} = an_{1243} + bn_{1234} + cn_{1324}. \tag{4.3}$$

Same τ_{1342} can also obtained from τ_{1234} by relabeling $2 \to 3, 3 \to 4, 4 \to 2$, thus we arrive following equation

$$\tau_{1342} = an_{1342} + bn_{1324} + cn_{1432} = an_{1243} + bn_{1324} + cn_{1234}. \tag{4.4}$$

By comparing the τ_{1342} in this two different ways, we can get

$$b = c. \tag{4.5}$$

Thus

$$\tau_{abcd} = an_{abcd} + b(n_{abdc} + n_{acbd}). \tag{4.6}$$

Substituting this into the relation between n and τ (3.15), we get

$$a = \frac{21}{62}, \quad b = -\frac{5}{31}. \tag{4.7}$$

Then

$$\tau_{abcd} = \frac{21}{62}n_{abcd} - \frac{5}{31}(n_{abdc} + n_{acbd}). \tag{4.8}$$

This is the same with the result obtained by imposing KK relation and then solving linear equations.

Five-point expansion

Similarly at five-points, we assume the dual trace can be expanded into the $(5-1)!/2 = 12$ basis numerators $n_{1,\sigma}$ discussed in section 3.3,

$$\tau_{12345} = \sum_{\sigma \in S_4/R} c_{1,\sigma} n_{1,\sigma} \tag{4.9}$$
$$= c_{12345} n_{12345} + c_{12354} n_{12354} + \cdots + c_{14325} n_{14325},$$

where R denotes reflection. Comparing the expansion expressions derived through permutating leg 1 with 2, 3, 4, 5 with the corresponding expressions obtained by relabeling, we get the following relations

$$\tau_{21345} = -\tau_{12543} \longrightarrow \quad c_{12453} = c_{12534}, \quad c_{12354} = c_{12435}, \tag{4.10}$$
$$c_{13254} = -c_{13425}, \; c_{13245} = -c_{14325},$$

$$\tau_{32145} = -\tau_{12354} \longrightarrow \quad c_{12534} = c_{14235}, \quad c_{12543} = -c_{13245}, \tag{4.11}$$
$$c_{12435} = -c_{14325}, \; c_{12453} = -c_{13254},$$

$$\tau_{42315} = -\tau_{13245} \longrightarrow c_{13254} = -c_{14235}, \; c_{12453} = c_{13425} \tag{4.12}$$
$$c_{12435} = -c_{12543}, \; c_{12354} = -c_{14325},$$

$$\tau_{52341} = -\tau_{14325} \longrightarrow c_{12534} = -c_{13254}, \; c_{13425} = c_{14235}, \tag{4.13}$$
$$c_{12354} = -c_{12543}, \; c_{12435} = c_{13245}.$$

Relabling symmetry therefore reduces the number of independent coefficients to four, yielding

$$\tau_{12345} = a\, n_{12345} + +b\, (n_{12453} + n_{12534} - n_{13254} + n_{13425} + n_{14235}) \tag{4.14}$$
$$c\, n_{13524} - d\, (n_{12354} + n_{12435} - n_{12543} + n_{13245} - n_{14325}),$$

while the other basis dual traces τs can be obtained by relabelings of legs 2, 3, 4 and 5. Substituting these expressions back to just one relation (3.27) allows us to fully determine the remaining all four coefficients. Again, we arrive at

$$\tau_{\{12345\}} = \frac{1}{62}\, (16 n_{12345} - 6 n_{12354} - 6 n_{12435} - 3 n_{12453} - 3 n_{12534} + 6 n_{12543}$$
$$-6 n_{13245} + 3 n_{13254} - 3 n_{13425} - 3 n_{14235} + 6 n_{14325}). \tag{4.15}$$

5. Conclusion

In this work, we have discussed two kinds of dual-color decompositions in Yang-Mills theory at one-loop level. These are the dual-DDM decomposition and the dual-trace decomposition. In both cases, the color-dressed Yang-Mills integrands can be decomposed in terms of color-ordered scalar amplitudes. We constructed the dual color factors in dual DDM-form by applying Jacobi-like identity for kinematic factors in double-copy formula. We also constructed the dual-trace factors by imposing KK relation, reflection relation and the relation with the kinematic factor in dual DDM-form.

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