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Mesonic low-energy constants

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Abstract

We review the status of the coupling constants of chiral Lagrangians in the meson sector, the so-called low-energy constants (LECs). Special emphasis is put on the chiral $SU(2)$ and $SU(3)$ Lagrangians for the strong interactions of light mesons. The theoretical and experimental input for determining the corresponding LECs is discussed. In the two-flavour sector, we review the knowledge of the $O(p^4)$ LECs from both continuum fits and lattice QCD analyses. For chiral $SU(3)$, NNLO effects play a much bigger role. Our main new results are fits of the LECs L_i both at NLO and NNLO, making extensive use of the available knowledge of NNLO LECs. We compare our results with available lattice determinations. Resonance saturation of LECs and the convergence of chiral $SU(3)$ to NNLO are discussed. We also review the status of predictions for the LECs of chiral Lagrangians with dynamical photons and leptons.

Contents

1 INTRODUCTION

Low-energy meson physics and the study of the strong interaction at low energies underwent a phase transition in the theoretical description with the introduction of Chiral Perturbation Theory (ChPT) in the early 1980s [1–3]. It allowed the theory of the lightest hadrons, the pions, kaons and eta, to be put on a solid theoretical footing. The main idea is that rather than a perturbative expansion in a small parameter like α or α_S, there is a well-defined perturbation theory as an expansion in orders of momenta and masses. ChPT was also the prototype effective field theory, showing how to make sense of nonrenormalizable theories in a well-defined fashion.

The predictions of ChPT are of twofold type. There are the loop contributions at each order and the contributions that involve the parameters of the higher-order Lagrangians. This review summarizes the present knowledge of the values of these parameters. The standard name for these parameters is low-energy constants (LECs). The main part of this review concerns the LECs of two- and three-flavour mesonic ChPT in the isospin limit. Section 2 gives an overview of the Lagrangians and serves to define our notation.

A first determination of the LECs was done in the papers where they were introduced [2, 3]. For those of the two-flavour or $n_f = 2$ case, the various LECs at next-to-leading order (NLO) can be determined in a rather straightforward fashion. The main analyses have been pushed to next-to-next-to-leading order (NNLO). This is reviewed in Sec. 3 where we discuss the theoretical and experimental input to determine them.

The three-flavour or $n_f = 3$ coefficients were first determined in [3] by using large-N_c arguments, the relation with the $n_f = 2$ LECs and the pseudoscalar masses and F_K/F_π. The next step was to determine them from K_{l4} decays at NLO [4, 5]. The first attempt at adding higher-order effects in determining the $n_f = 3$ LECs was Ref. [6]. The first full calculations at NNLO in $n_f = 3$ mesonic ChPT appeared in the late 1990s and a first fit using these expressions for the LECs was done in [7,8]. At this level, there was not sufficient information to really determine all LECs at NLO directly from data, in particular L_4 and L_6 are very difficult to obtain. The underlying reason for this is discussed in Sec. 4.2. Another difficulty is that quark masses and LECs cannot be disentangled without using more information [9]. We fix this ambiguity by using the quark mass ratio $m_s/\hat{m}$ as input. More calculations became available and partial analyses were performed but a new complete analysis was done in [10]. The main improvement of the refitting done in this review over [10] is a more extensive use of knowledge of the NNLO LECs as discussed in Sec. 4.6. A minor improvement is the inclusion of some newer K_{l4} data. The data and theoretical input used in the $n_f = 3$ fits beyond that already used for the $n_f = 2$ results are described in Sec. 4.3. Our fitting and the new central values for the LECs are given in Sec. 4.4. The evidence for resonance saturation of both NLO and NNLO LECs is discussed in Sec. 4.6. The quality of the fits and the convergence of the chiral expansion are discussed in Sec. 4.7.

The LECs that show up in extensions with dynamical photons and leptons cannot be determined from phenomenology directly but need further treatment. We collect the known results in Sec. 4.8 where we pay close attention to the correct inclusion of short-distance contributions. For those involving the weak nonleptonic interaction we only give a short list of the main references in Sec. 2.1 and refer to the recent review [11] for more references and details. Likewise, we remain very cursory with respect to the anomalous intrinsic parity sector in Sec. 2.1.

Lattice QCD has started to make progress in the determination of LECs, especially for those involving masses and decay constants. We rely heavily on the flavour lattice averaging group (FLAG) reports [12, 13]. Specific results are quoted and compared with our continuum results in Secs. 3.3 and 4.5. Some comments can also be found in Sec. 4.2.

A summary of the main results can be found in the conclusions.

2 CHIRAL PERTURBATION THEORY

ChPT dates back to current algebra but its modern form was introduced by the papers of Weinberg, Gasser and Leutwyler [1–3]. The underlying idea is to use the global chiral symmetry present in the QCD Lagrangian for two ($n_f = 2$) or three ($n_f = 3$) light quarks when the quark masses are put to zero. This symmetry is spontaneously broken in QCD. The Nambu-Goldstone bosons resulting from this breaking are identified with the pions ($n_f = 2$) or the lightest pseudoscalar octet, π, K and η ($n_f = 3$). The singlet axial symmetry is broken explicitly for QCD at the quantum level due to the $U(1)_A$ anomaly and we thus disregard it. A direct derivation of ChPT from the underlying assumptions is given by Leutwyler [14].

The perturbation in ChPT is not an expansion in a small coupling constant but an expansion in momenta and quark masses. Its consistency was shown in detail in [1] and is often referred to as Weinberg or p power counting.

A more extensive introduction to ChPT can be found in [15]. There are many reviews of ChPT. Those focusing on the meson sector are two at the one-loop level [16,17] and one at the two-loop level [18].

In terms of a quark field $\bar q = (\bar u\ \bar d)$ ($n_f = 2$) or $\bar q = (\bar u\ \bar d\ \bar s)$ ($n_f = 3$) the fermionic part of the QCD Lagrangian can be written as

$$\mathcal{L}_{QCD} = \bar q i\gamma^\mu \left(\partial_\mu - ig_s G_\mu - i(v_\mu + \gamma_5 a_\mu)\right) q - \bar q s q + i\bar q p \gamma_5 q\,. \tag{1}$$

The external fields or sources v_μ, a_μ, s and p are $n_f \times n_f$ matrices in flavour space. They were introduced in [2,3] to make chiral symmetry explicit throughout the calculation and to facilitate the connection between QCD and ChPT. For later use we define $l_\mu = v_\mu - a_\mu$ and $r_\mu = v_\mu + a_\mu$.

The degrees of freedom are the Goldstone bosons of the spontaneous breakdown of the $SU(n_f)_L \times SU(n_f)_R$ chiral symmetry of QCD with n_f massless flavours to the vector subgroup $SU(n_f)_V$. These are parametrized by a special $n_f \times n_f$ unitary matrix U. The transformations under a chiral symmetry transformation $g_L \times g_R \in SU(n_f)_L \times SU(n_f)_R$ are

$$\begin{array}{rclr} U & \to & g_R U g_L^\dagger\,, & s + ip \to g_R(s+ip)g_L^\dagger\,, \\ l_\mu & \to & g_L l_\mu g_L^\dagger - i\partial_\mu g_L g_L^\dagger\,, & r_\mu \to g_R r_\mu g_R^\dagger - i\partial_\mu g_R g_R^\dagger\,. \end{array} \tag{2}$$

In addition we define u with $u^2 = U$ and $h(u, g_L, g_R)$ transforming as

$$u \to g_R u h^\dagger = h u g_L^\dagger\,. \tag{3}$$

The easiest way to construct Lagrangians is to use objects X that transform under chiral symmetry as $X \to hXh^\dagger$. For the present paper these are $u_\mu, f_\pm^{\mu\nu}$, $\chi_\pm$ and χ_-^μ

defined by

$$
\begin{aligned}
u_\mu &= i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right]\,, \qquad \chi = 2B(s+ip)\,,\\
\chi_\pm &= u^\dagger\chi u^\dagger \pm u\chi^\dagger u\,, \qquad\qquad \chi_-^\mu = u^\dagger D^\mu\chi u^\dagger - uD^\mu\chi^\dagger u\,,\\
f_\pm^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u\,, \qquad\qquad D_\mu\chi = \partial_\mu\chi - ir_\mu\chi + i\chi l_\mu\,,\\
F_L^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i\left[l^\mu, l^\nu\right]\,, \qquad F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i\left[r^\mu, r^\nu\right]\,.
\end{aligned}
\tag{4}
$$

The Lagrangian at lowest order, p^2, is known since long ago and is in the present notation

$$
\mathcal{L}_2 = \frac{F^2}{4}\left\langle u_\mu u^\mu + \chi_+\right\rangle \tag{5}
$$

where $\langle\ldots\rangle$ denotes the n_f-dimensional flavour trace. The notation we will use for $n_f = 2$ is F and B, and for $n_f = 3$ F_0 and B_0 for the constants in (4) and (5). The Lagrangians at next-to-leading order, p^4, were constructed in [2,3] for $n_f = 2$ and $n_f = 3$. The $n_f = 2$ Lagrangian is

$$
\begin{aligned}
\mathcal{L}_4^{n_f=2} =\; & \frac{l_1}{4}\left\langle u_\mu u^\mu\right\rangle\left\langle u_\nu u^\nu\right\rangle + \frac{l_2}{4}\left\langle u_\mu u_\nu\right\rangle\left\langle u^\mu u^\nu\right\rangle + \frac{l_3}{16}\left\langle\chi_+\right\rangle^2 + \frac{il_4}{4}\left\langle u_\mu\chi_-^\mu\right\rangle\\
& + \frac{l_5}{4}\left\langle f_+^2 - f_-^2\right\rangle + \frac{il_6}{2}\left\langle f_{+\mu\nu}u^\mu u^\nu\right\rangle - \frac{l_7}{16}\left\langle\chi_-\right\rangle^2\\
& + \;3 \text{ contact terms}\,.
\end{aligned}
\tag{6}
$$

The $n_f = 3$ Lagrangian is

$$
\begin{aligned}
\mathcal{L}_4^{n_f=3} =\; & L_1\left\langle u_\mu u^\mu\right\rangle\left\langle u_\nu u^\nu\right\rangle + L_2\left\langle u_\mu u_\nu\right\rangle\left\langle u^\mu u^\nu\right\rangle + L_3\left\langle u_\mu u^\mu u_\nu u^\nu\right\rangle\\
& + L_4\left\langle u_\mu u^\mu\right\rangle\left\langle\chi_+\right\rangle + L_5\left\langle u_\mu u^\mu\chi_+\right\rangle + L_6\left\langle\chi_+\right\rangle^2 + L_7\left\langle\chi_-\right\rangle^2\\
& + \frac{L_8}{2}\left\langle\chi_+^2 + \chi_-^2\right\rangle - iL_9\left\langle f_{+\mu\nu}u^\mu u^\nu\right\rangle + \frac{L_{10}}{4}\left\langle f_+^2 - f_-^2\right\rangle\\
& + \;2 \text{ contact terms}\,.
\end{aligned}
\tag{7}
$$

The Lagrangian for the general n_f-flavour case can be found in [19]. Terms that vanish due to the equations of motion have been dropped. This is discussed in detail in [19].

The Lagrangians at $O(p^6)$ are of the form

$$
\mathcal{L}_6^{n_f=2} = \sum_{i=1,56} c_i o_i\,, \qquad \mathcal{L}_6^{n_f=3} = \sum_{i=1,94} C_i O_i\,, \tag{8}
$$

The classification was done in [19] after an earlier attempt [20]. The form of the operators o_i and O_i can be found in [19]. In [21] an extra relation for the $n_f = 2$ case was found reducing the number of terms there to 56.

Renormalization is done with a ChPT variant of $\overline{MS}$ introduced in [2]. A detailed explanation valid to two-loop order can be found in [22, 23].

The relevant subtraction coefficients for all cases are known. These are then used to split the coupling constants in the Lagrangian into an infinite and a renormalized part. This split is not unique, so below is the definition of the renormalized constants that we use:

$$\hat{L}_i = (c\mu)^{d-4}\left(\hat{\Gamma}_i\Lambda + \hat{L}_i^r(\mu)\right) . \tag{9}$$

The divergent part is contained in $\Lambda = 1/(16\pi^2(d-4))$ and in ChPT we use as a standard $\ln c = -(1/2)\left(\ln 4\pi + \Gamma'(1) + 1\right)$. For $n_f = 2$, $\hat{L}_i = l_i$ and $\hat{\Gamma}_i = \gamma_i$ are derived and listed in [2]. For $n_f = 3$, $\hat{L}_i = L_i$ and $\hat{\Gamma}_i = \Gamma_i$ are derived and listed in [3]. For $n_f = 3$ the convention is to directly list the L_i^r at a scale $\mu = 0.77$ GeV. For $n_f = 2$ the convention is to quote instead values for the μ-independent $\bar{l}_i$ which are defined as

$$\bar{l}_i = \frac{32\pi^2}{\gamma_i} l_i^r(\mu) - \ln\frac{M_\pi^2}{\mu^2} . \tag{10}$$

The definition of the renormalized couplings at $O(p^6)$ is

$$\hat{C}_i = \frac{(c\mu)^{2(d-4)}}{\hat{F}^2}\left(\hat{C}_i^r(\mu) - \hat{\Gamma}_i^{(2)}\Lambda^2 - \left(\hat{\Gamma}_i^{(1)} + \hat{\Gamma}_i^{(L)}(\mu)\right)\Lambda\right) . \tag{11}$$

The values for all quantities needed for the $n_f = 2$ and $n_f = 3$ cases can be found in [23]. We will below quote the c_i^r and C_i^r at a scale $\mu = 0.77$ GeV and use the physical pion decay constant $F_\pi = 0.0922$ GeV to make the $\hat{C}_i^r$ dimensionless.

In addition to the operators listed explicitly in the Lagrangians (5,6,7,8) there are also so-called contact terms. The corresponding coefficients cannot be directly measured in physical quantities involving mesons and are therefore not relevant for phenomenology. Nevertheless, they have in principle well-defined values from Green functions of currents, but depend on the precise definitions of these currents.

2.1 Other Lagrangians

In the treatment of radiative corrections for strong and semileptonic processes at low energies, photons and leptons enter as dynamical degrees of freedom. Consequently, additional effective Lagrangians are needed.

Leaving out the kinetic terms for photons and leptons, a single new term arises to lowest order, $O(e^2p^0)$ [24]:

$$\mathcal{L}_{e^2p^0} = e^2 {F_0}^4 Z \langle \mathcal{Q}_L^{\text{em}} \mathcal{Q}_R^{\text{em}} \rangle . \tag{12}$$

The spurion fields

$$\mathcal{Q}_L^{\text{em}} = u Q^{\text{em}} u^\dagger , \qquad \mathcal{Q}_R^{\text{em}} = u^\dagger Q^{\text{em}} u \tag{13}$$

are expressed in terms of the quark charge matrix

$$Q^{\text{em}} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} . \tag{14}$$

The lowest-order electromagnetic LEC Z can be determined either directly from the pion charge difference ($Z \simeq 0.8$) or, in principle more reliably, from a sum rule in the chiral limit ($Z \simeq 0.9$) [25].

Neglecting leptonic terms for the moment, the next-to-leading order Lagrangian of $O(e^2p^2)$ was constructed by Urech [26]:

$$
\begin{aligned}
\mathcal{L}_{e^2p^2} \;=\;& e^2 {F_0}^2 \Big\{ \frac{1}{2} K_1 \, \langle (\mathcal{Q}_L^{\rm em})^2 + (\mathcal{Q}_R^{\rm em})^2 \rangle \, \langle u_\mu u^\mu \rangle + K_2 \, \langle \mathcal{Q}_L^{\rm em} \mathcal{Q}_R^{\rm em} \rangle \, \langle u_\mu u^\mu \rangle \\
& - K_3 \, [\langle \mathcal{Q}_L^{\rm em} u_\mu \rangle \, \langle \mathcal{Q}_L^{\rm em} u^\mu \rangle + \langle \mathcal{Q}_R^{\rm em} u_\mu \rangle \, \langle \mathcal{Q}_R^{\rm em} u^\mu \rangle] \\
& + K_4 \, \langle \mathcal{Q}_L^{\rm em} u_\mu \rangle \, \langle \mathcal{Q}_R^{\rm em} u^\mu \rangle + K_5 \, \langle [(\mathcal{Q}_L^{\rm em})^2 + (\mathcal{Q}_R^{\rm em})^2] u_\mu u^\mu \rangle \\
& + K_6 \, \langle (\mathcal{Q}_L^{\rm em} \mathcal{Q}_R^{\rm em} + \mathcal{Q}_R^{\rm em} \mathcal{Q}_L^{\rm em}) u_\mu u^\mu \rangle + \frac{1}{2} K_7 \, \langle (\mathcal{Q}_L^{\rm em})^2 + (\mathcal{Q}_R^{\rm em})^2 \rangle \, \langle \chi_+ \rangle \\
& + K_8 \, \langle \mathcal{Q}_L^{\rm em} \mathcal{Q}_R^{\rm em} \rangle \, \langle \chi_+ \rangle + K_9 \, \langle [(\mathcal{Q}_L^{\rm em})^2 + (\mathcal{Q}_R^{\rm em})^2] \chi_+ \rangle \\
& + K_{10} \, \langle (\mathcal{Q}_L^{\rm em} \mathcal{Q}_R^{\rm em} + \mathcal{Q}_R^{\rm em} \mathcal{Q}_L^{\rm em}) \chi_+ \rangle - K_{11} \, \langle (\mathcal{Q}_L^{\rm em} \mathcal{Q}_R^{\rm em} - \mathcal{Q}_R^{\rm em} \mathcal{Q}_L^{\rm em}) \chi_- \rangle \\
& - i K_{12} \, \langle [(\widehat{\nabla}_\mu \mathcal{Q}_L^{\rm em}) \mathcal{Q}_L^{\rm em} - \mathcal{Q}_L^{\rm em} \widehat{\nabla}_\mu \mathcal{Q}_L^{\rm em} - (\widehat{\nabla}_\mu \mathcal{Q}_R^{\rm em}) \mathcal{Q}_R^{\rm em} + \mathcal{Q}_R^{\rm em} \widehat{\nabla}_\mu \mathcal{Q}_R^{\rm em}] u^\mu \rangle \\
& + K_{13} \, \langle (\widehat{\nabla}_\mu \mathcal{Q}_L^{\rm em}) (\widehat{\nabla}^\mu \mathcal{Q}_R^{\rm em}) \rangle \\
& + K_{14} \, \langle (\widehat{\nabla}_\mu \mathcal{Q}_L^{\rm em}) (\widehat{\nabla}^\mu \mathcal{Q}_L^{\rm em}) + (\widehat{\nabla}_\mu \mathcal{Q}_R^{\rm em}) (\widehat{\nabla}^\mu \mathcal{Q}_R^{\rm em}) \rangle \Big\} ,
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
\widehat{\nabla}_\mu \mathcal{Q}_L^{\rm em} &= u (D_\mu Q_L^{\rm em}) u^\dagger , \\
\widehat{\nabla}_\mu \mathcal{Q}_R^{\rm em} &= u^\dagger (D_\mu Q_R^{\rm em}) u ,
\end{aligned} \tag{16}
$$

with

$$
\begin{aligned}
D_\mu Q_L^{\rm em} &= \partial_\mu Q_L^{\rm em} - i [l_\mu, Q_L^{\rm em}] , \\
D_\mu Q_R^{\rm em} &= \partial_\mu Q_R^{\rm em} - i [r_\mu, Q_R^{\rm em}] \; .
\end{aligned} \tag{17}
$$

In the presence of dynamical photons and leptons, the external fields l_μ, r_μ are modified as

$$
\begin{aligned}
l_\mu \;&\longrightarrow\; v_\mu - a_\mu - e Q_L^{\rm em} A_\mu + \sum_{\ell=e,\mu} (\bar{\ell} \gamma_\mu \nu_{\ell L} Q_L^{\rm w} + \overline{\nu_{\ell L}} \gamma_\mu \ell Q_L^{\rm w\dagger}) , \\
r_\mu \;&\longrightarrow\; v_\mu + a_\mu - e Q_R^{\rm em} A_\mu
\end{aligned} \tag{18}
$$

where A_μ is the photon field and the weak charge matrix is defined as

$$
Q_L^{\rm w} = -2\sqrt{2}\, G_F \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; . \tag{19}
$$

G_F is the Fermi coupling constant and V_{ud}, V_{us} are Kobayashi-Maskawa matrix elements.

For radiative corrections in semileptonic processes, one needs in addition the leptonic Lagrangian [27]

$$\begin{aligned}
\mathcal{L}_{\text{lept}} \;=\; & e^2 \sum_{\ell} \Big\{ F_0{}^2 \Big[X_1 \bar{\ell}\gamma_\mu \nu_{\ell L} \langle u^\mu \{ \mathcal{Q}_R^{\text{em}}, \mathcal{Q}_L^{\text{w}} \} \rangle \\
& + X_2 \bar{\ell}\gamma_\mu \nu_{\ell L} \langle u^\mu [\mathcal{Q}_R^{\text{em}}, \mathcal{Q}_L^{\text{w}}] \rangle + X_3 m_\ell \bar{\ell} \nu_{\ell L} \langle \mathcal{Q}_L^{\text{w}} \mathcal{Q}_R^{\text{em}} \rangle \\
& + i X_4 \bar{\ell}\gamma_\mu \nu_{\ell L} \langle \mathcal{Q}_L^{\text{w}} \widehat{\nabla}^\mu \mathcal{Q}_L^{\text{em}} \rangle + i X_5 \bar{\ell}\gamma_\mu \nu_{\ell L} \langle \mathcal{Q}_L^{\text{w}} \widehat{\nabla}^\mu \mathcal{Q}_R^{\text{em}} \rangle + h.c. \Big] \\
& + X_6 \bar{\ell} (i \not{\partial} + e \not{A}) \ell + X_7 m_\ell \bar{\ell} \ell \Big\} . \qquad (20)
\end{aligned}$$

Estimates of the electromagnetic LECs K_i and X_i will be reviewed in Sec. 4.8.

In this review we restrict ourselves to mesonic Lagrangians for strong and semileptonic processes including radiative corrections. ChPT has been applied to many more cases even in the meson sector, which will not be treated in any detail here. Neglecting lattice actions altogether, two more classes of chiral Lagrangians have been considered for the treatment of odd-intrinsic-parity (anomalous) processes and of nonleptonic decays. We include some relevant references here for the Lagrangians and for estimates of the corresponding LECs.

Anomalous processes start at $O(p^4)$. The odd-intrinsic-parity Lagrangian of $O(p^4)$ is given by the Wess-Zumino-Witten Lagrangian [28, 29] that has no free parameters. The anomalous Lagrangian of $O(p^6)$ has 23 LECs [30, 31]. Only partial results are available for the numerical values of those constants. The most promising approach is based on a short-distance analysis with or without chiral resonance Lagrangians [32–34]. Electromagnetic corrections for anomalous processes ($n_f = 2$) have also been investigated [35].

The chiral Lagrangian for nonleptonic interactions of lowest order, $O(G_F p^2)$, contains two LECs g_8, g_{27} [36]. The most recent evaluation of these LECs, including isospin breaking corrections, can be found in the review [11].

The LECs of $O(G_F p^4)$ (22 couplings N_i in the octet and 28 couplings D_i in the 27-plet Lagrangians) [37, 38] are less known than their strong counterparts at $O(p^4)$. The most recent phenomenological analysis of those combinations that occur in the dominant $K \to 2\pi, 3\pi$ decays can be found in Ref. [39]. Some of the LECs appearing in rare K decays have also been analysed [11].

Resonance saturation of weak LECs [38, 40] suffers from the drawback that the weak resonance couplings are unknown and that short-distance constraints are missing. Nevertheless, resonance saturation provides at least a possible parametrization of the LECs. The most systematic approach is based on factorization (valid to leading order in $1/N_c$) [41, 42], but higher-order corrections in $1/N_c$ may well be sizeable.

A different approach is to use the $1/N_c$ arguments in conjunction with the underlying short-distance physics. This approach was pioneered by [43] and further pursued in [44]. A more recent discussion is in [45]. One main problem here is to make sure that short- and long-distance matching is performed in a clean fashion [46, 47].

Finally, there is a chiral Lagrangian for electromagnetic corrections to nonleptonic weak processes. The single LEC of lowest order, $O(G_F e^2 p^0)$, related to the electromagnetic penguin contribution [48,49], is reasonably well known [42,50,51]. The additional 14 LECs of $O(G_F e^2 p^2)$ [52] have again been estimated to leading order in $1/N_c$ (factorization). In this way, the LECs can be expressed in terms of Wilson coefficients, the strong LECs L_5, L_8 and the electromagnetic LECs K_i [41, 42].

2.2 Contributions at each order and terminology

ChPT has been an active field since the early 1980s. As a consequence, the same quantities are often denoted by different symbols and terminology. The actual constants are referred to as low-energy constants, parameters or coupling constants in the Lagrangians and sometimes even referred to as counterterms. For the purpose of this review, these are all equivalent.

Another source of confusion is the nomenclature used for the orders. The use in this review is that the contribution to tree-level diagrams from $O(p^2)$ Lagrangians only is called lowest order or order p^2 or tree-level. The next order, which consists of tree-level diagrams with one vertex of the $O(p^4)$ Lagrangian and the remaining vertices from the $O(p^2)$ Lagrangian and of one-loop diagrams with only $O(p^2)$ vertices, is called order p^4 or next-to-leading order or one-loop order. In the same vein, the third order is called next-to-next-to-leading order or p^6 or two-loop order.

When adding other Lagrangians, one needs in addition to specify to which order one has included electromagnetic or weak coupling constants and, if applicable, $m_d - m_u$.

3 TWO FLAVOURS

3.1 Continuum input

If one looks at the $n_f = 2$ Lagrangians we have two parameters at LO, F and B, $7+3$ at NLO and $52+4$ at NNLO. The $i+j$ notation refers to the number of LECs and the number of contact terms. For most processes of interest, the tree-level contributions at NNLO are small, since the relevant scale in most of these cases is M_π.

This has the advantage that the determination of the NLO LECs does not depend on how well we know the values of the c_i^r but it makes comparison with models for the c_i more uncertain.

All observables needed are known to NNLO. The NLO results are all present in [2]. The NNLO results for M_π^2 and F_π were done in [22,53]. $\pi\pi$ scattering was done to NNLO in [22, 54]. Finally, the scalar and vector form factors of the pion were obtained to two-loop order in [55] while the pion radiative decay can be found in [56].

The pion mass is experimentally very well known [57]. The larger question here is which pion mass to use, charged or neutral. When comparing theoretical results with experimental quantities it is usually better to use the charged pion mass since most experiments are performed with these. When extracting quark masses, it is better to use the neutral pion mass since this is expected to have only a very small contribution from electromagnetism. In the below we use the values

$$M_{\pi^+} = 139.57018(35)\ \text{MeV}\ , \qquad M_{\pi^0} = 134.9766(6)\ \text{MeV}\ . \tag{21}$$

The pion decay constant is measured in $\pi \to \mu\nu$ and the main uncertainty is the size of electromagnetic effects. V_{ud} is known to sufficient precision from neutron and nuclear decays. We will adopt the value of

$$F_\pi = 92.2 \pm 0.1\ \text{MeV} \tag{22}$$

from the PDG [57].

The next major input needed are the $\pi\pi$ scattering lengths and related quantities. The main theoretical underpinning of this are the Roy equations [58], a set of integral equations that the $\pi\pi$ scattering amplitude satisfies because of crossing and unitarity. These require as input two subtraction constants, or equivalently values of the scattering lengths a_0^0 and a_0^2, and phenomenological input for the higher waves and at short distances. A large (re)analysis was done in [59]. This analysis confirmed a number of results from the 1970s and sharpened them. In [60] the analysis was strengthened by two additional inputs: The scalar radius should have a value of about 0.6 fm^2 and the ChPT series for the $\pi\pi$ amplitude was found to converge extremely well in the center of the Mandelstam triangle. This allowed to make a rather sharp prediction for the two subtraction constants if one assumes ChPT. The resulting values for the scattering lengths a_0^0 and a_0^2 have since been confirmed by other theoretical analyses using the same or similar methods [61, 62]. There has since also been a large experimental effort to pin down these two scattering lengths by NA48/2 and others in both $K_{\ell 4}$ [63] and $K \to 3\pi$ [64] decays and by DIRAC [65]. The values we will use for the scattering lengths are those from [60]:

$$a_0^0 = 0.220 \pm 0.005\,, \qquad a_0^2 = -0.0444 \pm 0.0010\,. \tag{23}$$

The pion scalar form factor is not directly measureable but it can be obtained from dispersion relations. The main part is given by the Omnés solution using the $\pi\pi$ S-wave from the analysis above but improvements are possible by including also other channels, first and foremost the KK channel. The main conclusion from [66, 67] is that

$$\begin{aligned} F_S^\pi(q^2) &= F_S^\pi(0)\left(1 + \frac{\langle r^2\rangle_S^\pi}{6} q^2 + c_S^\pi q^4 + \ldots\right) \\ \langle r^2\rangle_S^\pi &= 0.61 \pm 0.04\ \text{fm}^2, \qquad c_S^\pi = 11 \pm 1\ \text{GeV}^{-4}\ . \end{aligned} \tag{24}$$

A more recent discussion can be found in [68].

The pion vector form factor can be measured directly as well as treated with dispersive methods. The direct fit to the low-energy data, which are dominated by [69], was done in [55]:

$$
\begin{aligned}
F_V^\pi(q^2) &= F_V^\pi(0)\left(1+\frac{\langle r^2\rangle_V^\pi}{6}q^2+c_V^\pi q^4+\dots\right)\\
\langle r^2\rangle_V^\pi &= 0.437\pm 0.016\ \text{fm}^2, \qquad c_V^\pi = 3.85\pm 0.60\ \text{GeV}^{-4}\,. \qquad (25)
\end{aligned}
$$

A more recent analysis using dispersive methods is [70] which reached $\langle r^2\rangle_V^\pi \in (0.42, 0.44)$ fm^2 and $c_V^\pi \in (3.79, 4.00)$ GeV^{-4} in good agreement with the above but somewhat smaller errors.

3.2 Values of the LECs

When looking at values of the LECs in two-flavour theory, the usual convention is to use the $\bar{l}_i$ defined in (10). These are independent of the subtraction scale but do depend instead explicitly on the pion mass. For a subtraction scale $\mu = 0.77$ GeV one should keep in mind that

$$
-\ln\frac{M_\pi^2}{\mu^2} = 3.42\,. \qquad (26)
$$

Values of the $\bar{l}_i$ around this value are thus dominated by the pion chiral logarithm.

The values of $\bar{l}_1$ and $\bar{l}_2$ were originally determined from the D-wave $\pi\pi$ scattering lengths in [2]. The analysis in [60] relies instead on the whole $\pi\pi$ scattering analysis and yields

$$
\bar{l}_1 = -0.4\pm 0.6\,, \qquad \bar{l}_2 = 4.3\pm 0.1\,. \qquad (27)
$$

The main sources of uncertainty are the input estimates of the c_i^r for both.

$\bar{l}_3$ is very difficult to get from phenomenology. For this one needs to know how the pion mass depends on higher powers of $\hat{m}$. Putting an upper limit on this leads to the estimate given in [2]:

$$
\bar{l}_3 = 2.9\pm 2.4\,. \qquad (28)
$$

The scalar radius was used as input in [60]. From the value above they derived

$$
\bar{l}_4 = 4.4\pm 0.2\,. \qquad (29)
$$

This value is in good agreement with the determination done in [55] directly from the scalar radius.

The constant $\bar{l}_5$ is quite well known. It can be determined from the difference between vector and axial-vector two-point functions. The analysis in [71] quotes

$$
\bar{l}_5 = 12.24\pm 0.21\,. \qquad (30)
$$

The constant $\bar{l}_6$ can be determined from the pion electromagnetic radius. This was done in [55] and gives the value

$$
\bar{l}_6 = 16.0\pm 0.5\pm 0.7\,. \qquad (31)
$$

The last error is mainly from the estimate of the c_i^r.

A combination of the latter two can be obtained from the axial form factor F_A in the decay $\pi^+ \to e^+\nu\gamma$. The two-loop calculation was done in [56] with the result

$$\bar{l}_5 - \bar{l}_6 = 3.0 \pm 0.3\,. \tag{32}$$

Ref. [56] used $F_A = 0.0116 \pm 0.0016$ from the measured value for $\gamma = F_A/F_V$ and the CVC prediction for F_V. F_V and F_A have since been measured with better precision with the result from [72] being $F_A = 0.0117 \pm 0.0017$ so the value in (32) does not change.

Note that the values in (30) and (32) can be combined to [71]

$$\bar{l}_6 = 15.24 \pm 0.39\,, \tag{33}$$

nicely compatible within errors with (31), thus providing proof that ChPT works well in this sector. Another check that ChPT works was done by looking at the relations for $\pi\pi$ scattering found in [73] independent of the values of the c_i^r. Those were fairly well satisfied.

The determination of the constants at order p^6 is in worse shape. Four combinations of the c_i^r are reasonably well known. These come from c_V^π, c_S^π [55] and $\pi\pi$ scattering [60]:

$$\begin{aligned}
r^r_{V2} &= -4c^r_{51} + 4c^r_{53} = (1.6 \pm 0.5)\cdot 10^{-4}\,,\\
r^r_{S3} &= -8c^r_6 \approx 1.5\cdot 10^{-4}\,,\\
r^r_5 &= -8c^r_1 + 10c^r_2 + 14c^r_r = (1.5 \pm 0.4)\cdot 10^{-4}\,,\\
r^r_6 &= 6c^r_2 + 2c^r_3 = (0.40 \pm 0.04)\cdot 10^{-4}\,.
\end{aligned} \tag{34}$$

r^r_{S3} is not so well known since c_S^π is dominated by the other contributions.

3.3 Including lattice results

In the last years the quark masses obtainable on the lattice have been coming closer to and even reaching the physical point. The situation relevant for the quantities considered in this review is presented in the FLAG reports [12,13]. In particular, Sec. 5.1 of [13] reviews the status as of 2013.

The value of $\bar{l}_6$ can be obtained from the lattice calculations of the electromagnetic pion radius and $\bar{l}_4$ from the scalar radius. These calculations have not yet reached a precision comparable to (31) and (29). The combination $\bar{l}_1 - \bar{l}_2$ can be obtained from higher-order effects in the pion form factors. Again this value is not yet competitive with the one of (27).

In the continuum we cannot vary the pion mass but lattice QCD calculations can easily do this by varying the quark masses. The constants that influence this behaviour are thus much easier to obtain from lattice calculations. The quantities that are measured here are the variation of F_π with the pion mass which gives $\bar{l}_4$ and the deviation of $M_\pi^2/(2B\hat{m})$ from unity as a function of M_π which gives $\bar{l}_3$. The lattice calculations are done in a number of physically different ways: with only up and down quarks, $N_f = 2$, including the strange quark, $N_f = 2+1$, and including the charm quark as well, $N_f = 2+1+1$. In addition one often uses partially quenched conditions where valence and sea quarks have different masses. The last case, $N_f = 2+1+1$, is pursued mainly by the ETM collaboration [74]. The case with $N_f = 2+1$ has many more contributors, RBC/UKQCD [75], MILC [76], NPLQCD [77] and BMW [78]. For a more complete reference list, including $N_f = 2$, see Ref. [13]. The FLAG results for $N_f = 2$ and $N_f = 2+1+1$ are dominated by the ETM results [79] and [74], respectively, while the $N_f = 2+1$ results are averaged over several collaborations, which are not always in good agreement. This is the reason why the errors for the $N_f = 2+1$ case as quoted below are larger. The results quoted are always those when the various N_f cases have been analysed with $n_f = 2$ ChPT. FLAG [13] gives the averages

$$\begin{aligned}
\bar{l}_3\big|_{N_f=2} &= 3.45 \pm 0.26\,, \quad \bar{l}_3\big|_{N_f=2+1} = 2.77 \pm 1.27\,, \quad \bar{l}_3\big|_{N_f=2+1+1} = 3.70 \pm 0.27\,,\\
\bar{l}_4\big|_{N_f=2} &= 4.59 \pm 0.26\,, \bar{l}_4\big|_{N_f=2+1} = 3.95 \pm 0.35\,, \quad \bar{l}_4\big|_{N_f=2+1+1} = 4.67 \pm 0.10\,.
\end{aligned} \tag{35}$$

Clearly, there is still a significant spread in central values. Not included in the FLAG averages (35) are the more recent $N_f = 2+1$ results [80] $\bar{l}_3 = 2.5 \pm 0.7$ and $\bar{l}_4 = 3.8 \pm 0.5$.

A rough estimate that covers the lattice range and the continuum results (28,29) is

$$\bar{l}_3 = 3.0 \pm 0.8\,, \quad \bar{l}_4 = 4.3 \pm 0.3\,. \tag{36}$$

These are what we will use in the fits for the $n_f = 3$ constants below.

4 THREE FLAVOURS

4.1 New issues

Among the issues specific to the three-flavour case are the size of NNLO corrections of $O(p^6)$ or alternatively the convergence of the low-energy expansion for chiral $SU(3)$. Related to this issue is a possible paramagnetic effect [81], which would manifest itself here by rather large values for L_4^r and L_6^r. The values we obtain are not fully conclusive but large N_c, Sec. 4.2, and the present lattice results, Sec. 4.5, support small values for L_4^r and L_6^r.

The convergence of several physical quantities is discussed in Sec. 4.7.

4.2 Large N_c

The expansion in the number of colours was defined in [82]. Leaving aside l_7 and L_7 because of the special large-N_c counting due to η' exchange [3,83], there is an important difference between the LECs of $O(p^4)$ for chiral $SU(2)$ and $SU(3)$.

The $SU(2)$ LECs $l_1,\ldots,l_6$ are all leading order in $1/N_c$, i.e. of $O(N_c)$. In the $SU(3)$ case, there are three (combinations of) LECs that are suppressed and of $O(1)$ at large N_c: $2L_1 - L_2$, L_4 and L_6 [3]. It is of special interest whether the phenomenological analysis respects this hierarchy.

In the original analysis of Gasser and Leutwyler [3] the large-N_c suppressed LECs were assumed to vanish at a scale $\mu = M_\eta$. More recent analyses showed (e.g., fit All in Ref. [10]) that it is difficult to verify the large-N_c suppression with global fits especially for L_4 and L_6 because of big errors.

Lattice determinations of L_4 and L_6, on the other hand, are quite consistent with the large-N_c suppression [13]. Why is it then notoriously difficult to extract meaningful values for L_4 and L_6 from global fits?

In the case of L_4, a partial explanation is the apparent anti-correlation of L_4 with the leading-order LEC F_0 in the fits of Ref. [10]: the bigger F_0, the smaller $L_4^r(M_\rho)$, and vice versa. This anti-correlation can be understood to some extent from the structure of the chiral $SU(3)$ Lagrangian up to and including NLO:

$$\begin{aligned}
\mathcal{L}_2 + \mathcal{L}_4^{n_f=3} &= \frac{F_0^2}{4}\langle u_\mu u^\mu + \chi_+\rangle + \; L_4\langle u_\mu u^\mu\rangle\langle\chi_+\rangle + \ldots \\
&= \frac{1}{4}\langle u_\mu u^\mu\rangle \left[F_0^2 + 8L_4\left(2M_K^{\circ 2} + M_\pi^{\circ 2}\right)\right] + \ldots
\end{aligned} \tag{37}$$

where $M_P^\circ(P = \pi, K)$ denotes the lowest-order meson masses. The dots refer to the remainder of the NLO Lagrangian (7) in the first line and to terms of higher order in the meson fields in the second line. Therefore, a lowest-order tree-level contribution is always accompanied by an L_4 contribution in the combination [84]

$$F(\mu)^2 := F_0^2 + 8L_4^r(\mu)\left(2M_K^{\circ 2} + M_\pi^{\circ 2}\right) \; . \tag{38}$$

Of course, there will in general be additional contributions involving L_4 at NLO, especially in higher-point functions. Nevertheless, the observed anti-correlation between F_0 and L_4 is clearly related to the structure of the chiral Lagrangian. Note that $F_\pi^2/16M_K^2 = 2\times 10^{-3}$ is the typical size of a NLO LEC. Although of different chiral order, the two terms in $F(\mu)^2$ could a priori be of the same order of magnitude. This makes it very difficult to disentangle F_0 and L_4 in phenomenological fits. At least in principle, the lattice is better off in this respect because the masses of quarks and mesons can be tuned on the lattice. In practice, most lattice studies employ strange quark masses of similar size corresponding to the actual kaon mass. As Eq. (38) indicates, the tuning of the light quark mass and thus of the pion mass is less effective for disentangling L_4 and F_0. Nevertheless, the uncertainties of L_4 from lattice evaluations are definitely smaller [13,84] than from continuum fits. Note also that the combination $F(\mu)$ defined in (38) can be much better determined than F_0 itself [84]. A similar discussion can clearly be done for L_6^r and the lowest-order mass term $(F_0^2/4)\,\langle\chi_+\rangle$.

In the following analysis, we are therefore not going to extract the large-N_c suppressed LECs directly from the global fits. However, it will turn out to be sufficient to restrict L_4^r

to a reasonable range suggested by large N_c and lattice results. The LECs L_6 and $2L_1 - L_2$ will then more or less automatically follow suit.

4.3 Continuum data

All ChPT results we use are known to NNLO and since long at NLO. The results for the masses and decay constants at NLO are from [3] and at NNLO from [85]. The scalar form factor of the pion was done at NLO in [86] and at NNLO in [87]. $\pi\pi$ scattering was done at NLO in [3] and at NNLO in [88]. πK scattering was done at NLO in [89] and at NNLO in [90]. Finally, the F and G form factors in $K_{\ell 4}$ were done at NLO in [4,5] and NNLO in [7].

We will use as input the values of F_π, M_π, $\langle r^2\rangle^\pi_S$, c^π_S, a^0_0 and a^2_0 as given in Sec. 3.1.

The remaining input has changed somewhat from [10] but the final effect of these changes on the fits discussed below is small. The main differences come from our different treatment of the C^r_i.

The value of F_K/F_π we take from the PDG [57]:

$$\frac{F_K}{F_\pi} = 1.198 \pm 0.006\,. \tag{39}$$

The error is dominated by the uncertainty of V_{us}. This value is in good agreement with the lattice determinations [13].

We also include the results on πK scattering from a Roy-Steiner analysis [91]:

$$a_0^{1/2} = 0.224 \pm 0.022\,, \quad a_0^{3/2} = -0.0448 \pm 0.0077\,. \tag{40}$$

The main change w.r.t. [10] is that we now use the normalization of the $K_{\ell 4}$ decay also from NA48/2. The summary of their results can be found in [92]. From those results we use for the two form factors F and G their slope and value at threshold defined as

$$\begin{aligned} F &= f_s + f'_s q^2 + \dots\,, \quad f_s = 5.705 \pm 0.035\,, \quad f'_s = 0.867 \pm 0.050\,,\\ G &= g_p + g'_p q^2 + \dots\,, \quad g_p = 4.952 \pm 0.086\,, \quad g'_p = 0.508 \pm 0.122\,, \end{aligned} \tag{41}$$

with $q^2 = s_{\pi\pi}/(4m_\pi^2) - 1$.

The final input we need is the quark mass ratio. We perform fits for several values of these but use as central value [13]

$$\frac{m_s}{\hat{m}} = 27.5 \pm 0.5\,. \tag{42}$$

The masses used are those from the PDG (with the PDG 2010 value for M_η, since rerunning all the input calculations would be very time consuming; this small difference is not of any relevance in the remainder):

$$M_{K^+} = 493.677(16)\ \text{MeV}, \quad M_{K^0} = 497.614(24)\ \text{MeV}, \quad M_\eta = 547.853(18)\ \text{MeV}\,. \tag{43}$$

When using the masses for the quark mass ratios and decay constants we use the neutral pion mass, the eta mass and the average kaon mass with the electromagnetic corrections removed using the estimate of [93]. This results in an average kaon mass of

$$M_K = 494.5 \text{ MeV}\,. \tag{44}$$

4.4 Continuum fits

The main principle of the fit is the same as in [10]. We calculate numerically the p^4 and p^6 corrections for all the quantities discussed above. In all cases we use the physical masses and F_π in the expressions. For F_K/F_π we use the form

$$\frac{F_K}{F_\pi} = 1 + F_K^{(4)} - F_\pi^{(4)} + \left[F_K^{(6)} - F_\pi^{(6)} - F_\pi^{(4)}\left(F_K^{(4)} - F_\pi^{(4)}\right)\right]\,. \tag{45}$$

The masses are included via $M_M^2 = M_{M0}^2 + M_M^{2(4)} + M_M^{2(6)}$ and we then add as a χ^2 that $m_s/\hat{m}$ obtained from $M_{K0}^2/M_{\pi 0}^2$ and $M_{\eta 0}^2/M_{\pi 0}^2$ should agree with (42) with an error of 5%. This was chosen as a reasonable compromise for the neglected higher-order terms. The exception to the errors quoted in Secs. 3.1 and 4.3 is that we double the errors for $a_0^{1/2}$ and $a_0^{3/2}$ in (40).

The fits are also done with the variable

$$L_A^r = 2L_1^r - L_2^r \tag{46}$$

to allow for a large-N_c test.

4.4.1 NLO fits

We first perform a number of fits at NLO with F_K/F_π, $m_s/\hat{m}$ from the masses M_K and M_η, the four scattering lengths, the pion scalar radius and the $K_{\ell 4}$ form factors of (41) as input. We thus have 12 inputs to determine the L_i^r for $i = 1, \ldots, 8$.

The results are shown in Table 1. The free fit in the second column has the smallest χ^2 but it does not exhibit the large-N_c hierarchy of the LECs. Moreover, the large values of L_4^r and L_6^r are in conflict with most lattice results [13]. The anti-correlation with F_0 discussed in Sec. 4.2 is manifest. We also note that the results of this fit and the last column of Table 5 in [10] are a little different but compatible. The small differences are due to the different errors that have been used here, rather than the small change in central values.

As emphasized in Sec. 4.2, the lattice is in a much better position to determine L_4. We have therefore restricted L_4^r to a range compatible with lattice studies. In columns 3, 4, 5 in Table 1, we display the results for $10^3 L_4^r = 0,\, 0.3,\, -0.3$, respectively. What is remarkable is the manifest positive correlation of L_4 with L_A and L_6: Enforcing large N_c on L_4^r makes also L_A^r and L_6^r small, in agreement with the large-N_c suppression.

The relatively large χ^2 in the restricted fits comes almost exclusively from the scattering lengths a_0^2 and a_0^0. These parameters were determined in a sophisticated dispersion theoretical analysis of pion-pion scattering [60] in the framework of chiral $SU(2)$. It is therefore not surprising that an NLO fit in chiral $SU(3)$ cannot really cope with the precision of a_0^2 and a_0^0 in (23). However, doubling the errors of a_0^2, a_0^0 does not really change the picture: The resulting fit values for the L_i^r (including errors) are almost unchanged, but the χ^2 decreases to values similar to the one of the free fit in column 2.

Therefore, we consider the combined fit values with restricted L_4^r in column 6 as the most realistic values of the L_i^r at NLO. The errors listed are based on the input errors only and do not reflect the uncertainties due to higher-order corrections. This statement does not apply to the original estimates of the L_i by Gasser and Leutwyler [3] reproduced in the last column for comparison. It is worth emphasizing that our p^4 fit values and the estimates in Ref. [3] from nearly 30 years ago (columns 6 and 7) are still compatible with each other.

There are two more LECs, L_9^r and L_{10}^r. These were determined via the pion electromagnetic radius and the axial form factor in $\pi \to e\nu\gamma$ in [3] with the results

$$L_9^r = (6.9 \pm 0.7) \cdot 10^{-3}\,, \qquad L_{10}^r = (-5.5 \pm 0.7) \cdot 10^{-3}\,. \tag{47}$$

The value for L_9^r was redone at NLO in [94] with essentially the same result. A more precise value for L_{10}^r was obtained in [71] as

$$L_{10}^r = (-5.22 \pm 0.06) \cdot 10^{-3}\,. \tag{48}$$

4.4.2 NNLO fits

We now add more input. In addition to c_S^π defined in (24), we include also the $\bar{l}_i$ discussed in Sec. 3.2. The latter can be calculated from the L_i at $O(p^6)$ using the results of Ref. [95]. As was already remarked in that reference, this will allow a handle on some of the large-N_c suppressed C_i^r. In addition, we require a not too badly behaved perturbative ChPT series for the masses. This we enforce by adding to the χ^2 a contribution of the type $f^\chi(M_M^{2(6)}/M_M^2/\Delta_M)$ with

$$f^\chi(x) = 2x^4/(1+x^2) \tag{49}$$

and $\Delta_M = 0.1$. This form was chosen to be one when x=1, quadratic for large x but turning on slower than x^2. If we had chosen $f^\chi(x) = x^2$ it would be like a normal χ^2.

NNLO fits of the L_i turn out to be very sensitive to the values of the C_i. The naive dimensional estimate[1] for the NNLO LECs is $C_0 = (4\pi)^{-4} = 40 \cdot 10^{-6}$.

To set the scene, we perform NNLO fits for two different scenarios. In the first case, all C_i are set to zero at the scale $\mu = 0.77$ GeV. For the second scenario, we take the predictions of a chiral quark model [96], mainly because it is the only model we are aware

[1] We shall always display the C_i in units of 10^{-6}.

Table 1: Fits at NLO with L_4^r free, 0, $0.3 \cdot 10^{-3}$ and $-0.3 \cdot 10^{-3}$, respectively. The last two columns contain the combined results for the range $-0.3 \cdot 10^{-3} \leq L_4^r \leq 0.3 \cdot 10^{-3}$ and for comparison the original values from Ref. [3]. Except for the last column [3], the errors are only due to the input errors, no estimate of the error due to higher orders is included. As always for $SU(3)$ LECs, the renormalization scale is $\mu = 0.77$ GeV.

	free fit				p^4 fit	Ref. [3]
$10^3 L_A^r$	1.17(27)	0.39(16)	0.52(16)	0.27(16)	0.4(2)	
$10^3 L_1^r$	1.11(10)	0.98(09)	1.00(09)	0.95(09)	1.0(1)	0.7(3)
$10^3 L_2^r$	1.05(17)	1.56(09)	1.48(09)	1.64(09)	1.6(2)	1.3(7)
$10^3 L_3^r$	−3.82(30)	−3.82(30)	−3.82(30)	−3.82(30)	−3.8(3)	−4.4(2.5)
$10^3 L_4^r$	1.87(53)	≡0	≡0.3	≡ −0.3	0.0(3)	−0.3(5)
$10^3 L_5^r$	1.22(06)	1.23(06)	1.23(06)	1.23(06)	1.2(1)	1.4(5)
$10^3 L_6^r$	1.46(46)	−0.11(05)	0.14(06)	−0.36(05)	0.0(4)	−0.2(3)
$10^3 L_7^r$	−0.39(08)	−0.24(15)	−0.27(14)	−0.21(17)	−0.3(2)	−0.4(2)
$10^3 L_8^r$	0.65(07)	0.53(13)	0.55(12)	0.50(14)	0.5(2)	0.9(3)
χ^2	3.8	16	12	20		
dof	4	5	5	5	5	
F_0 [MeV]	58	81	76	86	81(5)	

of that predicts all the C_i contributing to the observables in our fits. The results are displayed in Table 2. In both cases, the fit is not satisfactory: In addition to the large χ^2, the LECs L_A, L_4 and L_6 show no sign of large-N_c suppression.

Therefore, it is obvious that we have to make some assumptions about the NNLO LECs in order to proceed. There are altogether 34 (combinations of the) C_i that appear in our 16 input observables. For most of those C_i predictions are available in the literature, although in some cases contradictory. There are essentially three types of predictions. The estimates in the first group are mainly phenomenologically oriented [71, 87, 97–106]. A second class uses more theory in addition to phenomenological input, e.g., chiral quark models, resonance chiral theory, short-distance constraints, holography, etc. [33, 96, 107–116]. Finally, there are also some estimates from lattice studies [84, 117–120].

We use the available information to define priors for the C_i with associated ranges of acceptable values. The fits are then performed by two different methods leading essentially to the same results: minimization and a random-walk procedure in the restricted space of the C_i. If the resulting fit values for the L_i deviate too much from the p^4 values in Table 1 and/or if the χ^2 is too large, we modify the boundaries of the C_i space and start again.

Therefore, we cannot claim to have found the best values for the L_i to NNLO with this procedure, because the notion of "best values" is mathematically ill-defined with 16

Table 2: Fits at NNLO for two different choices of the C_i. The results in the second column were obtained by taking $C_i^r = 0$ $(\forall i)$, those in the third column are based on the C_i from a chiral quark model [96]. The renormalization scale is $\mu = 0.77$ GeV.

C_i	$C_i^r = 0$	Ref. [96]
$10^3 L_A^r$	1.17(12)	0.70(12)
$10^3 L_1^r$	0.67(06)	0.48(07)
$10^3 L_2^r$	0.17(04)	0.25(04)
$10^3 L_3^r$	−1.76(21)	−1.68(22)
$10^3 L_4^r$	0.73(10)	0.86(11)
$10^3 L_5^r$	0.65(05)	2.08(14)
$10^3 L_6^r$	0.25(09)	0.83(06)
$10^3 L_7^r$	−0.17(06)	−0.33(06)
$10^3 L_8^r$	0.22(08)	1.03(14)
χ^2	26	41
dof	8	8

input data and 8+34 parameters. On the other hand, our final results displayed in Table 3 exhibit the following attractive properties.

- As shown by the χ^2, especially in comparison with the test fits in Table 2, the quality of the fits is excellent.

- The values of the NLO and NNLO fits are of course different, but not drastically so. This has partly been enforced by requiring that the meson masses show a reasonable "convergence" (see Sec. 4.7).

- The results to NNLO are much more sensitive to L_4 than at NLO. We therefore present only two cases, one without restricting L_4^r and the other for fixed $L_4^r = 0.3 \cdot 10^{-3}$. In the latter case, as found already at NLO, the small L_4 guarantees that also L_A and L_6 are in accordance with large N_c.

In Table 3 we present both cases: our preferred fit (BE14) with $L_4^r = 0.3 \cdot 10^{-3}$ and the general fit without any restrictions on the L_i^r. It is clear that our preference is not based on χ^2 only, but on a good deal of theoretical prejudice as well. Moreover, our preferred fit BE14 must always be considered together with the values of the NNLO LECs collected in Table 4. We do not display the C_i for the unrestricted fit but the values are similar.

There have been some more studies of L_4^r, L_6^r and the C_i^r using the pion and kaon scalar form factor [87], $\pi\pi$ scattering [88] and πK scattering [90,101]. Most of these studies were very sensitive to the input values of the L_i^r assumed and the input data used are to a large

Table 3: NNLO fits for the LECs L_i^r. The second column contains our preferred fit (BE14) with fixed $L_4^r = 0.3 \cdot 10^{-3}$, the third one the general free fit without any restrictions on the L_i^r. No estimate of the error due to higher orders is included.

fit	BE14	free fit
$10^3 L_A^r$	0.24(11)	0.68(11)
$10^3 L_1^r$	0.53(06)	0.64(06)
$10^3 L_2^r$	0.81(04)	0.59(04)
$10^3 L_3^r$	−3.07(20)	−2.80(20)
$10^3 L_4^r$	≡0.3	0.76(18)
$10^3 L_5^r$	1.01(06)	0.50(07)
$10^3 L_6^r$	0.14(05)	0.49(25)
$10^3 L_7^r$	−0.34(09)	−0.19(08)
$10^3 L_8^r$	0.47(10)	0.17(11)
χ^2	1.0	0.5
F_0 [MeV]	71	64

Table 4: Preferred values of the NNLO LECs for fit BE14 in Table 3. As always, the renormalization scale is $\mu = 0.77$ GeV and the numerical values are in units of 10^{-6}.

LEC		LEC		LEC		LEC	
C_1	12	C_{11}	−4.0	C_{20}	1.0	C_{29}	− 20
C_2	3.0	C_{12}	−2.8	C_{21}	− 0.48	C_{30}	3.0
C_3	4.0	C_{13}	1.5	C_{22}	9.0	C_{31}	2.0
C_4	15	C_{14}	−1.0	C_{23}	− 1.0	C_{32}	1.7
C_5	−4.0	C_{15}	−3.0	C_{25}	− 11	C_{33}	0.82
C_6	−4.0	C_{16}	3.2	C_{26}	10	C_{34}	7.0
C_7	5.0	C_{17}	−1.0	C_{28}	− 2.0	C_{36}	2.0
C_8	19	C_{18}	0.63	$C_{63} - C_{83} + C_{88}/2$			− 9.6
C_{10}	−0.25	C_{19}	− 4.0	$C_{66} - C_{69} - C_{88} + C_{90}$			50

extent included in the fits discussed above. We therefore do not discuss those constraints. The constraints from the scalar $K\pi$ transition form factor, or f_0 in K_{l3} decays can in principle also be used. The ChPT result is in [99] and has been used in [100,121] to obtain constraints on C_{12}^r and C_{34}^r but the results depend again on the L_i^r input used.

The situation for the constants L_9^r, L_{10}^r and associated C_i^r is better. The pion electromagnetic radius is dominated by the L_9^r contribution. The determination in [94] to NNLO gives

$$L_9^r = (5.93 \pm 0.43) \cdot 10^{-3}\,, \qquad C_{88}^r - C_{90}^r = (-55 \pm 5) \cdot 10^{-6}\,. \tag{50}$$

The values of L_{10}^r and C_{87}^r can be obtained from sum rules for the difference between the vector and axial-vector current correlators. The required ChPT calculation was done in [85] and recent analyses of the spectral sum rules are [71, 106, 120]. The latter also use some lattice data. A reasonable average of the values for $10^3 L_{10}^r$ of -4.06 ± 0.39 [71], -3.1 ± 0.8 [106] and -3.46 ± 0.32 [120] is

$$L_{10}^r = (-3.8 \pm 0.4) \cdot 10^{-3} . \tag{51}$$

The NNLO ChPT calculation for $\pi \to l\nu\gamma$ was done [122] but has not been used to obtain a value for L_{10}^r. The values for C_{87}^r in the above references are compatible and give

$$C_{87}^r = (42 \pm 2) \cdot 10^{-6} . \tag{52}$$

The error in (52) is probably a little underestimated. It does not include higher-order ChPT effects. The large-N_c estimate of [104], $10^6 C_{87}^r = 48 \pm 5$, is quite compatible with the above.

A similar analysis in the scalar sector for the difference between scalar and pseudoscalar spectral functions can in principle be done. However, here one has to rely on much more theoretical input since direct data are not available. Bounds on L_6^r were derived in [67, 123] with values typically a little larger than those of the fits reported here. Results for L_8^r were derived in [109, 124] with values for $10^3 L_8^r$ of 1.0 ± 0.3 and 0.6 ± 0.4, which are again somewhat larger than our estimates. However, neither of these references included a dependence on the other L_i^r used as input.

4.5 Including lattice results

There have been a few papers combining continuum and lattice input [84, 105, 118, 120], but so far no major effort has been done to combine the two in a systematic fashion. The situation on the lattice side has been reviewed in the FLAG reports [12, 13].

One of the problems is that relatively few lattice collaborations actually use the full NNLO formulas to fit the data. Given that a typical $n_f = 3$ ChPT correction at NLO is about 25% and the expected NNLO correction is thus about 7%, it is clear that NLO ChPT will not be sufficient to analyse lattice data at the physical kaon and pion masses. On the other hand, using the same argument, a typical N^{3}LO correction would be of the order of 1.5%, which is more appropriate. The MILC collaboration [117, 125, 126] is here one of the exceptions. The formulas are known also at NNLO for all needed partially quenched cases [127–130]. It should be noted that the number of new parameters is not that large. Most lattice calculations use an analytic NNLO mass term in their fits.

We only quote here the results of MILC [125] and HPQCD [131]. The former are with a full NNLO ChPT analysis while the latter are with an NLO ChPT analysis augmented with analytic NNLO terms. It is therefore not clear whether the latter should be compared fully with our results. The results are given in Table 5.

Table 5: Lattice results from the two most complete analyses available. The lattice values at $\mu = M_\eta$ have been transformed to the usual scale $\mu = 0.77$ GeV.

	[125]	[131]
$10^3 L_4^r$	0.04(14)	0.09(34)
$10^3 L_5^r$	0.84(40)	1.19(25)
$10^3 L_6^r$	0.07(11)	0.16(20)
$10^3 L_8^r$	0.36(09)	0.55(15)

The results shown clearly live up to the large-N_c expectations that L_4^r and L_6^r should be small at $\mu = 0.77$ GeV. The values for L_5^r and L_8^r are compatible with the continuum estimates with small L_4^r and L_6^r enforced.

The lattice results for the other L_i^r have not yet reached the accuracy needed to compete with the continuum determinations.

4.6 Resonance saturation

In ChPT, LECs parametrize physics at shorter distances. Following the time-honoured notion of vector meson dominance, Gasser and Leutwyler [2] suggested that the ρ meson should play a special role in the LECs to which it contributes. This suggestion was later formulated in terms of a resonance Lagrangian for chiral $SU(3)$ [24, 132]. Saturating the NLO LECs L_i with the lowest-lying resonance nonets turned out to provide a qualitative understanding of the numerical values of the L_i^r for renormalization scales near 0.77 GeV.

In this subsection, we first review the status of resonance saturation in view of our fit results for the L_i^r, both at NLO and at NNLO. In addition, we then investigate the validity of resonance saturation also for some of the NNLO LECs C_i^r, which come with our preferred fit BE14 of the L_i^r (see Tables 3,4).

The expressions for the L_i in terms of the parameters of the lowest-lying vector, axial-vector and scalar multiplets (neglecting small contributions from pseudoscalar resonances) are reproduced in Table 6. The resonance couplings were introduced in Ref. [24]. For the numerical estimates, we use the well-known relations from short-distance constraints on spectral functions and form factors [24, 132–136]:

$$\begin{aligned} F_V G_V &= F_0^2 \ , & 4 c_d c_m &= F_0^2 \ , \\ F_V^2 - F_A^2 &= F_0^2 \ , & F_V^2 M_V^2 &= F_A^2 M_A^2 \ . \end{aligned} \tag{53}$$

For the actual numerical values, we have used $F_0 = F_\pi$, $c_d = c_m$, $F_V = 2G_V$, $M_V = 0.77$ GeV and $M_S = 1.4$ GeV. The qualitative features of the numerical values of the L_i^r are well reproduced with resonance saturation, as shown in Table 6. Two comments are in order.

Table 6: Comparison of the fitted L_i^r at NLO (Table 1) and NNLO (Table 3) with resonance saturation [24, 132]. The numerical estimates of the resonance contributions are based on Eqs. (53), with $F_0 = F_\pi$, $c_d = c_m$, $F_V = 2G_V$, $M_V = 0.77$ GeV and $M_S = 1.4$ GeV. All numerical values are in units of 10^{-3}.

	$O(p^4)$	$O(p^6)$	R exchange	num. estimate
L_1^r	1.0(1)	0.53(6)	$\frac{G_V^2}{8M_V^2}$	0.9
L_2^r	1.6(2)	0.81(4)	$\frac{G_V^2}{4M_V^2}$	1.8
L_3^r	− 3.8(3)	− 3.07(20)	$-\frac{3G_V^2}{4M_V^2} + \frac{c_d^2}{2M_S^2}$	− 4.8
L_4^r	0.0(3)	0.3	0	0
L_5^r	1.2(1)	1.01(6)	$\frac{c_d c_m}{M_S^2}$	1.1
L_6^r	0.0(4)	0.14(5)	0	0
L_7^r	− 0.3(2)	− 0.34(9)	$-\frac{F_0^2}{48M_{\eta'}^2}$	− 0.2
L_8^r	0.5(2)	0.47(10)	$\frac{c_m^2}{2M_S^2}$	0.54
L_9^r	6.9(7)	5.9(4)	$\frac{F_V G_V}{2M_V^2}$	7.2
L_{10}^r	−5.2(1)	−3.8(4)	$-\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2}$	− 5.4

- The absolute values of L_1, L_2, L_3 for the p^6 fit are a bit smaller than the vector meson contributions. As already noted in Refs. [24, 132], the agreement improves if instead of using the KSFR relation $F_V = 2G_V$, the decay widths $\Gamma(\rho^0 \to e^+e^-)$ and $\Gamma(\rho \to \pi\pi)$ are invoked to extract the coupling constants F_V and G_V. This type of fine-tuning is not our concern here.

- The much-debated scalar resonance dominance works very well for L_5 and L_8. Essential for this agreement is the notion of the lightest scalar nonet that survives in the large-N_c limit. We refer to Ref. [137] for a discussion in favour of an average scalar resonance mass in the vicinity of $M_S \simeq 1.4$ GeV.

We now turn to the issue of resonance saturation of the NNLO LECs C_i^r. The most general chiral resonance Lagrangian that can generate chiral LECs up to $O(p^6)$ was constructed in Ref. [110]. The corresponding chiral resonance theory generates Green functions that interpolate between QCD and ChPT. It is therefore natural to expect that resonance saturation works qualitatively also for the C_i^r.

However, the situation is more complicated than at $O(p^4)$. First, as discussed in the previous subsection, our knowledge of the numerical values of the C_i^r is still limited. Second, many more couplings arise at $O(p^6)$, many of them related to double-resonance exchange, which are essentially unknown. In the following, we will therefore investigate relations for

Table 7: Relations among LECs C_i^r from resonance exchange [110, 111] that are either parameter-free or depend only on parameters occurring already at $O(p^4)$. The values in the last column are taken from Table 4 with the usual renormalization scale $\mu = 0.77$ GeV. The numerical estimates are given in units of 10^{-6}, with the same input values as in Table 6.

LECs	R exchange	num. estimate	fit value
$C_1 + C_3 - C_4$	$-\frac{c_d^2 F_\pi^2}{4M_S^4}$	$-$ 1.2	1
$3C_3 + C_4$	$\frac{G_V^2 F_\pi^2}{8M_V^4}$	13	27
C_{12}	$-\frac{c_d c_m F_\pi^2}{2M_S^4}$	$-$ 2.4	$-$ 2.8
C_{18}	$-\frac{F_\pi^4}{48M_{\eta'}^4}$	$-$ 1.8	0.6
C_{19}	$-\frac{F_\pi^4}{144M_{\eta'}^4}$	$-$ 0.6	$-$ 4.0
C_{20}	$\frac{F_\pi^4}{96M_{\eta'}^4}$	0.9	1.0
$C_{20} + 3C_{21}$	0	0	$-$ 0.4
$C_{32} + 3C_{21}$	0	0	0.3
$C_{28} - C_{30}/2$	0	0	$-$ 3.5
$C_1/12 - C_{28} + \frac{c_d}{c_m} C_{32}$	$-\frac{7c_d^2 F_\pi^2}{144M_S^4} - \frac{G_V^2 F_\pi^2}{288M_V^4} + \frac{c_d}{c_m}\frac{F_\pi^4}{96M_{\eta'}^4}$	0.3	4.7

the C_i^r that fulfill two conditions: These LECs contribute to observables used in our fits of the L_i^r and, secondly, the relations either involve only resonance parameters that occurred already at $O(p^4)$ or they do not depend on any parameters at all. A list of such relations was given in Ref. [110] but not analysed there. Note that many of the studies in this area assume that short-distance constraints should be satisfied. This is not always possible with a finite number of resonances [138] and in that case a choice of what is implemented is necessary.

In Table 7 we display the relations involving only those C_i^r that contribute to our observables. We make several comments.

- The numerical estimates should be viewed with the naive dimensional estimate of NNLO LECs in mind ($C_0 = 40 \cdot 10^{-6}$).

- While the fit values in the last column of Table 7 refer to the usual renormalization scale ($\mu = 0.77$ GeV), the numerical estimates from resonance exchange do not carry a scale dependence (leading order in $1/N_c$). For instance, for a scale $\mu = 0.85$ GeV, the fit value for the combination $C_1 + C_3 - C_4$ moves from 1 to $-$ 1.4, practically coinciding with the estimate from resonance exchange.

- $C_1 + C_3 - C_4$ and C_{12} are only sensitive to scalar exchange. As for L_5 and L_8 (see Table 6), the predictions work surprisingly well. There is certainly no evidence for a failure of scalar resonance saturation.
- The parameter-free relations involving C_{20}, C_{21} and C_{32} are not only well satisfied, but the involved LECs also seem to be dominated by η' exchange in accordance with large N_c [111].
- Finally, we cannot compare our results directly with one of the most solid predictions for NNLO LECs, i.e. for $C_{88} - C_{90}$ in Eq. (50). However, as the last entry in Table 4 documents, our values for the C_i certainly do not contradict that prediction.

4.7 Convergence of ChPT for chiral $SU(3)$

We now study the convergence of a number of quantities for the fit BE14 and the free fit. This can be compared with the same study done in Ref. [10].

The quantities are always in the order LO+NLO+NNLO and the main number is fit BE14 while the number in brackets is from the free fit.

$$\begin{aligned}
F_K/F_\pi &= 1 + 0.176(0.121) + 0.023(0.077)\,, \\
F_\pi/F_0 &= 1 + 0.208(0.313) + 0.088(0.127)\,, \\
M_\pi^2/M_{\pi\text{phys}}^2 &= 1.055(0.937) - 0.005(+0.107) - 0.050(-0.044)\,, \\
M_K^2/M_{K\text{phys}}^2 &= 1.112(0.994) - 0.069(+0.022) - 0.043(-0.016)\,, \\
M_\eta^2/M_{\eta\text{phys}}^2 &= 1.197(0.938) - 0.214(-0.076) + 0.017(0.014)\,.
\end{aligned} \tag{54}$$

The LO contribution to the masses is calculated from our NLO and NNLO results. The total higher-order corrections are very reasonable for all the ratios listed above even though the NLO corrections are small in some cases.

The $\pi\pi$ scattering lengths show a very good convergence for both:

$$\begin{aligned}
a_0^0 &= 0.160 + 0.044(0.046) + 0.012(0.012)\,, \\
a_0^2 &= -0.0456 + 0.0016(0.0017) - 0.0001(-0.0003)\,.
\end{aligned} \tag{55}$$

The πK scattering lengths have a worse convergence:

$$\begin{aligned}
a_0^{1/2} &= 0.142 + 0.031(0.027) + 0.051(0.057)\,, \\
a_0^{3/2} &= -0.071 + 0.007(0.005) + 0.016(0.019)\,.
\end{aligned} \tag{56}$$

Finally, we present the convergence for the K_{l4} form factors at threshold:

$$\begin{aligned}
f_s &= 3.786 + 1.202(1.231) + 0.717(0.688)\,, \\
g_p &= 3.786 + 0.952(0.857) + 0.212(0.309)\,.
\end{aligned} \tag{57}$$

Note that we have fitted the observables to within their errors, so all higher-order contributions are included in the NNLO parts. The overall picture is that the convergence is in line with expectations for $n_f = 3$ ChPT.

4.8 Dynamical photons and leptons

As shown in Ref. [32], the electromagnetic LECs K_i of the Lagrangian (15) obey integral sum rule representations, generalizing the DGMLY sum rule [139] for the π^+-π^0 mass difference. The integral representations have the form of convolutions of pure QCD n-point functions ($n \leq 4$) with the free photon propagator. The representations serve several purposes [32]: They can be used to study the dependence of the K_i^r on the chiral renormalization scale, their gauge dependence and possible short-distance ambiguities. The representations also lead to model-independent relations among the LECs. Last but not least, they allow for approximate determinations of the LECs by saturating the integrals in terms of resonance exchanges. This is especially important in the present case because it is nearly impossible to determine the LECs K_i from phenomenology.

In Ref. [32] the method was applied to $K_7,\ldots,K_{13}$ involving two- and three-point functions only. K_{14} multiplies a pure source term and is therefore irrelevant for phenomenology. It turns out that K_7 and K_8 are large-N_c suppressed and are therefore set to zero at the scale $\mu = M_\rho$. The remaining LECs $K_9,\ldots,K_{13}$ are all gauge dependent. In fact, $K_9,\ldots,K_{12}$ also depend on the QCD renormalization scale $\mu_{\rm SD}$ [32, 93]. These LECs can therefore not be expressed separately in terms of physical quantities but will occur only in certain combinations in observables. For instance, the combination $K_{10} + K_{11}$ enters the corrections of $O(e^2 m_s)$ to Dashen's theorem [140]. Consequently, this combination is independent of the gauge parameter ξ and $\mu_{\rm SD}$, depending only on the chiral renormalization scale μ. With this proviso in mind, numerical values will be given in Feynman gauge ($\xi = 1$), for $\mu_{\rm SD} = 1$ GeV and for the usual chiral renormalization scale $\mu = M_\rho$. Note that in contrast to the strong LECs, the chiral scale dependence already appears at leading order in $1/N_c$. Other estimates exist in the calculations of the corrections to Dashen's theorem [141, 142]. These use other methods to estimate the intermediate- and short-distance momentum regimes. In particular, Ref. [93] treated the intermediate-distance dynamics with the ENJL model and the short-distance part with perturbative QCD and factorization. Using the latter method, it is also very clear why the LECs are gauge and QCD scale dependent [93, 143].

The more complicated case of four-point functions in the sum rule representations of $K_1,\ldots,K_6$ was investigated in Ref. [144]. It turns out that all these LECs are gauge independent. Independently of the single-resonance approximation, one can derive the following large-N_c relations [93, 144]:

$$K_3^r = -K_1^r \;, \qquad\qquad K_4^r = 2K_2^r \;. \tag{58}$$

In Table 8 we collect numerical results for the $K_i^r(M_\rho)$ on the basis of the sum rule representations of Refs. [32, 144]. As the authors emphasize, uncertainties of the numerical predictions are difficult to estimate quantitatively even for physically relevant combinations: Both the large-N_c approximation and the single-resonance assumption (except for the relations (58)) should be kept in mind. The values in the other approaches have the same order of magnitude but differ in the detailed predictions.

Finally, we turn to the case of dynamical photons and leptons for the calculation of

Table 8: Numerical values of the LECs $K_i^r(M_\rho)$ in units of 10^{-3} [32, 144]. The gauge-dependent LECs (see text) are given in Feynman gauge ($\xi = 1$). The QCD renormalization scale is set to $\mu_{\rm SD} = 1$ GeV.

LEC		LEC	
K_1^r	-2.7	K_7^r	$\simeq 0$
K_2^r	0.7	K_8^r	$\simeq 0$
K_3^r	2.7	K_{10}^r	7.5
K_4^r	1.4	K_{11}^r	1.3
K_5^r	11.6	K_{12}^r	-4.2
K_6^r	2.8	K_{13}^r	4.7

Table 9: Numerical values of the LECs $X_i^r(M_\rho)$ $(i = 1, 2, 3, 5)$ for $M_A = \sqrt{2} M_\rho$ [145].

$10^3 X_1^r$	$10^3 X_2^r$	$10^3 X_3^r$	$10^3 X_5^r$
-3.7	3.6	5.0	-7.2

radiative corrections in semileptonic meson decays. The corresponding LECs $X_1, \ldots, X_7$ are defined in the Lagrangian (20). Actually, neither X_4 nor X_7 are phenomenologically relevant. In analogy to the formalism set up for the K_i, Descotes-Genon and Moussallam [145] established integral representations for all X_i with the help of a two-step matching procedure (Standard Model $\rightarrow$ Fermi theory $\rightarrow$ ChPT). These representations furnish numerical estimates of the LECs, once the chiral Green functions are approximated with the help of large-N_c-motivated models.

Again as for the K_i, the integral representations also allow the derivation of non-trivial relations among the X_i. Independently of any model for the two- and three-point functions involved, the following relations hold [145]:

$$X_2^r = \left(X_3^r + \frac{3}{32\pi^2} \right) / 4, \qquad X_5^r = -2 X_2^r \tag{59}$$

so that in fact only three independent LECs remain to be estimated with specific models.

In Table 9 we collect the numerical estimates for the LECs X_1, X_2, X_3 and X_5, putting X_6 aside for the moment. Except for the model-independent relations (59), the estimates are based on a minimal resonance model with a single multiplet of vector and axial-vector resonances each (only V and A spectral functions are involved).

The LEC X_6 plays a special role because it cannot be determined from the matching conditions in the same way as X_1, X_2, X_3 and X_5. By looking at explicit calculations of radiative corrections of semileptonic decays, one verifies that X_6 and K_{12} always appear in the same combination $X_6^{\rm phys} := X_6 - 4K_{12}$ related to wave-function renormalization.

It is convenient to write $X_6^{\rm phys}$ as the sum of two contributions:

$$X_6^{\rm phys}(\mu) = X_{6,\rm SD}^{\rm phys} + \tilde{X}_6^{\rm phys}(\mu) \ , \tag{60}$$

where the short-distance contribution is given by [146]

$$X_{6,\rm SD}^{\rm phys} = -\frac{1}{2\pi^2}\log\frac{M_Z}{M_V} \tag{61}$$

and the remainder has the following form in the single-resonance approximation [145]:

$$\tilde{X}_6^{\rm phys}(\mu) = \frac{1}{16\pi^2}\left(3\log\frac{\mu^2}{M_V^2} + \frac{1}{2}\log\frac{M_A^2}{M_V^2} - \frac{3M_V^2+M_A^2}{(4\pi F_0)^2} + \frac{7}{2}\right) \ . \tag{62}$$

Summing up powers of electroweak logarithms and adding a small correction of $O(\alpha_s)$ [147],

$$X_{6,\rm SD}^{\rm phys} = -0.2419 \qquad \longrightarrow \qquad \bar{X}_{6,\rm SD}^{\rm phys} = -0.2527 \ . \tag{63}$$

By convention, the short-distance contribution is factorized and appears in a universal multiplicative factor

$$S_{\rm EW} = 1 - e^2 \bar{X}_{6,\rm SD}^{\rm phys} = 1.0232 \tag{64}$$

in all radiatively corrected semileptonic decay rates. The subdominant remainder $\tilde{X}_6^{\rm phys}(\mu)$, on the other hand, combines with other terms in the decay amplitude to yield a scale-independent expression. In the model of Ref. [145] one obtains (with $M_A = \sqrt{2}M_\rho$)

$$\tilde{X}_6^{\rm phys}(\mu = M_\rho) = 0.0104 \ . \tag{65}$$

5 CONCLUSIONS AND FINAL RESULTS

ChPT as a nonrenormalizable effective field theory requires reliable information on many of its coupling constants in order to arrive at meaningful predictions. In this review we have collected the available knowledge of the low-energy constants in mesonic ChPT, emphasizing the chiral Lagrangians for the strong interactions.

For $n_f = 2$ ChPT the NLO LECs are by now quite well known from phenomenology with the exception of $\bar{l}_3$, for which one should turn to the lattice. The values for the $\bar{l}_i$ are summarized in Eqs. (27,29,30,33,36). The convergence for most of the quantities studied is excellent. The phenomenological knowledge of the NNLO LECs given in (34) is still rather modest. Here we have restricted ourselves to quoting published results.

For $n_f = 3$ ChPT we have given a short overview of the different types of chiral Lagrangians. We then concentrated on a new fit of the NLO LECs in the strong sector, using all available information about the NNLO LECs. With reasonable values for the

C_i^r a good fit can be obtained with satisfactory convergence for many physical quantities. However, one should keep in mind that determining L_4^r from continuum phenomenology is very difficult. Lattice results and large-N_c arguments suggest $|L_4^r(M_\rho)| \leq 3 \cdot 10^{-3}$. This leads to our main new NNLO fit for the L_i^r given in column BE14 in Table 3. We have emphasized that fit BE14 should always be considered together with the associated set of C_i^r values in Table 4. Although the changes in the L_i^r are non-negligible when going from NLO to NNLO, the pattern is quite stable. Another interesting feature is that requiring a small $|L_4^r(M_\rho)|$ leads automatically to small values of $|2L_1^r(M_\rho) - L_2^r(M_\rho)|$ and $|L_6^r(M_\rho)|$ in accordance with large N_c. We have compared our findings with available results from lattice studies.

Quite generally, LECs parametrize the physics at shorter distances. We have taken a fresh look at the evidence for resonance saturation of the strong LECs, confirming the qualitative agreement for the L_i^r and finding new evidence also for some of the C_i^r. In the few cases where only scalar resonances contribute, resonance saturation seems to work as well at least qualitatively for both NLO and NNLO LECs. In addition, our preferred values of the C_i^r are consistent with η' exchange in accordance with large N_c.

Although not as impressive as for $n_f = 2$, most of the observables used for our fits show a reasonable "convergence" also for $n_f = 3$, once this pattern is enforced for the meson masses.

Finally, we have reviewed the status of the LECs occurring in the chiral Lagrangians with dynamical photons and leptons relevant for radiative corrections. Although phenomenology is not of much help for a determination of those LECs, different theoretical approaches have led to a consistent picture for all NLO LECs. The interplay between intermediate- and short-distance physics is well under control.

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References

[1] S. Weinberg, Physica A **96** (1979) 327.

[2] J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142.

[3] J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.

[4] C. Riggenbach, J. Gasser, J. F. Donoghue and B. R. Holstein, Phys. Rev. D **43** (1991) 127.

[5] J. Bijnens, Nucl. Phys. B **337** (1990) 635.

[6] J. Bijnens, G. Colangelo and J. Gasser, Nucl. Phys. B **427** (1994) 427 [hep-ph/9403390].

[7] G. Amorós, J. Bijnens and P. Talavera, Nucl. Phys. B **585** (2000) 293 [Erratum ibid. B **598** (2001) 665] [hep-ph/0003258].

[8] G. Amorós, J. Bijnens and P. Talavera, Nucl. Phys. B **602** (2001) 87 [hep-ph/0101127].

[9] D. B. Kaplan and A. V. Manohar, Phys. Rev. Lett. **56** (1986) 2004.

[10] J. Bijnens and I. Jemos, Nucl. Phys. B **854** (2012) 631 [arXiv:1103.5945 [hep-ph]].

[11] V. Cirigliano *et al.*, Rev. Mod. Phys. **84** (2012) 399 [arXiv:1107.6001 [hep-ph]].

[12] G. Colangelo *et al.*, Eur. Phys. J. C **71** (2011) 1695 [arXiv:1011.4408 [hep-lat]].

[13] S. Aoki *et al.*, arXiv:1310.8555 [hep-lat].

[14] H. Leutwyler, Annals Phys. **235** (1994) 165 [hep-ph/9311274].

[15] S. Scherer and M. R. Schindler, Lect. Notes Phys. **830** (2012) 1.

[16] G. Ecker, Prog. Part. Nucl. Phys. **35** (1995) 1 [hep-ph/9501357].

[17] A. Pich, Rept. Prog. Phys. **58** (1995) 563 [hep-ph/9502366].

[18] J. Bijnens, Prog. Part. Nucl. Phys. **58** (2007) 521 [hep-ph/0604043].

[19] J. Bijnens, G. Colangelo and G. Ecker, JHEP **9902** (1999) 020 [hep-ph/9902437].

[20] H. W. Fearing and S. Scherer, Phys. Rev. D **53** (1996) 315 [hep-ph/9408346].

[21] C. Haefeli, M. A. Ivanov, M. Schmid and G. Ecker, arXiv:0705.0576 [hep-ph].

[22] J. Bijnens *et al.*, Nucl. Phys. B **508** (1997) 263 [Erratum ibid. B **517** (1998) 639] [hep-ph/9707291].

[23] J. Bijnens, G. Colangelo and G. Ecker, Annals Phys. **280** (2000) 100 [hep-ph/9907333].

[24] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B **321** (1989) 311.

[25] B. Moussallam, Eur. Phys. J. C **6** (1999) 681 [hep-ph/9804271].

[26] R. Urech, Nucl. Phys. B **433** (1995) 234 [hep-ph/9405341].

[27] M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, Eur. Phys. J. C **12** (2000) 469 [hep-ph/9909284].

[28] J. Wess and B. Zumino, Phys. Lett. B **37** (1971) 95.

[29] E. Witten, Nucl. Phys. B **223** (1983) 422.

[30] T. Ebertshauser, H. W. Fearing and S. Scherer, Phys. Rev. D **65** (2002) 054033 [hep-ph/0110261].

[31] J. Bijnens, L. Girlanda and P. Talavera, Eur. Phys. J. C **23** (2002) 539 [hep-ph/0110400].

[32] B. Moussallam, Nucl. Phys. B **504** (1997) 381 [hep-ph/9701400].

[33] M. Knecht and A. Nyffeler, Eur. Phys. J. C **21** (2001) 659 [hep-ph/0106034].

[34] P. D. Ruiz-Femenía, A. Pich and J. Portolés, JHEP **0307** (2003) 003 [hep-ph/0306157].

[35] B. Ananthanarayan and B. Moussallam, JHEP **0205** (2002) 052 [hep-ph/0205232].

[36] J. A. Cronin, Phys. Rev. **161** (1967) 1483.

[37] J. Kambor, J. H. Missimer and D. Wyler, Nucl. Phys. B **346** (1990) 17.

[38] G. Ecker, J. Kambor and D. Wyler, Nucl. Phys. B **394** (1993) 101.

[39] J. Bijnens and F. Borg, Eur. Phys. J. C **40** (2005) 383 [hep-ph/0501163].

[40] G. D'Ambrosio and J. Portolés, Nucl. Phys. B **533** (1998) 494 [hep-ph/9711211].

[41] E. Pallante, A. Pich and I. Scimemi, Nucl. Phys. B **617** (2001) 441 [hep-ph/0105011].

[42] V. Cirigliano, G. Ecker, H. Neufeld and A. Pich, Eur. Phys. J. C **33** (2004) 369 [hep-ph/0310351].

[43] W. A. Bardeen, A. J. Buras and J.-M. Gérard, Nucl. Phys. B **293** (1987) 787.

[44] J. Bijnens and J. Prades, JHEP **9901** (1999) 023 [hep-ph/9811472].

[45] A. J. Buras, J.-M. Gérard and W. A. Bardeen, arXiv:1401.1385 [hep-ph].

[46] J. Bijnens and J. Prades, JHEP **0001** (2000) 002 [hep-ph/9911392].

[47] S. Peris, M. Perrottet and E. de Rafael, JHEP **9805** (1998) 011 [hep-ph/9805442].

[48] J. Bijnens and M. B. Wise, Phys. Lett. B **137** (1984) 245.

[49] B. Grinstein, S.-J. Rey and M. B. Wise, Phys. Rev. D **33** (1986) 1495.

[50] V. Cirigliano, J. F. Donoghue and E. Golowich, Phys. Rev. D **61** (2000) 093002 [hep-ph/9909473].

[51] J. Bijnens, E. Gámiz and J. Prades, JHEP **0110** (2001) 009 [hep-ph/0108240].

[52] G. Ecker *et al.*, Nucl. Phys. B **591** (2000) 419 [hep-ph/0006172].

[53] U. Bürgi, Nucl. Phys. B **479** (1996) 392 [hep-ph/9602429].

[54] J. Bijnens *et al.*, Phys. Lett. B **374** (1996) 210 [hep-ph/9511397].

[55] J. Bijnens, G. Colangelo and P. Talavera, JHEP **9805** (1998) 014 [hep-ph/9805389].

[56] J. Bijnens and P. Talavera, Nucl. Phys. B **489** (1997) 387 [hep-ph/9610269].

[57] J. Beringer *et al.* [Particle Data Group Collaboration], Phys. Rev. D **86** (2012) 010001.

[58] S. M. Roy, Phys. Lett. B **36** (1971) 353.

[59] B. Ananthanarayan, G. Colangelo, J. Gasser and H. Leutwyler, Phys. Rept. **353** (2001) 207 [hep-ph/0005297].

[60] G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. B **603** (2001) 125 [hep-ph/0103088].

[61] S. Descotes-Genon, N. H. Fuchs, L. Girlanda and J. Stern, Eur. Phys. J. C **24** (2002) 469 [hep-ph/0112088].

[62] R. Garcia-Martin *et al.*, Phys. Rev. D **83** (2011) 074004 [arXiv:1102.2183 [hep-ph]].

[63] J. R. Batley *et al.* [NA48/2 Collaboration], Eur. Phys. J. C **54** (2008) 411.

[64] J. R. Batley *et al.* [NA48/2 Collaboration], Phys. Lett. B **633** (2006) 173 [hep-ex/0511056].

[65] B. Adeva *et al.* [DIRAC Collaboration], Phys. Lett. B **619** (2005) 50 [hep-ex/0504044].

[66] J. F. Donoghue, J. Gasser and H. Leutwyler, Nucl. Phys. B **343** (1990) 341.

[67] B. Moussallam, Eur. Phys. J. C **14** (2000) 111 [hep-ph/9909292].

[68] B. Ananthanarayan *et al.*, Phys. Lett. B **602** (2004) 218 [hep-ph/0409222].

[69] S. R. Amendolia *et al.* [NA7 Collaboration], Nucl. Phys. B **277** (1986) 168.

[70] B. Ananthanarayan, I. Caprini, D. Das and I. Sentitemsu Imsong, Eur. Phys. J. C **73** (2013) 2520 [arXiv:1302.6373 [hep-ph]].

[71] M. González-Alonso, A. Pich and J. Prades, Phys. Rev. D **78** (2008) 116012 [arXiv:0810.0760 [hep-ph]].

[72] M. Bychkov *et al.*, Phys. Rev. Lett. **103** (2009) 051802 [arXiv:0804.1815 [hep-ex]].

[73] J. Bijnens and I. Jemos, Eur. Phys. J. C **64** (2009) 273 [arXiv:0906.3118 [hep-ph]].

[74] R. Baron *et al.* [ETM Collaboration], PoS LATTICE **2010** (2010) 123 [arXiv:1101.0518 [hep-lat]].

[75] R. Arthur *et al.* [RBC and UKQCD Collaborations], Phys. Rev. D **87** (2013) 094514 [arXiv:1208.4412 [hep-lat]].

[76] A. Bazavov *et al.*, PoS LATTICE **2010** (2010) 083 [arXiv:1011.1792 [hep-lat]].

[77] S. R. Beane *et al.*, Phys. Rev. D **86** (2012) 094509 [arXiv:1108.1380 [hep-lat]].

[78] S. Borsányi *et al.*, Phys. Rev. D **88** (2013) 014513 [arXiv:1205.0788 [hep-lat]].

[79] R. Baron *et al.* [ETM Collaboration], JHEP **1008** (2010) 097 [arXiv:0911.5061 [hep-lat]].

[80] S. Dürr *et al.*, arXiv:1310.3626 [hep-lat].

[81] S. Descotes-Genon, L. Girlanda and J. Stern, JHEP **0001** (2000) 041 [hep-ph/9910537].

[82] G. 't Hooft, Nucl. Phys. B **72** (1974) 461.

[83] S. Peris and E. de Rafael, Phys. Lett. B **348** (1995) 539 [hep-ph/9412343].

[84] G. Ecker, P. Masjuan and H. Neufeld, Eur. Phys. J. C **74** (2014) 2748 [arXiv:1310.8452 [hep-ph]].

[85] G. Amorós, J. Bijnens and P. Talavera, Nucl. Phys. B **568** (2000) 319 [hep-ph/9907264].

[86] J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 517.

[87] J. Bijnens and P. Dhonte, JHEP **0310** (2003) 061 [hep-ph/0307044].

[88] J. Bijnens, P. Dhonte and P. Talavera, JHEP **0401** (2004) 050 [hep-ph/0401039].

[89] V. Bernard, N. Kaiser and U. G. Meißner, Nucl. Phys. B **357** (1991) 129.

[90] J. Bijnens, P. Dhonte and P. Talavera, JHEP **0405** (2004) 036 [hep-ph/0404150].

[91] P. Büttiker, S. Descotes-Genon and B. Moussallam, Eur. Phys. J. C **33** (2004) 409 [hep-ph/0310283].

[92] J. R. Batley *et al.* [NA48/2 Collaboration], Phys. Lett. B **715** (2012) 105 [arXiv:1206.7065 [hep-ex]].

[93] J. Bijnens and J. Prades, Nucl. Phys. B **490** (1997) 239 [hep-ph/9610360].

[94] J. Bijnens and P. Talavera, JHEP **0203** (2002) 046 [hep-ph/0203049].

[95] J. Gasser, C. Haefeli, M. A. Ivanov and M. Schmid, Phys. Lett. B **652** (2007) 21 [arXiv:0706.0955 [hep-ph]].

[96] S.-Z. Jiang, Y. Zhang, C. Li and Q. Wang, Phys. Rev. D **81** (2010) 014001 [arXiv:0907.5229 [hep-ph]].

[97] S. Dürr and J. Kambor, Phys. Rev. D **61** (2000) 114025 [hep-ph/9907539].

[98] D. Boito *et al.*, arXiv:1311.6679 [hep-ph].

[99] J. Bijnens and P. Talavera, Nucl. Phys. B **669** (2003) 341 [hep-ph/0303103].

[100] M. Jamin, J. A. Oller and A. Pich, JHEP **0402** (2004) 047 [hep-ph/0401080].

[101] K. Kampf and B. Moussallam, Eur. Phys. J. C **47** (2006) 723 [hep-ph/0604125].

[102] J. Prades, PoS KAON (2008) 022 [arXiv:0707.1789 [hep-ph]].

[103] R. Unterdorfer and H. Pichl, Eur. Phys. J. C **55** (2008) 273 [arXiv:0801.2482 [hep-ph]].

[104] P. Masjuan and S. Peris, Phys. Lett. B **663** (2008) 61 [arXiv:0801.3558 [hep-ph]].

[105] V. Bernard and E. Passemar, JHEP **1004** (2010) 001 [arXiv:0912.3792 [hep-ph]].

[106] D. Boito *et al.*, Phys. Rev. D **87** (2013) 094008 [arXiv:1212.4471 [hep-ph]].

[107] V. Cirigliano *et al.*, Phys. Lett. B **596** (2004) 96 [hep-ph/0404004].

[108] V. Cirigliano *et al.*, JHEP **0504** (2005) 006 [hep-ph/0503108].

[109] I. Rosell, J. J. Sanz-Cillero and A. Pich, JHEP **0701** (2007) 039 [hep-ph/0610290].

[110] V. Cirigliano *et al.*, Nucl. Phys. B **753** (2006) 139 [hep-ph/0603205].

[111] R. Kaiser, Nucl. Phys. Proc. Suppl. **174** (2007) 97.

[112] A. Pich, I. Rosell and J. J. Sanz-Cillero, JHEP **0807** (2008) 014 [arXiv:0803.1567 [hep-ph]].

[113] J. J. Sanz-Cillero and J. Trnka, Phys. Rev. D **81** (2010) 056005 [arXiv:0912.0495 [hep-ph]].

[114] A. Pich, I. Rosell and J. J. Sanz-Cillero, JHEP **1102** (2011) 109 [arXiv:1011.5771 [hep-ph]].

[115] P. Colangelo, J. J. Sanz-Cillero and F. Zuo, JHEP **1211** (2012) 012 [arXiv:1207.5744 [hep-ph]].

[116] M. Golterman, K. Maltman and S. Peris, Phys. Rev. D **89** (2014) 054036 [arXiv:1402.1043 [hep-ph]].

[117] A. Bazavov *et al.* [MILC Collaboration], PoS LATTICE **2010** (2010) 074 [arXiv:1012.0868 [hep-lat]].

[118] G. Ecker, P. Masjuan and H. Neufeld, Phys. Lett. B **692** (2010) 184 [arXiv:1004.3422 [hep-ph]].

[119] A. Bazavov *et al.*, Phys. Rev. D **87** (2013) 073012 [arXiv:1212.4993 [hep-lat]].

[120] P. A. Boyle *et al.*, arXiv:1403.6729 [hep-ph].

[121] V. Bernard and E. Passemar, Phys. Lett. B **661** (2008) 95 [arXiv:0711.3450 [hep-ph]].

[122] C. Q. Geng, I.-L. Ho and T. H. Wu, Nucl. Phys. B **684** (2004) 281 [hep-ph/0306165].

[123] B. Moussallam, JHEP **0008** (2000) 005 [hep-ph/0005245].

[124] J. Bordes *et al.*, JHEP **1210** (2012) 102 [arXiv:1208.1159 [hep-ph]].

[125] A. Bazavov *et al.* [MILC Collaboration], PoS CD **09** (2009) 007 [arXiv:0910.2966 [hep-ph]].

[126] A. Bazavov *et al.* [MILC Collaboration], PoS LATTICE **2011** (2011) 107 [arXiv:1111.4314 [hep-lat]].

[127] J. Bijnens, N. Danielsson and T. A. Lähde, Phys. Rev. D **70** (2004) 111503 [hep-lat/0406017].

[128] J. Bijnens and T. A. Lähde, Phys. Rev. D **71** (2005) 094502 [hep-lat/0501014].

[129] J. Bijnens and T. A. Lähde, Phys. Rev. D **72** (2005) 074502 [hep-lat/0506004].

[130] J. Bijnens, N. Danielsson and T. A. Lähde, Phys. Rev. D **73** (2006) 074509 [hep-lat/0602003].

[131] R. J. Dowdall, C. T. H. Davies, G. P. Lepage and C. McNeile, Phys. Rev. D **88** (2013) 074504 [arXiv:1303.1670 [hep-lat]].

[132] G. Ecker *et al.*, Phys. Lett. B **223** (1989) 425.

[133] S. Weinberg, Phys. Rev. Lett. **18** (1967) 507.

[134] M. Knecht and E. de Rafael, Phys. Lett. B **424** (1998) 335 [hep-ph/9712457].

[135] M. Jamin, J. A. Oller and A. Pich, Nucl. Phys. B **622** (2002) 279 [hep-ph/0110193].

[136] A. Pich, Proc. Institute for Nuclear Theory (World Scientific) **12** (2002) 239 [hep-ph/0205030].

[137] V. Cirigliano, G. Ecker, H. Neufeld and A. Pich, JHEP **0306** (2003) 012 [hep-ph/0305311].

[138] J. Bijnens, E. Gámiz, E. Lipartia and J. Prades, JHEP **0304** (2003) 055 [hep-ph/0304222].

[139] T. Das *et al.*, Phys. Rev. Lett. **18** (1967) 759.

[140] R. F. Dashen, Phys. Rev. **183** (1969) 1245.

[141] J. F. Donoghue, B. R. Holstein and D. Wyler, Phys. Rev. D **47** (1993) 2089.

[142] J. Bijnens, Phys. Lett. B **306** (1993) 343 [hep-ph/9302217].

[143] J. Gasser, A. Rusetsky and I. Scimemi, Eur. Phys. J. C **32** (2003) 97 [hep-ph/0305260].

[144] B. Ananthanarayan and B. Moussallam, JHEP **0406** (2004) 047 [hep-ph/0405206].

[145] S. Descotes-Genon and B. Moussallam, Eur. Phys. J. C **42** (2005) 403 [hep-ph/0505077].

[146] A. Sirlin, Nucl. Phys. B **196** (1982) 83.

[147] W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **71** (1993) 3629.