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Some remarks on weak measurements and weak values [1].

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Abstract. After a brief review of how weak measurements plus post-selection give rise to the weak value concept, I concentrate on its application to the so called Leggett-Garg inequalities, speculating also on what relevance these results may have to the basic assumptions behind the Bell inequality. I then criticize an interpretation of a weak value as any sort of *bona fide* property of the system under investigation. I even question whether the system can at all be said to "carry" its weak value.

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1. Introduction

The pioneering work by Aharonov, Albert and Vaidman [1] introduced the concept of 'weak measurements' of an observable in quantum mechanics (QM) in combination with the procedure of 'post-selection', resulting in the concept of a 'weak value'. The basic idea of weak measurement + post-selection has since attracted much interest; for recent reviews with further references see, *e.g.*, [2-5].

Not the least has the field opened up new tools for experimentalists to investigate aspects of phenomena that were thought impossible earlier. I refer to any of the reviews in [2 – 5] for details.

Weak measurements have also been employed to illuminate the fundamental difference between classical and quantum mechanics exhibited by violation of Leggett-Garg inequalities [3, 6 - 10] [2], also called "Bell inequalities in time".

In addition, weak values of number operators have been invoked to revisit what are conceived as QM paradoxes [11,12], like the so called Three-Box Paradox [13 - 15] and Hardy's Paradox [16 - 19]

This paper reviews briefly – without any pretention of completeness – some of these applications. In particular, I point out the important use of weak measurement in analyzing the Leggett-Garg inequalities, relating it also to the Bell inequalities [20]. I also criticize the interpretation of a weak value as an ordinary property of the system under investigation, an assumption which lies behind the application of weak values to paradox-solving.

[1] Synopsis of talk given at the QTPA Conference in Växjö, Sweden, June 2013.
[2] Reference [23] is a review of the Leggett-Garg inequalities that became available to me only after the conference.

2. Operational definition of weak value

The weak value can be introduced using ideas on how to describe a measurement that go back to von Neumann [21]. Here, I give but a brief outline, referring to, *e.g.*, [3] for a more thorough treatment and for some other situations where a weak value applies.

The purpose is to describe a 'system' $\mathcal{S}$ that is subject to a 'measurement' by an apparatus, a 'meter', $\mathcal{M}$. Both $\mathcal{S}$ and $\mathcal{M}$ are described quantum-mechanically. Initially, the system is supposed to be in a 'pre-selected' state $|s>$, the meter in a state $|m>$, and the joint system in the product state $|s> \otimes |m>$. The system then interacts with the apparatus in a 'pre-measurement', described by a unitary operator $U = exp\ (-i \int H_{int}\ dt\ )$ in units such that $\hbar = 1$. It is taken for granted that there is no further time evolution. One assumes a short interaction time, and a coupling between the apparatus and the system given by $H_{int} \sim S \otimes P_M$. Here, S is the operator for the property of the system under investigation, while P_M is the momentum operator for the meter such that the corresponding position operator Q_M is the pointer operator, having the pointer states $|q>$ of the meter as its eigenstates. One may then write $U = exp\ (-\ i\ g\ S \otimes P_M)$ with g a measure of the strength of the interaction. Without reading off the meter, one next subjects the system to a 'post-selection', projecting it onto a chosen state $|f>$. This operation causes the meter into the state $< f\,|U\,|(|s> \otimes |m>)$. In the weak measurement limit, defined by letting g tend to zero, one deduces for the mean value $< Q >_f$ of the meter pointer variable Q, subject to the constraint that the system is in the state $|f>$,

$$< Q >_f\ =\ g\, S_w\ +\ O(g^2), \qquad (1)$$

where the weak value S_w is defined by

$$S_w\ :=\ <f\,|S|s>\ /\ <f\,|s>. \qquad (2)$$

Here, and throughout this paper, I assume S_w to be real; the more general case of complex S_w requires a slightly more general formal analysis (see , *e.g.*, [3]), but does not really add any essentials to the more fundamental aspects treated here.

The relations (1) and (2) are the basis for all my subsequent arguments in this paper.

3. Application to a Leggett-Garg inequality.

From a few reasonable assumptions on what characterizes the macro-world, Leggett and Garg [6] were able to deduce inequalities involving (in general different) measurements performed at successive times but on one and the same system. These LGI inequalities are then generalized to the micro-domain. The LGI are particularly well adapted for tests in weak measurements.

The assumptions invoked by Leggett and Garg are, slightly reformulated,

Macroscopic realism (MAR): A macroscopic system will at all times be in one or the other of the states available to it.

Noninvasive detection (NID): It is possible to determine the state of the system with arbitrary small perturbations on its subsequent dynamics.

Leggett and Garg than consider three successive measurements of observables A_0, A_1 and A_2 for a system with results a_i , $i = 0,1,2$, and with all the a_i assumed to be in the interval $[-1, +1]$. It is then straight-forward to show that the entity

$$B\ :=\ a_0\, a_1 + a_1\, a_2 - a_2 a_0 \qquad (4)$$

obeys the inequality

$$-\ 3\ \ \leq\ B\ \leq\ \ 1\ . \qquad (5)$$

From the assumptions MAR and NID it then follow that this inequality remains intact after averaging B over the probability distributions for obtaining the respective values. So with

$$< B >\ :=\ < A_0\, A_1 > + < A_1\, A_2 > - < A_2\, A_0 >, \qquad (6)$$

(here, bra-kets $< .. >$ denote averages over the respective probability distributions), one arrives at

$$-\,3 \;\leq\; < B > \;\leq\; 1. \tag{7}$$

This inequality is then taken over to the micro-domain by interpreting the averages as QM expectation values in an initial state prepared before the measurement of A_0 takes place. It constitutes (one of) the Leggett-Garg inequality (-ties), LGI.

In applications, one posits that a weak measurement (approximately) obeys the NID assumption. One may even relax a bit on the requirement of weak measurements. In fact, the main steps in the derivation of LGI really only require the middle measurement, that of A_1, to obey NID and, thus, to be weak. Both initially and finally, for A_0 respectively A_2, the measurement could be projective.

The actual applications [7 - 10] even choose $A_0 = |s >< s|$, the projection operator on the initial ('pre-selected') state. The Leggett-Garg entity then reads

$$< B > = < s|A_1|s > + ½ \; < s|\{A_1 \, , A_2\}|s > \; - < s|A_2|s > . \tag{8}$$

Here, I have also replaced the operator product $A_1\, A_2$ with half the anti-commutator; some such trick is needed to compensate for the fact that the order of two non-commuting operators matters, and to insure that $< B >$ will be a real number.

As said, such an LGI is aptly suited for studies using weak measurement: the condition NID of noninvasive measurement can be considered (approximately) fulfilled. Therefore, with weak measurement, LGI effectively only tests the macro-realism assumption, MAR.

In the experimental realizations, different choices for A_1 and A_2 have been investigated [7 - 10].

As in [3], one could fit the LGI even more closely to apply to the weak measurement + post-selection approach by making the substitutions

$$A_0 \hookrightarrow |s >< s\,| \tag{9a}$$

$$A_1 \hookrightarrow S \;, \tag{9b}$$

$$A_2 \hookrightarrow |f >< f\;| \;, \tag{9c}$$

which, after a trivial calculation, results in an Leggett-Garg inequality taking the form

$$-\,3 \quad\leq\; < B_w > \;\leq\; 1 \;, \tag{10}$$

with

$$< B_w > \; = \; < s\,|\,S|\,s > \; + \; |< f\,|s >|^{\,2} \; (S_w \; - \; 1 \;), \tag{11}$$

explicitly exhibiting the dependence of $< B_w >$ on the weak value S_w.

It is not difficult to find situations in which this inequity is violated. For example, if the system is a qubit in the state $|s > = \alpha\,|+> \; + \; \beta\,|->$, if S is chosen to be $= \sigma_z$, and with $|\,f> = cos\;\theta\;\;|+> \; + \; sin\;\theta\;\;|->$, one indeed find that $< B_w >$ takes a maximum value = 1.083 for $\beta \;= -\,1/6 = cos\;2\theta$, thus violating the Leggett-Garg inequality.

The conclusion from this deliberation, in combination with the experimentally found violations of other Leggett-Garg inequalities, is the following: Since in weak measurement the Leggett-Garg assumption of non-invasive detection is (approximately) fulfilled, the other assumption, that of macroscopic realism must be violated: quantum mechanics as well as nature do not respect the assumption of macroscopic definiteness.

The following speculation comes naturally. Let me combine the analysis of the Leggett-Garg inequalities just presented with that of the Bell inequality [20]. The latter requires two assumptions for its derivation, locality and no-hidden-variables (or something equivalent to that). To the extent that the no-hidden-variable assumption of the Bell inequality is equivalent to the macro-realism assumption of the Leggett-Garg inequalities –this is the crucial point – and since that latter assumption has been shown to be violated, this leaves room for blaming the no-hidden-variable assumption for the violations of Bell-s inequalities and thus leaving the locality assumption to be valid. It is important to analyze this situation further, in particular to find out whether the assumption MAR of the Leggett-

Garg inequalities is indeed logically the same as the no-hidden-variable assumption of the Bell inequality.

4. Deconstructing the weak value

Aharonov and his collaborators have made much an affair of using weak values to resolve some conceived paradoxes of quantum mechanics, among them the so called three-box paradox and Hardy's paradox [11 - 14,17]. They claim that some of the so called 'counter-factual' statements in these 'paradoxes' can be made 'factual' by using weak measurements, quantified in terms of weak values, instead of the ordinary 'strong' measurements. These claims rely on a straight-forward interpretation of a weak value on *a par* with, *e.g.*, a usual mean value. That is, they ascribe a meaning to a weak value as an ordinary property – like a probability, the number of particles, *etc.* – of the system under investigation.

In [3] and [22], I have criticized this use of the weak value; I refer to these articles for further details. My contention is that one is not allowed to ascribe to a weak value any such straight-forward interpretation.

My criticism contains two main points.

The first is that there is no support in the basic QM postulates for interpreting the weak value as an ordinary property. Indeed, as has been pointed out in particular by Kastner [23], the weak value is an amplitude (or, better, a ratio between two amplitudes) which, from the QM rules, cannot be interpreted as an ordinary property of the system.

The second criticism starts from the observation that a weak value S_w of an operator S depends not only on the initial, 'pre-selected' state $|s>$ but also on the final, 'post-selected' state $|f>$. And by varying this final state, it is possible, also in experimentally reasonable situations with S a number operator, to get essentially any value for S_w. And to accept a negative (or even a complex) value for a number operator requires strong deviations from orthodoxy, (as in , *e.g.*, [24] and references therein)! For a detailed analysis , see [22].

One may even deepen this argument somewhat. To start with, the weak value, in its definition (2), only refers to quantities pertaining to the system alone. In principle, then, one should be able to deduce a weak value from ordinary, projective measurements on the system by itself: both the nominator $<f|S|s>$ and the denominator $<f|s>$ in S_w are in principle amenable to experimental determination separately (at least statistically and possibly except for a phase factor). In principle, then, a weak value could be reconstructed from other measurement but weak ones. The virtue of a weak measurement may then be seen in the fact that it is a one-step procedure for determining S_w (and without phase ambiguities). But a weak measurement carries no preferential role when it comes to interpreting a weak value.

Let me look upon this issue from another angle. Does the system in any way "carries" its weak value in the weak measurement procedure? Put differently: Can one ascribe the weak value to the system during any part of the weak measurement process? These questions touch on deep QM interpretational issues. My doubt in answering them in an affirmative way is based on a few simple observations.

The first is that one essential point in the weak measurement procedure – as in any measurement – is that the system becomes entangled with the meter through the pre-measurement U of section 2 above. Another is that a *raison d'être* for the weak measurement is its leaving the system as undisturbed as possible; in fact, as is most clearly seen using a density matrix formulation [3], the system density matrix after the pre-measurement equals the initial density matrix up to terms of second order in the measurement strength g (with the diagonal elements being even more suppressed;

see, *e.g.*, eq. (99) in [3]). So, in this way, the state of the system by itself after the weak pre-measurement contains no reference to any weak value. Nor does it after the post-selection, which by definition projects the system state into the final state $|f>$. Nor, either, does the weak value sits unequivocally with the meter in between the pre-measurement and the post-selection. In fact, it appears for the first time in the meter after the system has been subjected to the post-selection. Does this mean that the system "has" or "know of" its weak value?

5. Summary

I have investigated how weak measurements may be employed in connection with a Leggett-Garg inequality. Such an inequality is based on two assumptions, *viz.*, macroscopic realism (MAR) and non-invasive detection (NID. The the fact that nature as well as quantum mechanics violate the inequality in weak measurements, approximately obeying the NID assumption as it does, means that MAR must be violated. I speculate that, to the extent that the MAR assumption is equivalent to the no-hidden-variable assumption of the Bell inequality, it is possible to uphold the locality assumption in drawing conclusions from the violation of that latter inequality. These matters will require further investigations.

I then criticize the interpretation of a weak value as a *bona fide* property of the system under investigation. In particular, it cannot be used in "explaining" quantum mechanical paradoxes. Indeed, interpreting a weak value as an ordinary property has support neither in the basic postulates of quantum mechanics nor in its dependence on the post-selection. I even argue that the very way a weak value is defined raises doubts on whether the system in any way can be said to "have" or "know" or "carry" its weak value.

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