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ColorFull – a C++ library for calculations in SU(Nc) color space

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Abstract: `ColorFull`, a C++ package for treating QCD color structure, is presented. `ColorFull`, which utilizes the trace basis approach, is intended for interfacing with event generators, but can also be used as a stand-alone package for squaring QCD amplitudes, calculating interferences, and describing the effect of gluon emission and gluon exchange.

Keywords: QCD, SU(N), Color, Colour

Contents

1 Introduction

The description of QCD color structure in the presence of many external colored partons is a field of increased importance. Some methods for performing automatic color summations of fully contracted vacuum bubbles, for example as implemented in `FeynCalc` [1], in the C program `COLOR` [2], or as presented in [3], have been around for a while, and recently a more flexible Mathematica package, `ColorMath` [4], allowing color structures with any number of open indices, has been published. Yet other general purpose event generator codes, such as `MadGraph` [5], have separate built in routines for dealing with the color structure.

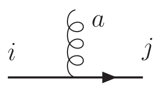
In the present paper a stand-alone C++ code, `ColorFull`, designed for dealing with color contraction using color bases is presented[1]. `ColorFull` is written with interfacing to event generators in mind, and is currently interfaced to `Herwig++` (2.7) [6, 7], but can also be used as a stand-alone package for investigations in color space.

`ColorFull` is based on trace bases [8–16], where the basis vectors are given by (products of) closed and open quark-lines, but the code also offers functionality for reading in and treating any (orthogonal) basis for color space, such as multiplet bases [17, 18].

The intent of this paper is to convey the underlying idea of `ColorFull`. For full technical details we refer to the online Doxygen documentation[2]. To set the stage, a brief introduction to the QCD color space is given in section 2 and the trace basis approach is presented in section 3. Following this, some remarks about the computational strategy are made in section 4 and the key design features are presented in section 5, whereas examples of stand-alone usage are given in section 6, and the interface to `Herwig++` is commented upon in section 7. The following section, section 8, describes the classes of `ColorFull` and code validation is discussed in section 9. Finally some concluding remarks are made in section 10.

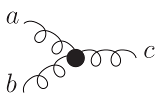
2 Color space

Apart from four-gluon vertices for which the color structure can be rewritten in terms of (one-gluon contracted) triple-gluon vertices, the QCD Lagrangian contains

quark-gluon vertices, 

$= (t^a)^i{}_j, \quad (2.1)$

and

triple-gluon vertices, 

$= i f^{abc}, \quad (2.2)$

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[1] `ColorFull` can be downloaded from `http://colorfull.hepforge.org/`.

[2] The automatically generated Doxygen documentation is available at `http://colorfull.hepforge.org/doxygen`.

where we follow the convention of reading the fully anti-symmetric structure constant indices in counter clock-wise order. The color structure of any amplitude, tree-level or beyond, pure QCD or not, can thus be expressed in terms of these objects alone. For observables we are – as QCD is confining – only interested in color summed/averaged quantities.

Letting $\mathbf{c}_1$ denote the color structure of the amplitude under consideration, we are thus interested in $|\mathbf{c}_1|^2$ where the scalar product[3] is given by summing over all external color indices, i.e.,

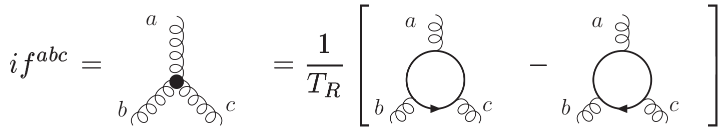
$$\langle \mathbf{c}_1 | \mathbf{c}_2 \rangle = \sum_{a_1,\, a_2,\, \ldots} \mathbf{c}_1^{*a_1\, a_2 \ldots}\, \mathbf{c}_2^{a_1\, a_2 \ldots} \tag{2.3}$$

with $a_i = 1, ..., N_c$ if parton i is a quark or an anti-quark and $a_i = 1, ..., N_c^2 - 1$ if parton i is a gluon.

Clearly, in any QCD calculation, the color amplitudes, $\mathbf{c}_1$ and $\mathbf{c}_2$, may be kept as they are, with color structure read off from the contributing Feynman diagrams. Alternatively – and this is likely to be the preferred solutions for more than a few partons – they may be decomposed into a color basis (spanning set), such as a trace basis [8–16], a color flow basis [19] or a multiplet basis [17, 18].

3 Trace bases

One way of organizing calculations in color space is to use trace bases [8–16]. To see that this is always possible, we note that the triple-gluon vertex can be expressed as

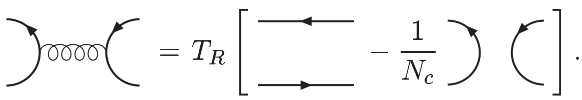


$$= \frac{1}{T_R}\left[(t^a)^i{}_j (t^b)^j{}_k (t^c)^k{}_i - (t^b)^i{}_j (t^a)^j{}_k (t^c)^k{}_i\right] = \frac{1}{T_R}\left[\mathrm{tr}[t^a t^b t^c] - \mathrm{tr}[t^b t^a t^c]\right]. \tag{3.1}$$

where T_R is the normalization of the $\mathrm{SU}(N_c)$ generators, $\mathrm{tr}(t^a t^b) = T_R \delta^{ab}$, commonly taken to be 1/2 or 1.

Using this relation on every triple-gluon vertex in any amplitude results in general in a sum of products of (connected) traces over $\mathrm{SU}(N_c)$ generators and open quark-lines. More specifically, there is one open quark-line for every incoming quark/outgoing anti-quark and outgoing quark/incoming anti-quark. (Note that, from a color perspective outgoing anti-quarks are equivalent to incoming quarks; we will here refer to them collectively as quarks. Similarly outgoing quarks are equivalent to incoming anti-quarks, and will be referred to as anti-quarks.)

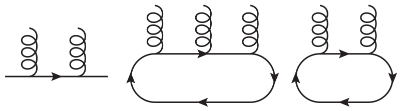
To further simplify the color structure, we may contract every internal gluon propagator (which after application of eq. (3.1) connects quark-lines) using the Fierz (completeness) relation



(3.2)

[3]It is not hard to prove that this actually is a scalar product.

From this we see that every amplitude in QCD, at any order, may be expressed as a sum of products of open and closed quark-lines, where *all* gluon indices are external indices. The set of all such products of quark-lines can thus be used as a spanning set for any QCD process, for example, for one $q\overline{q}$-pair and seven gluons, we may have color structures of form

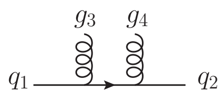


(3.3)

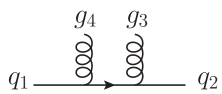
for all possible gluon permutations, as well as color structures with more or fewer traces. We here refer to this type of basis as a *trace basis*, although we remark that when the number of gluons, N_g, plus the number of $q\overline{q}$-pairs, N_q, exceeds N_c, this spanning set is overcomplete, and hence strictly speaking not a basis. For $N_g + N_q \leq N_c$, the bases are not overcomplete, but they are still non-orthogonal, having non-diagonal scalar products being suppressed by powers of N_c. Only in the $N_c \to \infty$ limit, are these bases minimal and orthogonal.

As a simple, but non-trivial example, we may consider the basis needed for $q_1\overline{q}_2 \to g_3 g_4$. The basis, which is constructed by connecting quarks and gluons in all allowed ways [16] is given by[4]

$$\mathbf{V}^0_{q_1,q_2,g_3,g_4} = (t^{g_3}t^{g_4})^{q_1}{}_{q_2} =$$

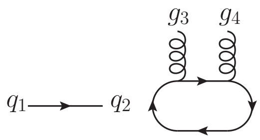


$$\mathbf{V}^1_{q_1,q_2,g_3,g_4} = (t^{g_4}t^{g_3})^{q_1}{}_{q_2} =$$



(3.4)

$$\mathbf{V}^2_{q_1,q_2,g_3,g_4} = \delta^{q_1}{}_{q_2}\mathrm{tr}(t^{g_3}t^{g_4}) =$$



.

To all orders in perturbation theory in the $N_c \to \infty$ limit, one can prove that the number of basis vectors can be found using the recursion relation [17]

$$N_{\mathrm{vec}}[N_q, N_g] = N_{\mathrm{vec}}[N_q, N_g - 1](N_g - 1 + N_q) + N_{\mathrm{vec}}[N_q, N_g - 2](N_g - 1)\,, \tag{3.5}$$

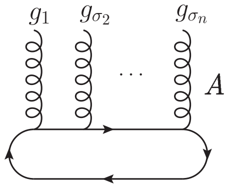
where

$$N_{\mathrm{vec}}[N_q, 0] = N_q!\,, \quad N_{\mathrm{vec}}[N_q, 1] = N_q N_q!\,. \tag{3.6}$$

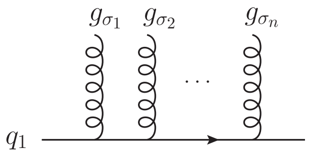
In the gluon-only case, at tree-level, the only color structures that can appear are traces

[4] To enhance the similarity with C++, the vector numbering starts at 0 here.

over generators, meaning that a general tree-level gluon amplitude can be decomposed as

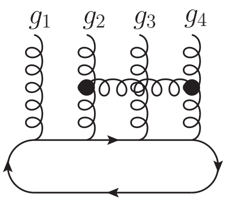
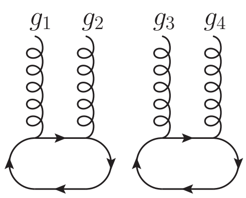
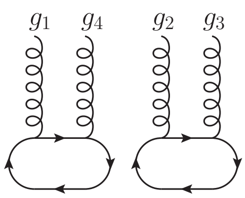
$$\mathcal{M}(g_1, g_2, \ldots, g_n) = \sum_{\sigma \in S_{N_g - 1}} \mathrm{tr}(t^{g_1} t^{g_{\sigma_2}} \ldots t^{g_{\sigma_n}}) A(\sigma) = \sum_{\sigma \in S_{N_g - 1}}$$  $$A(\sigma). \quad (3.7)$$

That only fully connected color structures enter in tree-level gluon amplitudes can easily be understood from the decomposition of Feynman diagrams into basis vectors; upon application of eq. (3.1) all external gluons remain attached to a quark-line, and – while contracting internal gluons using the Fierz identity, eq. (3.2) – they remain connected to the same quark-line, as the color suppressed terms cancel each other out. (This can be proved by a short calculation.) The same cancellation appears for gluon exchange between a quark and a gluon, meaning that also tree-level color structures for one $q\overline{q}$-pair and N_g gluons must be of the "fully connected" form of a trace that has been cut open, an open quark-line,

$$\mathcal{M}(q_1, g_3, \ldots, g_n, \overline{q}_2) = \sum_{\sigma \in S_{N_g}} (t^{g_{\sigma_1}} t^{g_{\sigma_2}} \ldots t^{g_{\sigma_n}})^{q_1}{}_{q_2} A(\sigma) = \sum_{\sigma \in S_{N_g}}$$  $$A(\sigma), \quad (3.8)$$

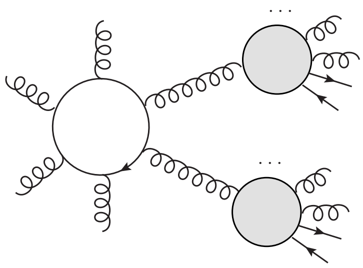
i.e., only the first two basis vectors in eq. (3.4) are needed. However, when the Fierz identity is applied directly to a gluon exchange between quarks, as in eq. (3.2), both terms do appear, and color structures with up to N_q disconnected quark-lines may appear even at tree-level.

Starting from a trace basis tree-level color structure, for example a single trace over gluons, and exchanging a gluon between two partons may split off a disconnected color structure, such as in

 $= -T_R$  $- T_R$  . (3.9)

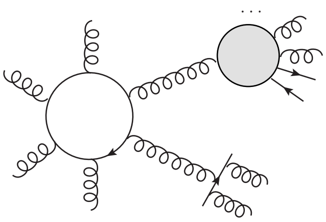
Thus, counting to l_g additional gluon exchanges (on top of a tree-level diagram), the color structures can not consist of more than $\max(1, N_q) + l_g$ open and closed quark-lines, two in the above case. In general, when any Feynman diagram is decomposed into a trace basis, there can be at most $N_q + \lfloor N_g/2 \rfloor$ quark-lines, since all gluons may be disconnected from the quarks, but no gluon can stand alone in a trace, giving the factor $\lfloor N_g/2 \rfloor$.

For NLO color structures having a quark-loop in the Feynman diagram, the quark-loop is necessarily connected to the remaining color structure via at least one gluon exchange, unless the Feynman diagram consists of only gluons attached to one quark-loop, in which case the trace basis decomposition is trivial. In general it may be connected to more tree-level color structures, for example of form

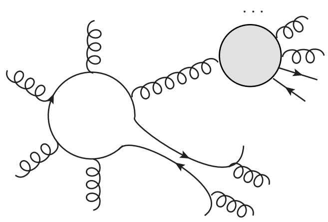


$$, \tag{3.10}$$

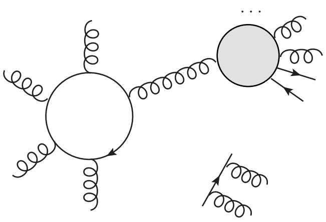
where the gray blobs denote any tree-level color structures. For each color structure connected to the quark-loop (gray blob), contracting the connecting gluon gives a term where the quark-loop is disconnected if the connecting gluon goes to a quark, the second term in



$$= T_R$$



$$- \frac{T_R}{N_c}$$



$$. \tag{3.11}$$

If the quark-loop is not connected to any other color structure (there is no remaining gray blob in the equation above), the resulting color structure thus contains one more single trace meaning that, decomposed in the trace basis, we get one more quark-line than at tree-level. (For purely gluonic processes, no additional trace will appear.) The same argument can be applied repeatedly if the Feynman diagram contains more than one closed quark-loop, giving (similar to the gluon case) no more than $\max(N_q, 1) + l_q$ quark-lines for l_q quark-loops.

Finally, we note that, if, on top of the quark-loops, there are also l_g loops from gluon exchanges in the Feynman diagram, these may contribute with up to one additional quark-line each (such as in eq. (3.9)), completing the argument that for $n_l = l_q + l_g$ loops, a general color structure in pure QCD can be written in terms of sums of products of closed and open quark-lines, where each product contains at most $\min[\max(1, N_q) + n_l, N_q + \lfloor N_g/2 \rfloor]$ traces and open quark-lines.

4 Computational strategy

The `ColorFull` strategy for treating color space is based on the above observations, i.e.,

- For given external particles (quarks, anti-quarks and gluons), we may always decompose the color space into a linear combination of closed and open quark-lines, as described in section 3. (Other color bases may be expressed in terms of linear combinations of these terms.)

- We can always evaluate scalar products by first replacing triple-gluon vertices using eq. (3.1), then removing all gluon propagators using eq. (3.2), and finally contracting the remaining product of $q\overline{q}$-delta functions.

The above outlined procedure for calculating scalar products has the advantages of being conceptually easy, and of covering all contractions occurring in QCD. It has the disadvantage of potentially giving a very large number of terms, since every replacement of a structure using eq. (3.1) may double the number of terms, and similarly, so does naive application of the Fierz identity, eq. (3.2).

In order to mitigate this potential explosion of terms upon index contraction, the `ColorFull` procedure for scalar product contraction is:

1. Contract all quark-ends, giving a sum of products of *closed* quark-lines.
2. On the individual quark-lines, contract all neighboring gluons (giving a factor C_F) and all next to neighboring gluons, giving a factor $-T_R/N_c$ each, as only the color suppressed term in the Fierz identity, eq. (3.2), survives. Also, contract traces of two gluons, $\mathrm{tr}(t^{g_1}t^{g_2}) = \delta^{g_1 g_2}/T_R$.
3. Look for gluons to contract within the quark-lines. Each such contraction may give rise to two terms, but at least the new traces tend to be shorter.
4. Look for gluon index contractions between the quark-lines.

In this way all color indices can be iteratively contracted. To further speed up calculations, `ColorFull` can use memoization to save calculated topologies, meaning that contractions that differ only by a relabeling of indices are performed only once. This significantly speeds up the calculations.

5 ColorFull at a glance

`ColorFull` is designed to handle the contraction of QCD color indices, to decompose the QCD color space using (trace) basis vectors, and to describe the effect of gluon exchange and gluon emission. In particular, `ColorFull` can, for arbitrary N_c:

- Square any QCD amplitude and calculate any interference term.
- Create a trace basis for any number of quarks and gluons, and to arbitrary order.
- Read in and write out color bases, including non-trace bases.
- Calculate scalar product matrices, i.e., the matrices of scalar products between the basis vectors, write these out and read them in again.
- Describe the effect of gluon exchange, including calculating the color soft anomalous dimension matrices.

- Describe the effect of gluon emission.

ColorFull can also be interfaced to Herwig++ ($\geq$ 2.7) via Matchbox [20] and can thus easily be used for event generation with Herwig++.

5.1 Technical overview

Table 1. Below is a complete list of the ColorFull classes, ordered by dependence, such that, a certain class only depends on classes standing above it in the list.

Class	Functionality
Monomial	Class for containing a single term of form $\texttt{Nc}^{\texttt{pow\_Nc}} \times \texttt{TR}^{\texttt{pow\_TR}} \times \texttt{CF}^{\texttt{pow\_CF}} \times$ int_part $\times$ cnum_part and associated functions.
Polynomial	Class for containing a sum of Monomials and associated functions.
Poly_vec	Class for containing a vector of Polynomials and associated functions.
Poly_matr	Class for containing a matrix of Polynomials and associated functions.
Quark_line	Class for containing a single closed or open quark-line multiplying a Polynomial and associated functions. For example a term of the form $\mathrm{tr}[t^{g_1}t^{g_2}t^{g_3}t^{g_4}] \sim$ (1,2,3,4) or of the form $(t^{g_3}t^{g_4})^{q_1}_{q_2} \sim$ {1,3,4,2} times a Polynomial.
Col_str	Class for containing a product of Quark_lines multiplying a Polynomial, and associated functions. For example a polynomial (N_c^2-1) may multiply the Quark_lines, $\delta^{q_1}{}_{q_2}\delta^{g_1g_4}$, in total giving the Col_str $\sim$ (Nc2-1) [{1,2}(3,4)]
Col_amp	Class for containing a sum of Col_strs such as $\mathrm{tr}[t^{g_1}t^{g_2}t^{g_3}t^{g_4}] + \mathrm{tr}[t^{g_1}t^{g_2}]\mathrm{tr}[t^{g_3}t^{g_4}]$ $\sim$(1,2,3,4)+(1,2)(3,4) and associated functions.
Col_functions	Library class containing functions for performing numerical evaluations, taking the leading N_c limit, evaluating scalar products, and describing gluon emission and exchange.
Col_basis	Base class for all basis classes, see below.
Trace_type_basis	Base class for Trace_basis and Tree_level_gluon_basis.
Trace_basis	Class for creating and using trace bases. Perhaps the most important basis class.
Tree_level_gluon_basis	Class for bases needed for tree-level gluon calculations, where each basis vector is a single trace plus an implicit complex conjugate.
Orthogonal_basis	Class for utilizing the benefits of orthogonal bases such as multiplet bases [17, 18].

This section is intended to give an overview of ColorFull. For examples, the reader is referred to section 6, for overviews of the classes to section 8, and for details, the tables in the appendix A, as well as the online Doxygen documentation.

As contraction of indices gives rise to polynomials in N_c, T_R and C_F, `ColorFull` by necessity needs minimal classes for dealing with such polynomials. This is implemented in the classes `Monomial` (a single term in a polynomial[5]), `Polynomial` (a sum of `Monomials`), `Poly_vec` (a vector of `Polynomials`) and `Poly_matr` (a matrix of `Polynomials`).

For the color structure itself, `ColorFull` uses the class `Quark_line` for treating an individual closed or open quark-line, a class `Col_str` for treating a product of `Quark_lines` and a class `Col_amp` for treating a general color amplitudes, i.e., a linear combination of `Col_strs`.

For performing numerical evaluations, taking the leading N_c limit, evaluating scalar products, and describing gluon emission and exchange, `ColorFull` has a library class `Col_functions`.

Finally, `ColorFull` has classes for describing color bases. The classes intended for the user, `Trace_basis`, `Tree_level_gluon_basis` and `Orthogonal_basis`, are derived from the base class `Col_basis` (in the case of the first two via the class `Trace_type_basis`).

All `ColorFull` classes are listed in table 1 according to dependencies, meaning that each class only depends on classes listed above it. The next section will give an introduction to using `ColorFull` in stand-alone mode.

6 Stand-alone usage

`ColorFull` is mainly designed to deal with relatively large color spaces where it is advantageous to use bases in which color structures coming from Feynman diagrams – or alternative recursive strategies – are decomposed.

For trace bases, the bases may be automatically created by `ColorFull`. For example, a basis for `1` $q\overline{q}$-pair, `3` gluons and `0` loops (in pure QCD) can be created using

```
Trace_basis MyBasis(1,3,0);.                                   (6.1)
```

The last argument is provided to avoid carrying around basis vectors for which the kinematic factors vanish to a certain order in perturbation theory. It can be skipped upon which an all order basis vector is created.

To view the resulting basis, it can be written out to a stream (`cout`) or to a file, either with default filename (no argument) or a user supplied name,

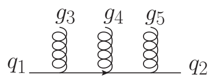
```
cout << MyBasis;
MyBasis.write_out_Col_basis("ColorResults/MyBasis");           (6.2)
```

[5]Throughout the `ColorFull` documentation terms of form constant$\times N_c^a\, T_R^b\, C_F^c$ will be referred to as monomials, despite the possible occurrence of negative powers. Similarly sums of such terms will be referred to as polynomials.

resulting in

```
0    [{1,3,4,5,2}]
1    [{1,3,5,4,2}]
2    [{1,4,3,5,2}]
3    [{1,4,5,3,2}]
4    [{1,5,3,4,2}]
5    [{1,5,4,3,2}]
```
(6.3)

where, for example,

$$\texttt{[\{1,3,4,5,2\}]} = (t^{g3}t^{g4}t^{g5})^{q1}{}_{q2} =$$ 

. (6.4)

If no argument is supplied to `MyBasis.write_out_Col_basis()`, the basis is written out to a file with a default filename, in this case `CF_TB_q_1_g_3`. The bases can thus be written out and saved for future purposes. For reading in a basis `MyBasis.read_in_Col_basis("path/to/filename")` can be used.

The option of reading in bases is particularly useful for the `Orthogonal_basis` class. Presently `ColorFull` can not automatically create orthogonal multiplet bases, such as in [17], but externally created (orthogonal) bases, with basis vectors expressed in terms of sums of products of traces, may be read in and used. For the `Orthogonal_basis` class, the orthogonality is then utilized to significantly speed up the the calculation of scalar products. For a non-orthogonal basis it is necessary to evaluate all scalar products between all basis vectors, which can be done as

```
MyBasis.scalar_product_matrix();.
```
(6.5)

`Polynomial` and `double` versions of the scalar product matrix are then calculated and saved in the member variables `P_spm` and `d_spm`. For larger bases, the evaluation can be sped up by only calculating the numerical version using `MyBasis.scalar_product_matrix_num()`. The content of `P_spm` and `d_spm` can be explored by using the `<<` operator, but it may also be saved to a file using the `Col_basis` member functions `write_out_P_spm()` and `write_out_d_spm()` which write out the result in `ColorFull` and Mathematica, readable format, by default to a file with a default filename in the directory `ColorResults`. Alternatively the user may supply a filename as argument.

While the intended usage of `ColorFull` is to use color bases, `ColorFull` can also directly define color amplitudes

```
Col_amp Ca1("[{1,3,2}(4,5)]");      // (t^{g3})^{q1}_{q2} tr(t^{g4} t^{g5})
Col_amp Ca2("[{1,3,4,5,2}]");       // (t^{g3} t^{g4} t^{g5})^{q1}_{q2}
```
(6.6)

and evaluate scalar products using the `scalar_product` member function in the `Col_functions` library class

```
Col_functions Col_fun;
Col_fun.scalar_product(Ca1,Ca1);                (6.7)
Col_fun.scalar_product(Ca1,Ca2);
```

giving the results `TR*Nc^2*CF^2` and `TR*Nc*CF^2`, respectively.

In several contexts, such as parton showers, cancellation of infrared singularities in NLO calculations, and recursive methods [21] for calculating amplitudes, it is of interest to know the effect of gluon emission on a color structure. This can be calculated by using the `Col_functions` member function `emit_gluon`

```
Col_fun.emit_gluon(Ca1,3,6);                (6.8)
```

resulting in

```
[{1,3,6,2}(4,5)]-[{1,6,3,2}(4,5)].                (6.9)
```

The sign conventions here and elsewhere are such that every gluon inserted after the emitter in the quark-line comes with a plus sign and every gluon inserted before comes with a minus sign. In a basis-decomposed calculation, one would be interested in this result decomposed in the large basis required for one additional gluon. Having a trace basis for this larger vector space, this decomposition can be calculated using

```
Trace_basis LargerBasis(nq,ng+1);                (6.10)
LargerBasis.new_vector_numbers(Cs, emitter);                (6.11)
```

where `Cs` is the `Col_str` (basis vector in the smaller trace basis) from which the parton `emitter` emits a new gluon.

Similarly the effect of gluon exchange on a color structure is of interest. `ColorFull` offers several functions for dealing with this. Starting with an amplitude, the new amplitude after gluon exchange can be obtained as

```
Col_fun.exchange_gluon(Ca1, 1, 4);                (6.12)
```

resulting in `TR[{1,4,5,3,2}] - TR[{1,5,4,3,2}]`. `ColorFull` also has a function for directly calculating the color correlator, i.e., given a color amplitude $|\mathbf{c}\rangle$, the quantity $\langle\mathbf{c}|\mathbf{T}_i \cdot \mathbf{T}_j|\mathbf{c}\rangle$ where $\mathbf{T}_i$ describes the effect of attaching a gluon to parton i. For example

```
Col_fun.color_correlator(Ca1, 1, 2);                (6.13)
```

results in the `Polynomial TR^(2)*Nc*CF^(2)`.

In some situations, such as soft gluon resummation, it is also useful to calculate the soft anomalous dimension matrix, i.e., to have the result of gluon exchange on any basis vector decomposed into the basis. This can be computed automatically using the `Col_basis` member function color_gamma. The result is contained in a Poly_matr. For this we need the full basis since color structures not present at LO will appear at NLO, etc. Coding

```
Trace_basis MyFullBasis(1,3);
MyFullBasis.color_gamma(1,4);                                    (6.14)
```

for gluon exchange between the partons 1 and 4 results in the matrix

```
{{ 0, 0, 0, 0, 0, 0, 0, 0, 1*-1 TR, 0, 0},
{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1*-1 TR},
{ 0, 0, 1 TR Nc, 0, 0, 0, 0, 0, 1 TR, 0, 1 TR},
{ 0, 0, 0, 1 TR Nc, 0, 0, 1 TR, 0, 0, 1 TR, 0},
{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1*-1 TR, 0},
{ 0, 0, 0, 0, 0, 0, 1*-1 TR, 0, 0, 0, 0},
{ 0, 0, 0, 0, 0, 1*-1 TR, 0, 0, 0, 0, 0},
{ 0, 1 TR, 0, 0, 1 TR, 0, 0, 1 TR Nc, 0, 0, 0},
{ 1*-1 TR, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 0, 0, 1*-1 TR, 0, 0, 0, 0, 0, 0},
{ 0, 1*-1 TR, 0, 0, 0, 0, 0, 0, 0, 0, 0}}.                       (6.15)
```

In this way all the soft anomalous dimension matrices needed in [22] can easily be recalculated.

While the number of colors in QCD is three, ColorFull can deal with any N_c, both algebraically and numerically. Numerical evaluation is handled by the Col_functions class, using the (private) member variables Nc, TR and CF. Thus also T_R and C_F can be changed independently. The reason for keeping C_F as a parameter technically independent of N_c is that this allows for keeping the color suppressed part of C_F, $-T_R/N_c$, in a consistent way. This choice has proved useful for accounting for sub-leading N_c effects in several phenomenological studies [23, 24].

To numerically evaluate a Monomial, Polynomial, Poly_vec or Poly_matr, the Col_functions member functions double_num are used, for example we may want the $N_c = 3$ version of a scalar product

```
Col_fun.double_num(Col_fun.scalar_product(Ca1,Ca1));             (6.16)
```

giving 8.

For comparison, it is of interest to evaluate (squared) amplitudes in the limit $N_c \to \infty$. For taking the leading N_c limit of any of the polynomial classes Polynomial, Poly_vec or Poly_matr, the Col_functions member function(s) leading can be used. For example we may take the leading N_c limit of the matrix in eq. (6.15),

```
Col_fun.leading(MyFullBasis.color_gamma(1,4));                   (6.17)
```

resulting in the diagonal matrix

```
{{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 1 TR Nc, 0, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 0, 1 TR Nc, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 0, 0, 0, 0, 0, 1 TR Nc, 0, 0, 0},
{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}.
```
(6.18)

For an expression containing C_F, the color suppressed term $-T_R/N_c$ can be kept in numerical evaluation by setting `full_CF` to `true`, `Col_fun.set_full_CF(true)`.

7 Interfacing to Herwig via Matchbox

`ColorFull` can also be interfaced to `Herwig++` ($\geq$2.7) [7] via the `Matchbox` component [20], and can be used to treat the hard interaction, as in for example [25], as well as the parton shower itself [23]. When linked to `Herwig++`, `ColorFull` is hooked into the `boost` linear algebra package, enabling a very efficient treatment of numerical linear algebra. From the next major release of `Herwig++`, `ColorFull` will be directly shipped with the `Herwig++`, code.

8 ColorFull code

8.1 Operators

In order to simplify calculations and increase readability, `ColorFull` defines a standard set of operators, listed in table 2. This includes the arithmetic operators `+`, `-` and `*` as well as the comparison operators, `==` and `!=`, and the stream operator `<<` . The comparison operators, `==` and `!=`, work by comparing data entries term by term, both for the polynomial classes and for the color carrying classes. Thus, for example, two `Polynomials` only differing by the order of terms are not considered equal.

8.2 Classes for Polynomials

The result of color index contraction can always be written as a sum of terms of the form ${N_c}^a {T_R}^b {C_F}^c$ $\times$ constant where we allow for negative integers a, b, c. For the purpose of treating contracted color structures, `ColorFull` has a minimalistic set of classes for basic manipulation of color factors arising when contracting color indices. One such term is

Table 2. Standard operators for color structure and polynomial classes. (Operators for both orders of arguments are defined for the arithmetic operators +, - and *.)

Operator	Classes
+ and -	Polynomial +/- (Monomial, Polynomial) Col_amp +/- (Col_str, Col_amp)
+=	Polynomial += (Monomial, Polynomial) Col_amp += (Col_str, Col_amp)
*	Monomial*(int, cnum, double, Monomial) Polynomial*(int, cnum, double, Monomial, Polynomial) Quark_line*(int, cnum, double, Monomial, Polynomial, Quark_line) Col_str*(int, cnum, double, Monomial, Polynomial, Quark_line, Col_str) Col_amp*(int, cnum, double, Monomial, Polynomial, Quark_line, Col_str, Col_amp)
*=	Monomial*=(int, cnum, double, Monomial) Polynomial*=(int, cnum, double, Monomial, Polynomial) Col_amp*=(Col_str, Col_amp)
==	Monomial == Monomial Polynomial == Polynomial Poly_vec == Poly_vec Poly_matr == Poly_matr Quark_line == Quark_line Col_str == Col_str Col_amp == Col_amp
!=	Monomial != Monomial Polynomial != Polynomial Poly_vec != Poly_vec Poly_matr != Poly_matr Quark_line != Quark_line Col_str != Col_str Col_amp != Col_amp
<	Monomial < Monomial The Monomials are ordered first according to pow_Nc + pow_CF, then according to pow_Nc (for same pow_Nc + pow_CF), then according to int_part*abs(cnum_part), then according to int_part and finally according to pow_TR. This is thus *not* a magnitude operator.
<<	Monomial, Polynomial, Poly_vec, Poly_matr, Quark_line, Col_str, Col_amp, Col_basis

defined as a Monomial, and a sum of such terms as a Polynomial. To decompose vectors in a color space we also need a vector of Polynomials, contained in a Poly_vec, and for a scalar product matrix (or soft anomalous dimension matrix) we need a matrix of polynomials, a `Poly_matr`.

Apart from the `Monomial` class, these classes have the actual polynomial information contained in uncapitalized typedefs carrying the same name as the class in question. There are thus `polynomial`, `poly_vec`, and `poly_matr` typedefs. For manipulating the polynomial classes, the operators listed in table 2 may be used. For example, we may multiply a `Polynomial` and a `Monomial`.

8.2.1 Monomial

The simplest class for containing contracted color information is a `Monomial`

$$\texttt{Monomial} \sim \texttt{Nc}^{\texttt{pow\_Nc}}\,\texttt{TR}^{\texttt{pow\_TR}}\,\texttt{CF}^{\texttt{pow\_CF}} \times \texttt{int\_part} \times \texttt{cnum\_part}. \tag{8.1}$$

The exponents `pow_Nc`, `pow_TR` and `pow_CF`, as well as `int_part` are of type `int` and `cnum_part` is a `cnum`, an `std::complex<double>`. These member variables carry the information about the actual monomial.

By default, using the standard constructor, `Monomial()`, a `Monomial` is set to 1, by letting `cnum_part=1`, `int_part=1` and setting all powers to 0[6]. A `Monomial` with different integer part can be obtained by using the constructor taking an `int` or as argument, and a general `Monomial` can be constructed using the string constructor (see table 3 in appendix A).

8.2.2 Polynomial

A sum of `Monomials` is contained in the class `Polynomial`

$$\texttt{Polynomial} \quad \sim \quad \underbrace{\texttt{Monomial}_0 + \texttt{Monomial}_1 + \ldots}_{\texttt{poly}} \tag{8.2}$$

where, technically, the polynomial information is stored in the public member `poly`, of type `polynomial=std::vector<Monomial>`. `ColorFull` is not designed for manipulating polynomials but `Polynomial` nevertheless has a minimalistic `simplify()` function which collects terms with equal power of N_c, T_R and C_F. It also has a `remove_CF()` function for replacing C_F with $T_R N_c - T_R/N_c$. Details about the `Polynomial` class can be found in table 4 in appendix A.

8.2.3 Poly_vec

For dealing with vectors of `Polynomials`, `ColorFull` has a class `Poly_vec`

$$\texttt{Poly\_vec} \quad \sim \quad \underbrace{(\texttt{Polynomial}_0, \texttt{Polynomial}_1, \ldots)}_{\texttt{pv}}\,. \tag{8.3}$$

`Poly_vec` is a container class for functions acting on vectors of `Polynomials` whereas the actual information is stored in the `poly_vec=typedef std::vector<Polynomial>` data member, `pv`. The member functions of `Poly_vec`, which include `simplify()` and `remove_CF()`, can be found in table 5 in appendix A.

[6] A default `Monomial` is thus the neutral element under multiplication, rather than under addition.

8.2.4 Poly_matr

Finally, there is a class for treating matrices of `Polynomial`s, a `Poly_matr`,

$$\begin{array}{lcl} \texttt{Poly\_matr} & \sim & \{\{\texttt{Polynomial}_{00}, \texttt{Polynomial}_{01}, ..., \texttt{Polynomial}_{0n}\} \\ & & \qquad\qquad\vdots \\ & & \{\texttt{Polynomial}_{m0}, \texttt{Polynomial}_{m1}, ..., \texttt{Polynomial}_{mn}\}\} \\ & & \qquad\qquad\texttt{pm} \end{array} \tag{8.4}$$

which stores the matrix information as a vector of vectors of `Polynomial`s, a `poly_matr=typedef std::vector<Poly_vec>`. Again, details can be found in the appendix, in table 6.

8.3 Classes containing the color structure

This section describes the building blocks used by `ColorFull` to treat the color structure. *All* classes used to carry the color structure, `Quark_line`, `Col_str`, `Col_amp`, including the basis classes `Col_basis` – from which `Trace_type_basis`, `Trace_basis`, `Tree_level_gluon_basis` and `Orthogonal_basis` are derived – have the property that the actual information of the color structure is contained in a type with corresponding name, whereas the class acts as a container for related functions. Thus, for example, the color amplitude information in a `Col_amp` is contained in a `col_amp` variable.

8.3.1 Quark_line

The most basic color carrying class is a `Quark_line`. Loosely speaking, a `Quark_line` is a `quark_line` times a `Polynomial`,

$$\begin{array}{lcccc} \texttt{Quark\_line} & \sim & \texttt{Polynomial} & \times & \texttt{quark\_line} \\ & & \texttt{Poly} & \times & \texttt{ql}, \end{array} \tag{8.5}$$

where the `quark_line` (a `std::vector<int>`) `ql` contains the actual quark-line information, together with the boolean variable `open` which is `true` if the quark-line is open, and `false` for a trace over gluon indices. The `quark_line` is multiplied by a `Polynomial`. For example, we may have the closed `Quark_line`

$$\texttt{(Nc\^{}2-1) (1,2,3,4)} \sim (N_c^2 - 1)\ \mathrm{tr}(t^{g_1} t^{g_2} t^{g_3} t^{g_4}) \tag{8.6}$$

or the open `Quark_line`

$$\texttt{Nc TR \{1,3,4,2\}} \sim N_c T_R\ (t^{g_3} t^{g_4})^{q_1}{}_{q_2}. \tag{8.7}$$

`Quark_line`s may be created using the `Quark_line(std::string)` constructor. Closed quark-lines are denoted by standard parenthesis, whereas curly brackets represent open quark-lines. For example, we may write

```
Quark_line Ql1("(Nc^2-1)(1,2,3,4)");
Quark_line Ql2("Nc TR {1,3,4,2}");
```
(8.8)

for the above quark-lines. Among the `quark_line` member functions we especially note the functions for contracting neighboring and next to neighboring gluon indices, `contract_neighboring_gluons()` and `contract_next_neighboring_gluons()`, as well as the `normal_order()` member function which orders a closed `Quark_line` such that the smallest gluon index is written first. Thus, for example

```
Quark_line Ql("(2,3,4,4,5,1)");
Ql.contract_neighboring_gluons();
Ql.normal_order();
```
(8.9)

results in `CF(1,2,3,5)`. The other public member functions are listed in table 7 in appendix A.

8.3.2 Col_str

A general color amplitude consists not only of one quark-line, but of a linear combination of products of `Quark_lines`. One term in this linear combination, a `Polynomial` times a product of closed and open quark-lines is contained in a `Col_str`,

$$\begin{array}{cccccc} \texttt{Col\_str} & \sim & \texttt{Polynomial} & \times & \texttt{col\_str} & \\ & & \texttt{Poly} & \times & \texttt{cs} & . \end{array} \tag{8.10}$$

Here the actual information about the color structure is carried by a `col_str`, (technically a vector of `Quark_lines`), for example we may thus have

$$\texttt{(Nc\^{}2-1) \{1,3,4,2\}(5,6)(7,8)} \sim (N_c^2-1)\,(t^{g_3}t^{g_4})^{q_1}{}_{q_2}\mathrm{tr}(t^{g_5}t^{g_6})\,\mathrm{tr}(t^{g_7}t^{g_8}). \tag{8.11}$$

Like the other color classes, `Col_strs` may be created using a string constructor

```
Col_str Cs("(Nc^2-1)[{1,3,4,2}(5,6)(7,8)]");
```
(8.12)

where we note that the `col_str` is written inside square brackets. The `Polynomial` should multiply the whole `col_str`, rather than individual `quark_lines`.

The `Col_str` member functions, listed in table 8, overlap to a high degree with the `Quark_line` member functions. In particular there are functions for contracting neighboring and next to neighboring gluons, and a `normal_order` member function. Apart from normal ordering the individual `Quark_lines`, the `Col_str` `normal_order()` member function sorts the `Quark_lines` as described in table 8 in appendix A, where the public members of `Col_str` are listed.

For `Col_str` (and `Quark_line` and `Col_amp`), there is also a `simplify()` member function. This function removes quark-lines with 0 quarks, (these are `Nc` for closed `Quark_lines` and `1` for open `Quark_lines`), normal orders the `col_strs`, and simplifies the `Polynomials`. Potential `Polynomials` multiplying individual `Quark_lines` are also moved to the `Col_str` member variable `Poly`.

8.3.3 Col_amp

The linear combination of `Col_strs` which in general is needed to keep track of a color structure is contained in a `Col_amp`. In order to contain scalar results, i.e., terms not containing any color structure (which arise in the process of color contraction) a `Col_amp` also contains a `Polynomial` part, `Scalar`,

$$\begin{array}{lcccl} \texttt{Col\_amp} & \sim & \texttt{Polynomial} & + & \texttt{col\_amp} \\ & & \texttt{Scalar} & + & \texttt{ca} \\ \text{where} & & & & \\ \texttt{ca} & \sim & \texttt{Col\_str}_0 + \texttt{Col\_str}_1 + \texttt{Col\_str}_2 + \ldots & & \end{array}.$$

For example, we may have the `Col_amp`

$$\texttt{(1,2)(3,4) - Nc\^{}(-1)(1,2,3,4)} \sim \mathrm{tr}(t^{g_1}t^{g_2})\,\mathrm{tr}(t^{g_3}t^{g_4}) - \frac{1}{N_c}\mathrm{tr}(t^{g_1}t^{g_2}t^{g_3}t^{g_4}), \quad (8.13)$$

where `Scalar` is `0`, and the `Polynomials` multiplying the `Col_strs` `(1,2)(3,4)` and `(1,2,3,4)` are `1` and `1/Nc` respectively.

Also the `Col_amp` class has a string constructor which reads in to the `col_amp` part according to the syntax

$$\texttt{Col\_amp Ca("[(1,2,3,4)]-1/Nc[(1,2)(3,4)]");}. \quad (8.14)$$

As for `Quark_line` and `Col_str`, `Col_amp` has member functions for contracting all neighboring and next to neighboring gluons. Gluon contraction between gluons which are further away on the `Quark_line` may result in a sum of `Quark_lines`. Such contractions cannot be seen as actions on a single `quark_line` and are therefore not implemented in the `Quark_line` and `Col_str` classes. Instead the `Col_amp` class contains functions for contracting a gluon or contracting all gluons in the `Col_amp`. These functions are intended for `Col_amps` with closed `Quark_lines`, i.e., quarks should be contracted first, using `contract_quarks(Col_amp1, Col_amp2)`. When calculating a scalar product, all quark indices are thus contracted first, followed by all the gluon indices. The complete list of public member functions is given in table 9 in appendix A.

8.4 Col_functions – a library class

`Col_functions` is a library class containing functions which cannot, in a natural way, be attributed to one class, or functions which act on many classes and are therefore conveniently collected into one class.

In particular, `Col_functions` contains classes for evaluating scalar products, for numerical evaluation, for taking the leading N_c limit, and for describing the effect of gluon emission or gluon exchange.

8.4.1 Numerical evaluation

`Col_functions` is the class which carries numerical values of N_c, T_R and C_F, stored in the private members `Nc`, `TR` and `CF` respectively. To numerically evaluate a `Monomial` to a double or a complex number (cnum) the function cnum_num or double_num is used

```
Col_functions  Col_fun;
Monomial  Mon;
Col_fun.cnum_num(Mon);                    (8.15)
Col_fun.double_num(Mon);
```

The syntax for numerical evaluation of Polynomials, Poly_vecs and Poly_matrs is identical.

8.4.2 Leading N_c evaluation

Col_functions also contains functions for taking the leading N_c limit of the classes Polynomial, Poly_vec and Poly_matr. The leading N_c terms can be evaluated in two different ways, either by taking the strict $N_c \to \infty$ limit and dropping all color suppressed terms (default), or by keeping the color suppressed terms in $C_F = T_R(N_c^2 - 1)/N_c$, while dropping other color suppressed terms. For keeping the full C_F in numerical evaluations the member variable full_CF must be set to true using set_full_CF(true).

8.4.3 Scalar products

Scalar products are evaluated using the Col_functions member functions scalar_product,

```
Col_str  Cs1("[{1,3,4,2}(5,6)]");
Col_str  Cs2("[{1,3,4,5,6,2}]");
Col_amp  Ca1(Cs1);
Col_amp  Ca2(Cs2);
Col_fun.scalar_product(Ca1,Ca2);          (8.16)
Col_fun.scalar_product(Cs1,Cs2);
```

in both cases resulting in $T_R N_c C_F^3$.

8.4.4 Gluon exchange and gluon emission

For soft gluon resummation, for NLO calculations, and for the cancellation of real emissions and virtual corrections in the soft limit, the color structure associated with gluon exchange plays an important role. The Col_functions class therefore has functions for describing the effect of gluon exchange on a Col_str or on a Col_amp. For example, we may exchange a gluon between parton 3 and 6 in Cs1

```
Col_fun.exchange_gluon(Cs1,3,6);          (8.17)
```

resulting in the Col_amp

```
-TR[{1,5,6,3,4,2}]+TR[{1,3,5,6,4,2}]+TR[{1,6,5,3,4,2}]-TR[{1,3,6,5,4,2}].
```

To understand the signs we note that each time a gluon is inserted before the emitter on a (closed or open) quark-line there is a minus sign, and each time a gluon is inserted after, there is a plus sign. We also note that in this case, the result of gluon exchange on a single color structure gave rise to a linear combination of four color structures, the maximal possible number of color structures from a single `col_str` [16].

In the context of gluon exchange we also remark that `Col_functions` can calculate the "color correlator", $\langle \mathbf{c}|\mathbf{T}_i \cdot \mathbf{T}_j|\mathbf{c}\rangle$ arising when coherently emitting a gluon from parton i and j in an amplitude $|\mathbf{c}\rangle$, or when exchanging a gluon between the partons i and j in $|\mathbf{c}\rangle$. For example we can calculate the color correlator for exchanging a gluon between parton 1 and 4 in `Ca1`,

$$\texttt{Col\_fun.color\_correlator(Ca1,1,4);} \tag{8.18}$$

resulting in a `Polynomial` with value $T_R^3 C_F^2 + T_R^2 N_c C_F^3$.

Sometimes, for example in the context of a parton shower, one may be interested in the effect of gluon emission itself. Starting in a `Col_amp Ca1` and emitting a gluon, 7, from parton 3 this may be found using

$$\texttt{Col\_fun.emit\_gluon(Ca1,3,7);} \tag{8.19}$$

giving the `Col_amp [{1,3,7,4,2}(5,6)]-[{1,7,3,4,2}(5,6)]`.

8.5 Classes for color bases

Although `ColorFull` may perform calculations with individual `Quark_lines`, `Col_strs` and `Col_amps`, the intended usage is via the color basis classes `Trace_basis` (in particular), `Tree_level_gluon_basis` and `Orthogonal_basis`.

As these classes share much of the most important functionality, they all inherit from one base class, `Col_basis`. The `Trace_basis` and `Tree_level_gluon_basis` classes inherit from `Col_basis` via `Trace_type_basis`, whereas `Orthogonal_basis` inherits directly from `Col_basis`.

8.5.1 Col_basis

The `Col_basis` class has a `col_basis` member variable `cb` for containing the basis vectors,

$$\texttt{Col\_basis} \sim \underset{\texttt{cb}}{\texttt{col\_basis}} . \tag{8.20}$$

`Col_basis` also carries information about the number of quarks and the number of gluons, in the public members `nq` and `ng`, and (if it has been calculated) the scalar product matrix in polynomial form, `P_spm`, and in double form, `d_spm`, as well as the leading (see section 8.4.2) scalar product matrix in polynomial and double versions, `leading_P_spm` and `leading_d_spm`.

The most important `Col_basis` member function is probably the `scalar_product_matrix()` function which calculates the matrix of scalar products between the basis vectors. In its default from, this function uses memoization, as this speeds up the calculations, but this may be circumvented in using the `scalar_product_matrix_no_mem()` version.

In order not to have to calculate scalar product matrices over and over again `Col_basis` also has functions for reading in and writing out scalar product matrices, both in numerical and polynomial form.

`Col_basis` also contains functions for reading in and writing out the basis itself. This is essential for the `Orthogonal_basis` class which cannot (presently) construct the orthogonal bases, but may read them in.

The decomposition of color amplitudes into bases is done with the (virtual) `decompose(Col_amp)` member function, which does the decomposition by exploring the coefficients in front of traces and products of traces for `Trace_basis` and `Tree_level_gluon_basis` and by evaluating scalar products for the `Orthogonal_basis` case.

Another important function is the `color_gamma` function which (using `decompose`) calculates the soft anomalous dimension matrix, i.e., calculates the matrix describing the effect of gluon exchange between two partons. The result is returned as a `Poly_matr` where component i, j gives the amplitude for ending up in vector i, if starting in vector j, see also section section 6. A list of public members and functions for `Col_basis` is found in table 11.

8.5.2 Trace_type_basis

`Trace_type_basis` is a small helper class for keeping track of functions which are similar for `Trace_basis` and `Tree_level_gluon_basis`. It inherits from `Col_basis` and is inherited from by `Trace_basis` and `Tree_level_gluon_basis`. Most importantly, this is where `decompose` is implemented for these two classes, see table 12.

8.5.3 Trace_basis

Although the observation that each color amplitude may be decomposed into products of open and closed quark-lines is a guiding principle for `ColorFull`, the `Trace_basis` class itself is rather small, containing mainly functions for creating bases.

A trace basis is created by first contracting all $q\overline{q}$-pairs in all $N_q!$ ways, and then attaching gluons to these closed quark-lines, and to additional closed quark-lines, such that at least two gluons are attached to each closed quark-line.

This can be done either directly using a constructor

```
Trace_basis Tb(2,2);        (8.21)
```

or as

```
Trace_basis Tb;
Tb.create_basis(2,2);       (8.22)
```

The result, which can be written out to a user given or default file (`write_out_Col_basis`) or to `cout` (using `<<`) has the basis vectors

$$\begin{array}{cc} 0 & \texttt{[\{1,5,6,2\}\{3,4\}]} \\ 1 & \texttt{[\{1,5,6,4\}\{3,2\}]} \\ 2 & \texttt{[\{1,6,5,2\}\{3,4\}]} \\ 3 & \texttt{[\{1,6,5,4\}\{3,2\}]} \\ 4 & \texttt{[\{3,5,6,2\}\{1,4\}]} \\ 5 & \texttt{[\{3,5,6,4\}\{1,2\}]} \\ 6 & \texttt{[\{3,6,5,2\}\{1,4\}]} \\ 7 & \texttt{[\{3,6,5,4\}\{1,2\}]} \\ 8 & \texttt{[\{1,5,2\}\{3,6,4\}]} \\ 9 & \texttt{[\{1,5,4\}\{3,6,2\}]} \\ 10 & \texttt{[\{1,6,2\}\{3,5,4\}]} \\ 11 & \texttt{[\{1,6,4\}\{3,5,2\}]} \\ 12 & \texttt{[\{1,2\}\{3,4\}(5,6)]} \\ 13 & \texttt{[\{1,4\}\{3,2\}(5,6)]} \end{array} \tag{8.23}$$

At tree-level, in pure QCD, the last two basis vectors vanish. To create a basis which is only valid up to order `n_loop` in pure QCD, we may use the `create_basis(n_quark,n_gluon, n_loop)` member function. See table 13 for member functions.

8.5.4 Tree_level_gluon_basis

In the case of gluon-only color structures, charge conjugation implies that each trace must appear with its conjugate, in linear combinations of the form $\mathrm{tr}[t^{g1}t^{g2}...t^{gn}] + (-1)^{N_g}\mathrm{tr}[t^{gn}...t^{g2}t^{g1}]$. In the trace type basis class `Tree_level_gluon_basis` this is used to reduce the number of basis vectors and speed up calculations. More information can be found in table 14.

8.5.5 Orthogonal_basis

`ColorFull` can not – in its present form – automatically create multiplet bases. However, orthogonal bases may be read in using the (`Col_basis`) function `read_in_Col_basis(std::string)` member function.

For dealing with orthogonal bases, `ColorFull` offers special functions for calculating the matrix of scalar products and decomposing vectors. The syntax for basis reading is the same as for basis writing. Bases thus appear much as as in eq. (8.23), see table 11 and table 15.

9 Validation

For a code with order 10 000 lines, validation is essential. For this reason `ColorFull` is continuously validated using a test suite, which aims at testing all the various components. The applied tests starts with checking basic functions for reading in and writing out files, and dealing with polynomials. After this, the creation of bases is tested, and scalar products are tested by changing the order of index contraction, and by switching on and off memoization. The scalar product matrices have further been tested against `ColorMath` [4] and, for processes occurring in the context of [25], also against another private Mathematica code. The functions describing gluon emission and gluon exchange are cross-checked against each other.

10 Conclusion and outlook

`ColorFull`, a C++ stand-alone QCD color algebra package, designed for interfacing with event generators, has been presented.

`ColorFull` is based on trace bases, which can automatically be created, and color contraction is performed by repeated usage of the Fierz identity. Employing these bases, one can in principle describe any QCD process. In reality, the scalar product matrices, which may be calculated once and for all, become hard to manage for more than approximately 8 gluons plus $q\overline{q}$-pairs. This limitation is inherent for the trace bases, since they are non-orthogonal, and for this reason `ColorFull` is written to be able to load and use orthogonal (multiplet) bases.

`ColorFull` does, however, not – in its present form – perform index contraction in terms of group invariants as described in for example [18, 26]. Extending `ColorFull` to inherently construct orthogonal multiplet bases and efficiently perform index contraction using $3j$ and $6j$ coefficient may speed up the treatment of QCD color space very significantly.

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A Class member functions

Table 3. Public data members and functions of the class `Monomial`.

Data members	**Content**
`cnum cnum_part`	Complex number multiplying the monomial.
`int int_part`	Integer multiplying the monomial, can be 0.
`int pow_CF`	Power of `CF=TR*(Nc^2-1)/Nc`.
`int pow_Nc`	Power of the number of colors.
`int pow_TR`	Power of `TR` in the `Monomial`.
Constructors	**Effect**
`Monomial()`	Default constructor that sets `int_part=cnum_part=1`, and `pow_Nc=pow_TR=pow_CF=0`.
`Monomial(double dnum)`	Constructor using a `double`. The `cnum_part` member is set to contain the value.
`Monomial(int num)`	Constructor using an `int`. The `int_part` member is set to contain the value.
`Monomial(std::string str)`	Constructor taking a string as argument. The argument should be of the form in for example "`-(20*TR^5)/Nc`" or "`-20 TR^(5)/Nc`" or "`20 / TR^(-5)Nc^(1) CF^(3)`". *Note*: All spaces and all `*` are ignored, except in `*(-1)` and `*-1`, which are understood as `*(-1)`. *Everything* standing after `/` is divided out, whereas everything standing before `/` is multiplied with. Parentheses are ignored unless they appear in powers, directly after `^`. No spaces are allowed inside the powers. If the string contains no information or is empty, the `Monomial` is set to 1, i.e., `pow_TR = pow_Nc = pow_CF = 0`, `int_part = 1`, `cnum_part = 1.0`. (Expanded Mathematica expressions are of this form.)
Member functions	**Effect**
`void conjugate()`	Take the complex conjugate. Note that this changes the `Monomial` itself.
`void read_in_Monomial(std::string filename)`	Function for reading in the `Monomial` from the file `filename`. The syntax is as for `Monomial(std::string)`, and comments starting with `#` are allowed at the top of the file.
`void write_out_Monomial(std::string filename) const`	Function for writing out the `Monomial` to a file with name `filename`.

Table 4. Public data members and functions of the class Polynomial.

Data member	**Content**
polynomial poly where polynomial is std::vector<Monomial>	Contains the polynomial, a sum of Monomials. An empty polynomial is defined as 1, to get 0, multiply with 0.
Constructors	**Effect**
Polynomial()	Default constructor, leaves polynomial empty (=1).
Polynomial(const std::string str)	Constructor allowing setting the Polynomial by using a string, should be used as for example: ”Polynomial Poly("(-20*TR^(5))/Nc + 28*Nc*TR^(5) - 10*Nc^3*TR^(5)")”. The Monomials should be separated by + or -, see also the Monomial string constructor.
Polynomial(double dnum)	Constructor allowing setting the Polynomial using a double. The Polynomial gets one Monomial where the real part of cnum_part equals dnum.
Polynomial(int num)	Constructor allowing setting the Polynomial using an int. The Polynomial gets one Monomial where int_part has the value num.
Member functions	**Effect**
void append(const Monomial Mon)	Adding a Monomial term.
const Monomial& at(int i) const	Returns the Monomial at place i.
Monomial& at(int i)	Returns the Monomial at place i.
void clear()	Erases the information in polynomial.
void conjugate()	Takes the complex conjugate of the polynomial. Note that this changes the Polynomial itself.
bool empty()	Is the polynomial empty?
void erase(int i)	Erase the Monomial at place i.
void normal_order()	Orders terms in Polynomial in a unique form, first according to pow_Nc+pow_CF, then according to pow_Nc (for same pow_Nc+pow_CF) then according to int_part*cnum_part, then according to int_part, and finally according to pow_TR.
void read_in_Polynomial(std::string filename)	Function for reading in the Polynomial from the file filename. Comments starting with # are allowed at the top of the file.
void remove_CF()	Replaces CF with TR*Nc-TR/Nc.
void simplify()	Collects terms with the same power of TR, Nc and CF.
int size()	Returns the number of terms in the Polynomial.
void write_out_Polynomial(std::string filename) const	Function for writing out the Polynomial to a file with name filename.

Table 5. Public members and functions in the class Poly_vec.

Data member	Content
poly_vec pv where poly_vec is: typedef std::vector<Polynomial>	To actually contain the vector of Polynomials.
Constructors	**Effect**
Poly_vec()	Default constructor, leaves pv empty.
Poly_vec(poly_vec poly_v)	Makes a Poly_vec of a poly_vec.
Member functions	**Effect**
void append(Polynomial Poly)	Appends a Polynomial to data member pv.
const Polynomial& at(int i) const	Returns the Polynomial at place i.
Polynomial& at(int i)	Returns the Polynomial at place i.
void clear()	Erases the information in the vector.
void conjugate()	Conjugates the Poly_vec.
bool empty() const	Is the vector empty?
void normal_order()	Normal orders all Polynomials in the poly_vec member pv (uses the Polynomial::normal_order function).
void read_in_Poly_vec(std::string filename)	Reads in a Polynomial vector of the form {Poly1, Poly2,...} to the member pv from the file filename. Comments starting with # are allowed at the top of the file.
void remove_CF()	Remove CF in the poly_vec member pv, i.e., replace CF by TR*Nc-TR/Nc.
void simplify()	Simplifies all polynomials in the poly_vec member pv (uses the simplify member function in Polynomial).
uint size() const	Returns the number of Polynomials in the vector, i.e., the size of the member pv.
void write_out_Poly_vec(std::string filename) const	Writes out the vector to the file filename.

Table 6. Public members and functions of the class `Poly_matr`.

Data member	**Content**
`poly_matr pm` where `poly_matr` is: typedef std::vector<Poly_vec>	To actually contain the matrix of `Polynomials`.
Constructor	**Effect**
`Poly_matr()`	Default constructor, leaves `pm` empty.
Member functions	**Effect**
`Poly_vec& at(int i)`	Returns the `Poly_vec` at place `i`.
`const Poly_vec& at(int i)`	Returns the `Poly_vec` at place `i`.
`Polynomial& at(int i, int j)`	Returns the matrix element at `i, j`.
`const Polynomial& at(int i, int j)`	Returns the matrix element at `i, j`.
`void append(Poly_vec Pv)`	Appends a `Poly_vec` to the data member `pm`.
`void clear()`	Erases the matrix information.
`void conjugate()`	Conjugates the matrix.
`bool empty()`	Is the matrix, stored in `pm`, empty?
`void normal_order()`	Normal orders all `quark_lines` in the `poly_matr` member `pm` (uses `Polynomial::normal_order`.)

`void read_in_Poly_matr(std::string filename)`	Reads in the matrix from the file filename.
	The file should be of the format
	`{{Poly11,...,Poly1n},`
	`...,`
	`{Polyn1,...,Polynn}},`
	and may contain comment lines starting with `#` at the top.

`void remove_CF()`	Removes `CF` in the `poly_matr` member `pm`, i.e., replaces `CF` by `TR*Nc-TR/Nc`.
`void simplify()`	Simplifies all `quark_lines` in the `poly_matr` member `pm`, (uses `Polynomial.simplify`).
`uint size()`	Returns the size of the matrix, the number of `Poly_vecs` in the member `pm`.
`void write_out_Poly_matr( std::string filename) const`	Writes out the matrix to the file `filename`.

Table 7. Public members and functions of the class `Quark_line`.

Data members	Content

`quark_line ql`
where `quark_line` is:
`typedef std::vector<int>`
	To actually contain the color information, in order $\{q, g_1, g_2, ...g_n, \overline{q}\}$ or $(g_1, g_2, ...g_n)$.

`bool open`	Is the string open, with a quark in the beginning and an antiquark in the end, or not?
`Polynomial Poly`	`Polynomial` factor, multiplying the `quark_line`.
Constructors	**Effect**
`Quark_line( )`	Default constructor, leaves `ql` empty.
`Quark_line(const std::string)`	Constructor used to set the color structure using a string. The string should be of form `Polynomial*quark_line`, used as `Quark_line Ql("5*TR*Nc^2 {1,6,7,2}");`, for an open `Quark_line` with a quark with number `1`, two gluons with number `6` and `7`, and a $\overline{q}$ with number `2`. For a closed `Quark_line` the syntax is `Quark_line Ql("(1,2,3)");` The integers should be positive. The `Polynomial` should be in a format which it is readable by the `Polynomial(std::string)` constructor.
Member functions	**Effect**
`Quark_line after(int j) const`	Returns a `Quark_line` where the `ql` member is changed to contain only partons after place `j`.
`void append(int p)`	Appends parton `p` to the `Quark_line`.
`void append(std::vector<int> in_ql)`	Appends a whole `quark_line` to the `Quark_line`.
`int at(int j) const`	Returns the parton at place `j`. For closed `quark_lines` `j` may be between -size and 2*size.
`Quark_line before(int j) const`	Returns a `Quark_line` where the `ql` member is changed to contain only partons before place `j`.
`void clear()`	Erase the information in `quark_line ql`.
`void conjugate()`	Conjugates the `Quark_line` by reversing the `quark_line ql` and conjugating the `Polynomial Poly`.
`void contract_neighboring_gluons(int j)`	Contracts neighboring gluons in the `Quark_line` starting at `j`, only intended for closed `Quark_lines`.
`void contract_neighboring_gluons()`	Function for contracting neighboring gluons in a `Quark_line` starting at place 0, and looking everywhere, only intended for closed `Quark_lines`.

void contract_next_neighboring_gluons(int j)	Contracts next to neighboring gluons in the Quark_line, starting at place j (i.e. checking first gluon j and j+2). Also looks for new neighbors, only intended for closed Quark_lines.
void contract_next_neighboring_gluons()	Contracts neighboring and next to neighboring gluons in the Quark_line, starting with contracting neighbors, only intended for closed Quark_lines.
bool empty() const	Is the quark_line empty?
void erase(int i)	To erase the parton at place i.
void insert(int j, int p)	Inserting parton p at place j.
void normal_order()	Orders a closed quark_line, such that the smallest gluon index stands first (i.e., use that the trace is cyclic).
void prepend(int p)	Prepends parton p to the Quark_line.
void prepend(std::vector<int> in_ql)	Prepends a whole quark_line to the Quark_line.
void read_in_Quark_line(std::string filename)	Function for reading in the Quark_line from the file filename. The syntax for the Quark_line is the same as for the Quark_line string constructor.
uint size() const	The size of the quark_line.
int smallest(const Quark_line& Ql1, const Quark_line& Ql2) const	Function for finding the "smallest" Quark_line of Ql1 and Ql2, used for deciding which Quark_line should stand first when normal ordering Col_strs. This function does *not* first normal order the Quark_lines. If only one is open, that Quark_line should stand first. If both are open or both are closed, the longest Quark_line should stand first. If the size is the same, the Quark_line with smallest starting number should stand first. If the first number is the same, check the second number, then the third etc. 1 is returned if Ql1 should stand first, and 2 if Ql2 should stand first. If Ql1==Ql2, 0 is returned.
std::pair<Quark_line, Quark_line> split_Quark_line(int j1, int j2) const	Function for splitting a closed Quark_line into two Quark_lines. The gluons at j1 and j2 are removed in the split. May create 1-rings and 0-rings.
void write_out_Quark_line(std::string filename) const	Function for writing out the Quark_line to a file with name filename.

Table 8. Public data members an functions of the class `Col_str`.

Data members	Content

`col_str cs`
where `col_str` is:
`typedef std::vector<Quark_line>`
	For containing the information about the color structure, a product of `Quark_lines`, contained in a vector of `quark_lines`.

`Polynomial Poly`	`Polynomial` factor multiplying the whole product of `Quark_lines`.
Constructors	**Effect**
`Col_str()`	Default constructor, leaves `cs` empty.
`Col_str(Quark_line Ql)`	Make a `Col_str` of a `Quark_line`.
`Col_str(const std::string str)`	Constructor for setting the color structure using a string. Should be used as: `Cs("Nc*TR^(3) [{1,2,3,4}(5,6)(7,8)]")`, i.e., the argument should be `Polynomial * col_str`. (The `Polynomial` should multiply the whole `col_str`, i.e., stand outside the `[]`-brackets.
Member functions	**Effect**
`const Quark_line& at(int i)`	Returns the `Quark_line` at place `i`.
`Quark_line& at(int i)`	Returns the `Quark_line` at place `i`.
`int at(int i, int j) const`	Returns the parton at place `j` in in `Quark_line i`.
`void append(Quark_line Ql)`	Appends a `Quark_line` to the data member `cs`.
`void append(col_str cs_in)`	Append the content of a `col_str` to the `cs` of the `Col_str`.
`void clear()`	Erase information in `col_str`.
`void conjugate()`	Function for conjugating the `Col_str` by conjugating each `Quark_line` in `cs`, as well as the `Polynomial Poly`.
`void contract_2_rings()`	Function for contracting gluon indices in closed `Quark_lines` with only 2 gluons. This removes the 2-ring, replaces one of the gluon indices, and multiplies with a factor $\mathrm{tr}[t^a t^a]$ = `TR` (no sum), only intended for fully contractable `Col_strs`.
`void contract_next_neighboring _gluons( )`	Contracts neighboring and next to neighboring gluons in each `Quark_line` in the `Col_str`, starting with contracting neighbors. This function should only be used on `Col_strs` with only closed `Quark_lines`.
`void contract_quarks( const Col_str Cs1, const Col_str Cs2)`	Function for contracting quarks between two color structures `Cs1` and `Cs2`. The result is stored in the `Col_str` itself.
`bool empty() const`	Is the `col_str` empty?

`void erase(int i)`	Erases the `Quark_line` at place `i`.
`void erase(int i, int j)`	Erases the parton at position `j` in `Quark_line i`.
`void erase(std::pair<int, int> place)`	Erases the parton located at `place`.
`std::string find_kind(int p) const`	Finds out if a parton is a quark, anti-quark or gluon, returns "q", "qbar" or "g" respectively. This function does *not* loop over all partons, but assumes that the parton is a gluon if it is in a closed `Quark_line`, or if the `Quark_line` is open but the parton cannot be found in the ends.
`std::pair<int, int> find_parton(int part_num) const`	Locates the parton with number `part_num` in a `col_str`.
`bool gluons_only() const`	Checks if the amplitude only has gluons, i.e., if all `Quark_lines` are closed.
`void insert(int i, int j, int part_num)`	To insert the parton `part_num` in `quark_line i` at place `j`.
`bool left_neighbor(int p1, int p2) const`	Function for telling if parton `p2` stands to the left of parton `p1`.
`int longest_quark_line() const`	Returns the length of the longest `Quark_line` in the `col_str`.
`int n_gluon() const`	Counts the number of gluons in a `Col_str`. Counts all gluon indices, both free and contractable.
`int n_quark() const`	Counts the number of quarks (=number of anti-quarks) in a `Col_str`. Counts all quark indices, both free and contracted.
`bool neighbor(int p1, int p2) const`	Function for telling if the partons `p1` and `p2` are neighbors.
`void normal_order()`	Normal orders the `Col_str` by first normal ordering individual `Quark_lines` and then normal ordering different `Quark_lines` in `cs`. For the ordering, see the member function `smallest` in this class and in the `Quark_line` class.
`void read_in_Col_str(std::string filename)`	Function for reading in the `Col_str` from the file `filename`.
`void remove_0_rings()`	Removes `Quark_lines` without partons, equal to `Nc` (closed) or 1 (open).
`void remove_1_rings()`	Removes `Quark_lines` with only one gluon as $\mathrm{tr}(t^a) = 0$.
`void replace(int old_ind, int new_ind)`	Replaces the parton index `old_ind` with `new_ind`.
`bool right_neighbor(int p1, int p2) const`	Function for telling if parton `p2` stands to the right of parton `p1`.

`void simplify()`	Removes 0- and 1-rings, moves factors multiplying the individual `Quark_lines` to multiply the `col_str` instead (i.e., being stored in `Poly`), simplifies the `Polynomial` and normal orders the `quark_lines`.
`uint size()`	The size of the `col_str`, number of `quark_lines`.
`int smallest(const Col_str& Cs1, const Col_str& Cs2) const`	Finds out the "smallest" `Col_str` of two `Col_strs`, i.e., which `Col_str` should stand first in a normal ordered `Col_amp` or basis. Returns `1`, if `Cs1` should stand before `Cs2` and `2` if `Cs2` should stand before `Cs1`. Both `Col_strs` have to be normal ordered for the result to be unique. The `Col_strs` are ordered by (1) number of `Quark_lines` (2) if the `Quark_line` at place 0,1,2... is open or not (3) the size of the `Quark_line` at place 1,2,3... (4) the parton numbers in the `Quark_lines` at place 1,2,3..., i.e., first the first parton in the first `Quark_line` is checked and last the last parton in the last `Quark_line`. The function returns `0` if `Cs1==Cs2`.
`void write_out_Col_str( std::string filename) const`	Function for writing out the `Col_str` to a file with name `filename`.

Table 9. Public members and functions of the class `Col_amp`.

Data members	**Content**

`col_amp ca`
where `col_amp` is:
`typedef std::vector<Col_str>`
	To actually contain the information about the `Col_strs`, `ca=Cs1+Cs2+Cs3+...` .

`Polynomial Scalar`	`Scalar` is a `Polynomial` for collecting color factors appearing when the color structure has been fully contracted. The full color amplitude is `Scalar+Cs1+Cs2+Cs3....` `Scalar` should thus be non-zero only if all indices can be contracted.
Constructors	**Effect**
`Col_amp()`	Default constructor, sets `Scalar = 0`, and leaves `ca` empty.
`Col_amp(Col_str Cs)`	Constructor converting a `Col_str` to a `Col_amp`.
`Col_amp(const std::string str)`	Constructor taking a string as argument. The string should be of the form `Polynomial1*col_str1+Polynomial2*col_str2`, for example: `Ca("Nc*[(1,3,4,2)]+1/Nc [(1,4)(3,2)]")`. (The `Polynomials` should multiply the whole `col_strs`, and thus stand outside the `[]`-brackets.)
Member functions	**Effect**
`void append(col_amp ca_in)`	Appends the `Col_strs` in `ca_in` to the `col_amp` member `ca`.
`const Col_str & at(int i)`	Returns the `Col_str` at place `i`.
`Col_str & at(int i)`	Returns the `Col_str` at place `i`.
`void clear()`	Erases the information about the color amplitude, stored in `ca`.
`void collect_col_strs()`	Compares `col_strs` in the `Col_amp` to collect similar `col_strs` and only store once in `ca`.
`void conjugate()`	Function for taking the conjugate of the `Col_amp` by conjugating each `Col_str` in `ca` and the `Polynomial` member `Scalar`.
`void contract_2_rings()`	Contract closed `Quark_lines` with only 2 gluons in each `Quark_line` in each `Col_str` in the `Col_amp`. This removes the 2-ring, replaces one of the gluon indices and multiplies with a factor $\mathrm{tr}[t^a t^a]$ = `TR` (no sum), only intended for fully contractable `Col_amps`.
`void contract_a_gluon()`	Contracts one gluon, the first gluon in the first `Quark_line` (in each `Col_str`), only intended for closed `Quark_lines`.

`void contract_all_gluons()`	Function for contracting all gluon indices in a `Col_amp`, only intended for closed `Quark_lines`.
`void contract_next_neighboring _gluons()`	Contracts up to next to neighboring gluons in each `Quark_line` in each `Col_str` in each `Col_amp`, only intended for closed `Quark_lines`.
`void contract_quarks(const Col_amp Ca1, const Col_amp Ca2)`	Function for contracting the (anti-)quarks in `Ca1` with those in `Ca2`. The results is saved in the `Col_amp` itself.
`void contract_Quark_line_gluons()`	Function for contracting gluon indices within the `Quark_lines`. Checks only for *one* pair in each `Quark_line`.
`bool empty() const`	Is the `col_amp ca` empty?
`void erase(int i)`	Erases the `Col_str` at place `i`.
`bool gluons_only() const`	Checks if the `Col_amp` only contains gluons, i.e., if all `Quark_lines` are closed.
`int longest_quark_line() const`	Returns the length of the longest `Quark_line` in any `Col_str`.
`int n_gluon() const`	Returns the number of gluons in the `Col_amp` as the number of gluons in the first `Col_str`. Note that the other `Col_strs` could have a different number of (contracted) gluons. (Intended for tree-level `Col_amps` with only one `Col_str`.)
`int n_gluon_check() const`	Returns the number of gluons in the `Col_amp` after checking that each `Col_str` has the same number of gluons.
`int n_quark() const`	Returns the number of quarks in the `Col_amp` as the number of quarks in the first `Col_str`. Note that the other `Col_strs` could have a different number of (contracted) quarks. (Intended for tree-level `Col_amps` with only one `Col_str`.)
`int n_quark_check() const`	Returns the number of quarks in the `Col_amp` after checking that each `Col_str` has the same number of quarks.
`void normal_order()`	Normal orders the individual `col_strs` and then orders the `Col_strs` using the order defined in the `Col_str` member function `smallest`.
`void normal_order_col_strs()`	Normal orders all `col_strs` in `ca`.
`void read_in_Col_amp( std::string filename)`	Reads in the `Col_amp` to the member `ca` from the file `filename`. (This is intended for reading in an actual color amplitude, nothing is read in to the `Polynomial` member `Scalar`.) Comments starting with `#` are allowed at the top of the file.
`void remove_0_rings()`	Removes `quark_lines` with no gluons, these are `Nc` if closed, and defined to be 1 if open.

`void remove_1_rings()`	Removes `Col_strs` with `quark_lines` with only 1 gluon, these are 0 as $\mathrm{tr}[t^a] = 0$.
`void remove_empty_Col_strs()`	Removes empty `Col_strs`. An empty `Col_str` means that all indices have been contracted, so the `Col_str` is equal to its `Polynomial`, which is moved to the `Scalar` part of the `Col_amp`.
`void simplify()`	Function for simplifying an amplitude. Removes 0 and 1-rings, compares `col_strs`, removes `Col_strs` multiplying 0, and simplifies `Polynomials` of the individual `Col_strs`.
`uint size() const`	The size of the `col_amp ca`.
`void write_out_Col_amp( std::string filename) const`	Function for writing out the `Col_amp` to a file with name `filename`.

Table 10. Some (private) data members and the public functions of the library class Col_functions.

Data members (private)	Content
double CF	The value of $C_F = T_R(N_c^2 - 1)/N_c$, changed by the set_CF function. Note that CF can be changed independently of Nc.
bool full_CF	While finding the leading terms in a Polynomial one may want to keep the full value of CF, TR*Nc-TR/Nc, or only keep the leading Nc term TR*Nc (default). The switch full_CF is used by the Polynomial version of leading (and hence also by the Poly_vec and Poly_matr versions etc.). The leading functions replaces CF by TR*Nc if full_CF is false (default) while evaluating the leading terms. If full_CF is true, CF is replaced by TR*Nc-TR/Nc. Clearly this affects the result of subsequent numerical evaluation. In the Col_basis class (and derived) the matrix version of leading is used to evaluate scalar product matrices.
double Nc	The number of colors, used in numerical results, changed by the set_Nc function.
double TR	The trace convention $\mathrm{tr}(t^a t^a)$=TR (no sum), the normalization of the SU(N_c) generators, to be used in numerical evaluation. This value can be changed by the set_TR function.
Constructor	**Effect**
Col_functions()	Default constructor, sets Nc=3, TR=0.5, CF=4.0/3.0 and full_CF=false.
Member functions	**Effect**
cnum cnum_num(const Monomial& Mon) const	Numerically evaluates a Monomial using the Nc, TR and CF variables.
cnum cnum_num(const Polynomial& Poly) const	Numerically evaluates a Polynomial, using the Nc TR and CF variables.
cvec cnum_num(const Poly_vec& Pv) const	Numerically evaluates a Poly_vec (vector of Polynomials), using cnum_num(Polynomial).
cmatr cnum_num(const Poly_matr& Pm) const	Numerically evaluates a Poly_matr (vector of Poly_vec), using cnum_num(Poly_vec) for each Poly_vec.

`Polynomial color_correlator(const Col_amp Ca, int i, int j) const`	Calculates $\langle \mathbf{c}\vert\mathbf{T}_i\mathbf{T}_j\vert\mathbf{c}\rangle$, the "color correlator" relevant for coherent gluon emission from parton `i` and parton `j`, or gluon exchange between `i` and `j`. The `Ca` thus corresponds to $\vert\mathbf{c}\rangle$, and `i` and `j` are the partons involved in the emission (exchange).
`double double_num(const Monomial & Mon) const`	Numerically evaluates a `Monomial` to a `double`, using the data members `Nc`, `CF` and `TR`.
`double double_num(const Polynomial & Poly) const`	Numerically evaluates a `Polynomial` to a `double`, using the data members `Nc`, `CF` and `TR`.
`dvec double_num(const Poly_vec & Pv) const`	Numerically evaluates a `Poly_vec`, vector of `Polynomial`.
`dmatr double_num(const Poly_matr & Pm) const`	Numerically evaluates a `Poly_matr` (vector of `Poly_vec`).
`Col_amp emit_gluon(const Col_str & in_Col_str, int emitter, int g_new) const`	Function for emitting a gluon from a `Col_str`. When the gluon is inserted before the emitter in a `Quark_line`, the amplitude comes with a minus sign.
`Col_amp emit_gluon(const Col_amp & Ca_in, int emitter, int g_new) const`	Function for emitting a gluon from a `Col_amp`. When the gluon is inserted before the emitter in a `Quark_line`, the amplitude comes with a minus sign.
`Col_amp exchange_gluon(const Col_str& Cs, int p1, int p2) const`	Function for exchanging a gluon between the partons `p1` and `p2` in the `Col_str Cs`. When the gluon is inserted before the emitter in a `Quark_line`, the amplitude comes with a minus sign.
`Col_amp exchange_gluon(const Col_amp & Ca, int p1, int p2) const`	Function for exchanging a gluon between two partons `p1` and `p2` in the `Col_amp Ca`. When the gluon is inserted before the emitter in a `Quark_line`, the amplitude comes with a minus sign.
`double get_CF() const`	Returns the value of `CF`.
`double get_Nc() const`	Returns the number of colors.
`double get_TR() const`	Returns the normalization of the generators, $\mathrm{tr}(t^a t^b) = \mathtt{TR}\delta^{ab}$.
`bool get_full_CF() const`	Returns `true` if `full_CF` is true and `false` otherwise.
`int factorial(int i) const`	The factorial of an `int`, 0! is defined as 1.

Polynomial leading(const Polynomial & Poly) const	Takes the leading Nc terms of a Polynomial, i.e., keeps the Monomials with the highest power of Nc plus CF. If full_CF is false (default), CF is replaced by TR*Nc. If full_CF is true CF is replaced by TR*Nc-TR/Nc.
Poly_vec leading(const Poly_vec& Pv) const	Takes the leading part of a Poly_vec. Keeps only Monomials with maximal power of CF plus Nc, uses leading(const Polynomial & Poly). If full_CF is false (default), CF is replaced by TR*Nc. If full_CF is true, CF is replaced by TR*Nc-TR/Nc. Note that taking the leading terms of a Poly_vec is not necessarily the same as taking the leading terms of each Polynomial.
Poly_matr leading(const Poly_matr& Pm) const	Takes the leading part of a matrix of Polynomials, keeping only those with maximal power of CF plus Nc. If full_CF is false (default), CF is replaced by $T_R N_c$. If full_CF is true CF is replaced by TR*Nc-TR/Nc. Note that taking the leading terms of a Poly_matr is not necessarily the same as taking the leading terms in each Poly_vec.
int leading_Nc_pow(const Polynomial & Poly) const	Function for finding the leading power of Nc in a Poly_vec, i.e., the power of Nc plus the power of CF.
int leading_Nc_pow(const Poly_vec & Pv) const	Function for finding the leading power of Nc in a Poly_vec.
Polynomial Polynomial_cnum_num(const Polynomial & Poly) const	Numerically evaluates a Polynomial using the value of the data member Nc, and stores in the format of a Polynomial with only one term with only a numerical part.
Poly_vec Poly_vec_cnum_num(const Poly_vec & Pv) const	Numerically evaluates a Poly_vec (vector of Polynomial) and stores in the form of a Poly_vec, uses polynomial_cnum_num(Pv.at(p)) for each Polynomial.
Poly_matr Poly_matr_cnum_num(const Poly_matr & Pm) const	Numerically evaluates a Poly_matr (vector of Poly_vec) and stores in the form of a Poly_matr.

`dvec read_in_dvec(std::string filename) const`	Reads in a numerical vector and saves it as a `double` matrix, `dmatr`. The file should be of the format `{d1,...,dn}`, and may contain comment lines starting with `#` at the top.

`dmatr read_in_dmatr(std::string filename) const`	Reads in a numerical matrix and save it as a `double` matrix, `dmatr`. The file should be of the format
	`{{d11,...,d1n},`
	`...,`
	`{dn1,...,dnn}}`
	and may contain comment lines starting with `#` at the top.

`Polynomial scalar_product(const Col_amp & Ca1, const Col_amp & Ca2) const`	Function for calculating the scalar product between two `Col_amps`.
`Polynomial scalar_product(const Col_str & Cs1, const Col_str & Cs2) const`	Function for calculating the scalar product between two `Col_strs`.
`void set_Nc(double n)`	Set the number of colors. The value of `CF` is adjusted accordingly.
`void set_TR(double tr)`	Sets the normalization of the generators. The value of `CF` is adjusted accordingly.
`void set_CF(double cf)`	Sets the value of `CF`. The value of `Nc` is *not* adjusted accordingly.
`void set_full_CF(bool is_full)`	Switch on/off `full_CF`.
`void write_out_dmatr(const dmatr & matr, std::string filename) const`	Writes out the double version of a (scalar product) matrix to the file `filename`.
`void write_out_dvec(const dvec & dv, std::string filename) const`	Function for writing out a numerical vector, to the file `filename`.

Table 11. Public members and functions of the class Col_basis.

Data members	**Content**
col_basis cb where col_basis is: typedef std::vector<Col_amp>	Contains the information about the basis vectors cb= vector1, vector2....
Col_functions Col_fun	Contains the set of Col_functions used.
dmatr d_spm	To contain the double version of the scalar product matrix.
dmatr leading_d_spm	To contain the double version of the leading part of the scalar product matrix.
Poly_matr leading_P_spm	To contain the Polynomial version of the leading part of the scalar product matrix.
int ng	The number of gluons, initially set to 0 and (possibly) changed with create_basis, or while reading in the basis.
int nq	The number of $q\bar{q}$-pairs, initially set to 0 and (possibly) changed with create_basis, or while reading in the basis.
Poly_matr P_spm	To contain the Polynomial version of the scalar product matrix.
Constructor and destructor	**Effect**
Col_basis()	Default constructor, sets nq=ng=0, and the private data members trace_basis = tree_level_gluon_basis = orthogonal_basis = false.
virtual ~Col_basis()	Destructor.
Member functions	**Effect**
Col_amp& at(const int i)	Returns the Col_amp (i.e. basis vector) at place i.
const Col_amp& at(const int i) const	Returns the Col_amp (i.e. basis vector) at place i.
void append(Col_amp Ca)	Appends a Col_amp to the basis stored in cb.
std::string basis_file_name() const	Returns a standard filename, used for writing out the basis to a file.
void clear()	Erase the basis, stored in cb.

`Poly_matr color_gamma(int p1, int p2)`	Function for calculating the color structure part of the soft anomalous dimension matrix. First calculates the effect of gluon exchange on a basis vector and then decomposes the result into the basis. For this to be possible, the basis must clearly contain all resulting basis vectors, meaning for example that it can not be used for `Tree_level_gluon_basis`. The function is only available for the `Trace_basis` and the `Orthogonal_basis` classes.
`virtual Poly_vec decompose( const Col_amp & Ca)`	Each type of color basis has to implement a function for decomposing an amplitude in the color basis.
`bool empty() const`	Is the `col_basis` empty?
`Col_amp exchange_gluon( uint vec, int p1, int p2)`	Function for finding the resulting `Col_amp` after exchanging a gluon between parton `p1` and parton `p2` in the basis vector vec.
`bool is_Orthogonal_basis() const`	Is it an `Orthogonal_basis`?
`bool is_Trace_basis()`	Is it a `Trace_basis`?
`bool is_Tree_level_gluon_basis()`	Is it a `Tree_level_gluon_basis`?
`void leading_scalar_product_matrix()`	Finds the leading `Nc` scalar product matrices, `leading_P_spm` and `leading_d_spm`. If the polynomial scalar product matrix, `P_spm` has been calculated, `P_spm` is used, otherwise `P_spm` is first calculated and the leading `Nc` limit is then taken of `P_spm`.
`int n_gluon_check() const`	Returns the number of gluons in the `Col_basis` after checking that each Col_str in each Col_amp has the same number of gluons.
`int n_quark_check() const`	Returns the number of quarks in the `Col_basis` after checking that each `Col_str` in each `Col_amp` has the same number of quarks.
`virtual void read_in_Col_basis()`	Function for reading in the basis from the default filename (see `basis_file_name`).

virtual void read_in_Col_basis(std::string filename)	Function for reading in the basis from the file filename. The basis should be in human readable format, of form: 0 [{1,3,4,2}] 1 [{1,4,3,2}] 2 [{1,2}(3,4)] i.e. first the basis vector number 0,1,2..., etc, then the Col_amp corresponding to the basis vector in question. The Col_amps may consist of several Col_strs, for example 0 [(1,2,3,4)]+[(1,4,3,2)] 1 [(1,2,4,3)]+[(1,3,4,2)] 2 [(1,3,4,2)]+[(1,2,4,3)] and each Col_str may also contain a Polynomial. (The Polynomial should multiply the whole col_str, rather than a quark_line, i.e., the Polynomial should stand outside the [] -brackets.)
void read_in_d_spm( )	Reads in a numerical matrix from a file with default filename (see spm_file_name) and saves it as a double matrix, dmatr, in the member variable d_spm. The file should be of the format {{d11,...,d1n}, ..., {dn1,....dnn}}, and may contain comment lines starting with # at the top.
void read_in_d_spm(std::string filename)	Reads in a numerical matrix from a file filename and saves it in the double matrix, dmatr, in the member variable d_spm. For format, see read_in_d_spm( ).
void read_in_leading_d_spm(std::string filename)	Reads in a numerical matrix from the file filename and saves it as a double matrix, dmatr, in the member variable leading_d_spm. The format should be as for read_in_d_spm( ), with 0 for non-diagonal elements.
void read_in_leading_d_spm( )	Reads in a numerical matrix from a file with default filename (see spm_file_name) and saves it as a double matrix, dmatr, in the member variable leading_d_spm. The format should be as for read_in_d_spm( ), with 0 for non-diagonal elements.

`void read_in_leading_P_spm( )`	Reads in a `Polynomial` matrix from default filename (see `spm_file_name`) and saves it as a `Poly_matr` in the member variable `leading_P_spm`. The format should be as for `void read_in_P_spm()`, with 0 for non-diagonal elements.
`void read_in_leading_P_spm( std::string filename)`	Reads in a `Polynomial` matrix from a file with default filename (see `spm_file_name`) and saves it as a `Poly_matr` in the member variable `leading_P_spm`. The format should be as for `void read_in_P_spm()`, with 0 for non-diagonal elements.

`void read_in_P_spm( )`	Reads in a `Polynomial` matrix from a file with default filename (see `spm_file_name`) and saves it as a `Poly_matr` in the member variable `P_spm`. The file should be in the format
	`{{Poly11,...,Poly1n},`
	`...,`
	`{Polyn1,...,Polynn}},`
	and may contain comment lines starting with `#` at the top.

`void read_in_P_spm(std::string filename)`	Reads in a `Polynomial` matrix from the file `filename` and saves it as a `Poly_matr` in the member variable `P_spm`. The format should be as for `void read_in_P_spm()`.
`void rename_indices( Col_str & Cs1, Col_str & Cs2) const`	A function to rename the indices in two `Col_strs`, such that in the first they are called 1,2,3..., and in the second the relative order is kept.
`virtual Polynomial scalar_product( const Col_amp & Ca1, const Col_amp & Ca2)`	Function for calculating scalar products algebraically using the basis and the scalar product matrix (`Poly_matr P_spm`) in the basis. (Does add implicit conjugated part for `Tree_level_gluon_basis`, as these terms are contained in the matrix of scalar products.)
`void scalar_product_matrix()`	Function for calculating the scalar products matrix. This function loops over all basis vectors and stores the value of the scalar product between basis vector `i` and basis vector `j` in the `i,j` -entry in `P_spm` and `d_spm`. The calculation is done using memoization. The symmetry of the scalar product matrix is not used for the calculation, instead it is checked that the resulting matrix is indeed symmetric.

`void scalar_product_matrix_num()`	As `scalar_product_matrix`, but does the calculation numerically. It hence only calculates `d_spm`.
`virtual void scalar_product_matrix_no_mem()`	As `scalar_product_matrix`, but without memoization.
`virtual cnum scalar_product_num( const Col_amp & Ca1, const Col_amp & Ca2)`	Function for calculating scalar products numerically, knowing the basis and the scalar product matrix in numerical form. (Does add implicit conjugated part for `Tree_level_gluon_basis`, as these terms are contained in the matrix of scalar products.)
`virtual cnum scalar_product_num( const cvec & v1, const cvec & v2 )`	Calculates the scalar product between numerical (complex) amplitudes `v1`, `v2` using the numerical scalar product matrix, `d_spm`. The vectors thus needs to be expressed in the basis contained in `cb`. (Does add implicit conjugated part for `Tree_level_gluon_basis`, as these terms are contained in the matrix of scalar products.)
`void scalar_product_matrix _num_no_mem()`	As `scalar_product_matrix_num`, but without memoization.
`cnum scalar_product_num_diagonal( const cvec & v1, const cvec & v2)`	Calculates the scalar product between numerical (complex) amplitudes `v1`, `v2` using the numerical scalar product matrix, `d_spm`. Assumes that there are only diagonal contributions. This is useful for calculations in leading `Nc` limit. (Does add implicit conjugated part for `Tree_level_gluon_basis`, as these terms are contained in the matrix of scalar products.)
`void simplify()`	Simplifies all the basis vectors by using `Col_amp.simplify()` on the individual `Col_amps` in the basis.
`const uint size() const`	Returns the number of basis vectors.
`std::string spm_file_name( const bool leading, const bool poly) const`	Returns a standard filename, used for writing out scalar product matrices. If `leading` is true, "`_l`" is appended to the filename. If "`poly`" is true "`P_`" is added to the filename, and if it is false "`d_`", as in `double`, is added to the filename.
`virtual void write_out_Col_basis() const`	Function for writing out the basis to a file with default name (see `basis_file_name`).
`virtual void write_out_Col_basis( std::string filename) const`	Function for writing out the basis to a file with name `filename`.

`void write_out_d_spm( std::string filename) const`	Writes out the `d_spm` to file `filename`.
`void write_out_d_spm( ) const`	Writes out `d_spm` to the standard filename, see `spm_file_name`.
`void write_out_leading_d_spm( std::string filename) const`	Writes out `leading_d_spm` to the file `filename`.
`void write_out_leading_d_spm( ) const`	Writes out `leading_d_spm` to the standard filename, see `spm_file_name`.
`void write_out_leading_P_spm( std::string filename) const`	Writes out `leading_P_spm` to the file `filename`.
`void write_out_leading_P_spm( ) const`	Writes out `leading_P_spm` to the standard filename, see `spm_file_name`.
`void write_out_P_spm( std::string filename) const`	Writes out `P_spm` to the file `filename`.
`void write_out_P_spm( ) const`	Writes out `P_spm` to the standard filename, see `spm_file_name`.

Table 12. Data member and public functions of the class `Trace_type_basis`. As `Trace_type_basis` inherits from `Col_basis` all the public members of `Col_basis` are also available, see table 11.

Data member (protected)	**Content**
`int max_ql`	The maximal number of quark-lines allowed in the basis. This is used for constructing bases that only are valid up to a certain order in QCD, such that unused information need not be carried around.
Member functions	**Effect**
`Trace_type_basis():Col_basis()`	Default constructor, uses the `Col_basis` constructor and sets the variable `max_ql` to `0`.
`Poly_vec decompose(const Col_amp& Ca)`	A function for decomposing the color amplitude `ca` in the basis, returning the result as a `Polynomial`.
`std::pair<int, int> new_vector_numbers(const Col_str & Cs, int emitter)`	Function for finding the new vector numbers in the new basis (this basis) after radiating a new gluon from the parton `emitter` in the old color structure is `Cs`. After emission a linear combination of new basis vectors is obtained. For emission from a quark or an antiquark there is only one resulting color structure, and -1 is returned in the place of the absent color structure. The second vector, where the new gluon is inserted before the `emitter`, comes with a minus sign in the new total amplitude.
`std::pair<int, int> new_vector_numbers(int old_num, int emitter, int n_loop) const`	This function is intended for tree-level processes with at most 2 $q\overline{q}$-pairs. It finds the new vector numbers in the basis for N_q+N_g+1 partons after radiating a new gluon from the parton `emitter`. This function does not actually use the `cb`, but only calculates the basis vector number, which makes it much quicker than the general version. The old vector has number `old_num`, and there were, before emission, N_q quarks (+ N_q antiquarks) and $N_g - 1$ gluons. For emission from a quark or an antiquark there is only one resulting color structure, and `-1` is returned in the place of the absent color structure. The second vector, where the new gluon is inserted before the emitter comes with a minus sign in the new total amplitude.

Table 13. Public member functions implemented in the class `Trace_basis`. As `Trace_basis` inherits from `Trace_type_basis` all the public members of `Trace_type_basis` and `Col_basis` are also available, see table 12 and table 11.

Constructors	**Effect**
`Trace_basis()`	Default constructor, calls the `Trace_type_basis` constructor and sets `nq` = `ng` = 0.
`Trace_basis(int n_quark, int n_gluon)`	Constructor for creating a trace basis for `n_quark` $q\bar{q}$-pairs and `n_gluon` gluons.
`Trace_basis(int n_quark, int n_gluon, int n_loop)`	Constructor for creating a trace basis for `n_quark` $q\bar{q}$-pairs and `n_gluon` gluons, keeping only those color structures that can appear to order `n_loop` in pure QCD. (Note: For electroweak interactions more color structures may be needed.)
Member functions	**Effect**
`void create_basis(int n_q, int n_g)`	Creates a trace basis with basis vectors saved in the `cb` member. Keeps all possible basis vectors, i.e., the basis is valid to any order in perturbation theory.
`void create_basis(int n_q, int n_g, int n_loop)`	Creates a trace basis with basis vectors saved in the cb member. Keeps only basis vectors consisting of at most `n_q` plus `n_loop` `quark_lines`.

Table 14. Public member functions of the class `Tree_level_gluon_basis`. As `Tree_level_gluon_basis` inherits from `Trace_type_basis` which inherits from `Col_basis`, all the public members of `Col_basis` and `Trace_type_basis` are also available, see table 11 and table 12.

Constructors	**Effect**
`Tree_level_gluon_basis()`	Default constructor.
`Tree_level_gluon_basis(int n_g)`	Constructor for creating a tree-level gluon basis with `ng` gluons.
Member functions	**Effect**
`void create_basis(int n_g)`	Creates a basis with basis vectors saved in the `cb` member. Each basis vector is a sum of two traces, of form $\mathrm{tr}[t^a t^b....t^z] + (-1)^{N_g}\mathrm{tr}[t^z....t^b t^a]$. The charge conjugated trace is implicit, and only one trace is actually carried around which speeds up calculations.
`void read_in_Col_basis( )`	Function reading in the basis from default name (see `basis_file_name` in the `Col_basis` class). The full basis, including the charge conjugated part should be contained in the file. (This is to simplify comparison with other programs, such as `ColorMath`.)
`void read_in_Col_basis(std::string filename)`	Function for reading in the basis from a file. The file should contain the whole basis, including the charge conjugated part.
`void write_out_Col_basis( ) const`	Function for writing out the basis to default name, (see `basis_file_name` in the `Col_basis` class). The full basis, including the charge conjugated part is written out. (This is to simplify comparison with other programs, such as `ColorMath`.)
`void write_out_Col_basis(std::string filename) const`	Function for writing out the basis to `filename`, (see `basis_file_name` in the `Col_basis` class). The full basis, including the charge conjugated part is written out. (This is to simplify comparison with other programs, such as `ColorMath`.)

Table 15. Public member functions of the class Orthogonal_basis. Orthogonal_basis inherits directly from Col_basis, meaning that public members of Col_basis are also available, see table 11.

Data members	**Content**
dvec diagonal_d_spm	To contain information about scalar products as a dvec, i.e., entry i is the square of vector i.
Poly_vec diagonal_P_spm	To contain information about scalar products as a Poly_vec, i.e., entry i is the square of vector i.
Constructor	**Effect**
Orthogonal_basis():Col_basis()	Default constructor, puts private variable orthogonal_basis =true and calls the constructor of Col_basis.
Member functions	**Effect**
Poly_vec decompose(const Col_amp & Ca)	The decomposition of a Col_amp in an orthogonal basis is done by calculating scalar products and dividing out the norm. The norm is evaluated numerically.
void diagonal_scalar_product_matrix(bool save_P_diagonal_spm, bool save_d_diagonal_spm, bool use_mem)	Calculates the diagonal entries in the scalar product matrix, and (depending on arguments), saves them to the member variables diagonal_P_spm and diagonal_d_spm. This function is used by the Orthogonal_basis version of scalar_product_matrix.
std::string diagonal_spm_file_name(const bool leading, const bool poly) const	Creates a default filename for writing out diagonal scalar products. The boolean variable leading should be true if the name is for a leading Nc variable. The filename is then modified accordingly.
Polynomial scalar_product(const Col_amp & Ca1, const Col_amp & Ca2)	Function for calculating scalar products given the information about the basis and the scalar product matrix in the basis. The Col_amps are first decomposed using decompose, and then squared using the scalar product matrix P_spm. An orthogonal scalar product matrix is assumed.
void scalar_product_matrix()	Calculates the scalar product matrix assuming the basis to be orthogonal. Calculates both the double (d_spm) and the Polynomial (P_spm) matrices and saves to default filenames.
cnum scalar_product_num(const Col_amp & Ca1, const Col_amp & Ca2)	Function for calculating scalar products given the information about the basis and the scalar product matrix in numerical form. The Col_amps are first decomposed using decompose, and then squared using diagonal_d_spm. For Orthogonal_basis an orthogonal scalar product matrix is assumed.

`cnum scalar_product_num(const cvec & v1, const cvec & v2)`	Calculates the scalar product between decomposed amplitudes `v1`, `v2` using the `diagonal_d_spm` diagonal numerical scalar product matrix. The vectors needs to be expressed in the basis contained in `cb`, i.e., the decomposition has to be known.
`void write_out_diagonal_d_spm( std::string filename) const`	Writes out `diagonal_d_spm` to the file `filename`.
`void write_out_diagonal_d_spm( ) const`	Writes out `diagonal_d_spm` to the standard filename, see `diagonal_spm_file_name`.
`void write_out_diagonal_P_spm( std::string filename) const`	Writes out `diagonal_P_spm` to the file `filename`.
`void write_out_diagonal_P_spm( ) const`	Writes out `diagonal_P_spm` to the standard filename, see `diagonal_spm_file_name`.
`void write_out_diagonal_spm(const dvec & dv, const bool leading) const`	Function for writing out a `dvec` (the diagonal scalar products, `diagonal_d_spm`) to a file with standard filename given by `diagonal_spm_file_name`. The boolean variable `leading` should be true if `dv` only has leading `Nc` contributions. The filename is then modified accordingly.
`void write_out_diagonal_spm(const Poly_vec & pv, const bool leading) const`	Function for writing out a `Poly_vec` (the diagonal scalar products, `diagonal_d_spm`) to a file with standard filename given by `diagonal_spm_file_name`. The boolean variable `leading` should be true if `Poly_vec only` has leading `Nc` contributions. The filename is then modified accordingly.

References

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