# Wilson lines <br> lightcone generating function 

Viktor Svensson<br>Department of Astronomy and Theoretical Physics, Lund University

Master thesis supervised by Alexey Vladimirov and Roman Pasechnik


LUND
UNIVERSITY


#### Abstract

In the thesis, we consider basic properties of Wilson lines, special attention is devoted to the renormalization and exponentiation property. We describe the generating function approach to non-abelian exponentiation and perform the calculation of the correlator of two and three semi-infinite Wilson lines meeting at a single point to two-loop order. We consider Wilson lines on lightcone and regularize infrared divergences with a $\delta$-regulator. The three line calculation is done within the generating function approach. Using this calculation, we derive the cusp anomalous dimension and soft anomalous dimension at two-loop order. The obtained result to be used for the higher order analysis of soft factor and can be used for application in multi-hadron factorization theorems, threshold resummation, etc.


## Contents

1 Introduction ..... 3
2 ..... 6
2.1 Renormalization ..... 6
2.1.1 Of Wilson loops ..... 7
$2.2 \quad$ Feynman diagrams ..... 8
3 Wilson lines ..... 10
3.1 Definitions, elementary properties ..... 10
3.2 Exponentiation ..... 11
3.2.1 Non-abelian exponentiation ..... 12
3.2.2 Generating Function approach ..... 12
4 Cusp for two lines on light cone ..... 15
4.1 One-loop ..... 15
4.2 Two-loop ..... 16
4.2.1 In generating function approach ..... 18
5 Three cusp ..... 20
6 Conclusion ..... 22
A Sample calculations ..... 23
A. $1 W_{3 \mathrm{~g}}$ calculation ..... 23
A. 2 Calculation of 3cusp with 3gluon vertex ..... 24
B Feynman rules ..... 27
B. 1 QCD ..... 27
B. 2 Semi-infinite Wilson lines ..... 28
C Dimensional reduction formula ..... 29
D Algebra ..... 30
E Various formulas ..... 31

## 1 Introduction

Wilson lines are natural objects in gauge theories. They are a function of the gauge field along some specified path. This path is arbitrary, it may have endpoints or it may be a loop, a particularly important example. In this thesis, we will consider properties of Wilson lines within perturbation theory.

In this paragraph, we write down the definition of a Wilson line and introduce necessary notation. A Wilson line between points $a$ and $b$ along the path $C$ looks like

$$
\begin{align*}
\Phi(a, b ; C) & =\mathcal{P} \exp \left(-i g \int_{a}^{b} d z^{\mu} A_{\mu}(z)\right)  \tag{1.1}\\
& =1-i g \int_{a}^{b} d z^{\mu} A_{\mu}(z)-g^{2} \int_{a}^{b} d z_{1}^{\mu_{1}} \int_{z_{1}}^{b} d z_{2}^{\mu_{2}} A_{\mu_{1}}\left(z_{1}\right) A_{\mu_{2}}\left(z_{2}\right)+\ldots \tag{1.2}
\end{align*}
$$

where the symbol $\mathcal{P}$ denotes path-ordering, $A_{\mu}$ is the gauge field and $g$ is the coupling constant. The gauge field $A_{\mu}$ is a matrix valued field, it can be written in terms of a basis of matrices $t^{a}$, called generators, as $A_{\mu}^{a} t^{a}$. The generators $t^{a}$ can belong to any representation of the gauge group. In general, generators do not commute. Path-ordering specifies that the fields further along the path should be written to the right of earlier fields. The function $\mathcal{P} \exp$ should be interpreted as the application of $\mathcal{P}$ to the power series of the exponential.

Wilson lines possess important properties under gauge transformations. The corresponding transformation for the gauge field has the form

$$
A_{\mu}(z) \rightarrow U(z) A_{\mu}(z) U^{\dagger}(z)-\frac{i}{g}\left(\partial_{\mu} U(z)\right) U^{\dagger}(z)
$$

where $U(z)$ is a matrix of the gauge group. It implies that the Wilson line transforms as

$$
\Phi(a, b) \rightarrow U(a) \Phi(a, b) U^{\dagger}(b)
$$

which we prove in section 3. In particular, it implies that Wilson loops (lines where the endpoints coincide) are gauge invariant. A reformulation of gauge theories in terms of loops rather than the gauge field has been sought [1, 2, 3, 4, [5]. Nowadays, Wilson lines are used in nearly all branches of quantum field theory (QFT), from the study of confinement [6] to lattice calculations.

Wilson lines and loops have a lot of applications in description of hadronic processes in quantum chromodynamics (QCD). In QCD, many processes factorize into a soft and a hard part, or a long-distance and a short-distance part [7] 8]. The hard part can be treated within perturbation theory. The soft part involves non-perturbative effects and usually parametrized by phenomenological functions (parton distributions). Wilson lines are an important ingredient in factorization theorems, since they help to reconstruct the infrared divergences of hadronic processes [9]. Wilson lines encode the effect of soft radiation, the emission of low energy gauge bosons, in any gauge theory. The detailed description of how


Figure 1: Some important classes of loops. A smooth, a cusped, and a cusped crossed loop.
soft emissions of a highly energetic particle is described by a Wilson line along the path of the particle, can be found in introductory textbooks on QFT such as [7, 10, 11].

One of the most involved problems in perturbative descriptions of Wilson lines is the consideration of their renormalization properties. The main difficulty is that Wilson lines do not a natural scale. Therefore, infrared singularities mix with ultraviolet singularities. On the other hand, ultraviolet singularities can be described with the renormalization group equation, which give us hope to resolve the infrared singularities as well [12].

It is known that smooth Wilson loops (left diagram in figure 1) are ultraviolet finite in the renormalized theory [4, 13]. However, the cusped Wilson line or self-intersecting Wilson lines (center and right diagram of figure 1, correspondingly) needs additional renormalization constant [14]. The renormalization constants are governed by the cusp anomalous dimension. The cusp anomalous dimension has been known to two loop-order for a long time [15]. Recently, the three-loop cusp anomalous dimension was found [16].

Wilson lines on lightcone are of special interest. Wilson lines of such configurations appear in the description of hard processes where one can neglect the mass of partons, which is the most typical case. The lightcone Wilson lines has additional infrared divergences [15]. Although this problem has been known for 30 years, there is no clear description of renormalization in this case. This thesis is devoted to the derivation of the anomalous dimension of several lightlike Wilson lines meeting at a point.

In this thesis, we consider the basic properties of Wilson lines. Our main interest is devoted to the renormalization of Wilson lines, especially to the case of several lightlike Wilson lines meeting at a single point. Such a configuration is called soft anomalous dimension matrix, and is of great importance in the phenomenology of high energy processes with many jets (for recent reviews, see [17, 18]).

We describe a generating function approach to non-abelian exponentiation, which was recently presented in [19, 20]. With the help of that method, we perform detailed two-loop analysis of soft anomalous dimension for lightlike Wilson loops. As regulator for soft divergences, we use a modified $\delta$-regularization. For regulation of ultraviolet (UV) divergences, we use dimensional regularization. The two-loop calculation of this combination of regularizations is novel. We have shown that the $\delta$-regulator has a number of problems arising at two-loops, such as violation of gauge invariance and scale invariance. These problems haven't been observed earlier in the one-loop calculations. We present solutions to these problems. Although we observe artificial terms in the two-loop cusp anomalous dimension. The dipole factorization of soft anomalous dimension at two-loops is confirmed [21, 22, 23].

The structure of the thesis is the following. In the first section, we review neccessary
elements of non-abelian gauge theories and perturbative expansion. In section two, we present derivations of important properties of Wilson lines and following that we introduce generating function for Wilson line [19, 20]. Section 3 is the main section of the thesis where we present our approach to lightlike Wilson lines and give details on the two-loop calculation of the soft anomalous dimension. The collection of equations needed are given in the appendices.

## 2

### 2.1 Renormalization

Two kinds of divergences are encountered in QFT, infrared and ultraviolet. UV singularities come from the high energy regime. This may signal that the theory is incomplete at high energies, and can be handled through renormalization. IR divergences in QCD may be soft or collinear. Soft divergences come from the emission of low energy gluons, while collinear divergences arise when a particle emits a gluon in the direction it travels. IR divergences should cancel in sums of diagrams corresponding to well-defined observables.

In a sense, UV divergences are also solved by careful analysis of what is observable. The lagrangian specifying the theory includes several parameters, such as coupling constants and masses. The quantities calculated from the theory with these parameters will often be infinite. Through renormalization, one can find finite relations between physical values.

The first step is to regularize the integrals, so that we may manipulate these results algebraically. This can be done in many ways. The conceptually simplest regularization is to cut the integral off at some value $\lambda$. The integrals now converge and we calculate observables as a function of $\lambda$ and the Lagrangian parameters. If we add this $\lambda$-dependence into the parameters, observables are finite functions of the new parameters. The new parameters are finitely related to experimental outcomes and can be measured.

The cut-off $\lambda$ may be unphysical, or it may represent a true physical cut-off, signaling that the theory is incomplete. Renormalization can be done in either case. We can choose other types of regularization. The final result does not depend on the specific way.

Dimensional regularization is based on the fact that integrals may be divergent in some dimensions, but convergent in others. In dimensional regularization, the integral is considered as a function of it's dimension $d$. This function can be evaluated for $d$ where it converges and analytically continued to other values. Typically, we start with 4 -dimensional integrals. By analytically continuing to $d=4-2 \epsilon$, where $\epsilon$ is small, the divergence of the original integral is represented by terms like $\epsilon^{-1}$.

This shift in dimension also modifies the dimension of your Lagrangian parameters. Previously dimensionless quantities, like the coupling constant $g_{0}$, acquire a fractional dimension. By introducing a new parameter $\mu$, of some appropriate dimension, we can define a dimensionless coupling constant by

$$
\begin{equation*}
\frac{\alpha_{s}}{4 \pi}=\frac{\mu^{-2 \epsilon} g_{0}^{2}}{(4 \pi)^{d / 2} \Gamma(1+\epsilon)} . \tag{2.3}
\end{equation*}
$$

The factors in the denominator simplify expressions by canceling out several artificial terms. This is called an $\overline{M S}$-scheme.

The independence of the original parameters on $\mu$ leads to the renormalization group equation (RGE), a differential equation for how parameters depend on $\mu$. We show how in the next section on the renormalization of Wilson lines. The value of the renormalization group is in handling different scales of $\mu$ within perturbation theory. The value of $\mu$ will impact the convergence properties of the perturbative series. The same series may not be
valid for values of $\mu$ that are too different. The RGE solves this problem by only using differential increments of $\mu$.

An important property of dimensional regularization is the way it handles scaleless integrals. Since the integral should produce an expression of fractional dimension, it must contain dimensional parameters. If it does not, it can be set to zero. A more detailed analysis would show that it in fact contains both UV and IR poles, but that they cancel each other.

If the modification of the parameters can be expressed with a renormalization factor, i.e,

$$
\begin{equation*}
g_{0}=Z_{g}(\epsilon, g(\mu)) g(\mu) \tag{2.4}
\end{equation*}
$$

where $g$ is the renormalized parameter, the theory is multiplicatively renormalizable. We will make use of the renormalization factors for the coupling constant $g$ and the gauge field $A_{\mu}$. these are well known in QCD to the order we need.

### 2.1.1 Of Wilson loops

The renormalization of Wilson loops depends on their path. The case of a smooth nonintersecting loop is the simplest. In dimensional regularization, their renormalization is complete when the coupling constant and the gauge field have been renormalized [4, 13. The renormalized loop takes the form

$$
\begin{equation*}
\Phi_{R}(C)=\mathcal{P} \exp \left(-i Z_{g} Z_{A}^{1 / 2} g \oint d z^{\mu} A_{\mu}(z)\right) . \tag{2.5}
\end{equation*}
$$

Wilson loops containing cusps, i.e. points where the contour is not smooth, have additional divergences. A cusp is characterized by its two tangent vectors $\nu_{1}$ and $\nu_{2}$. The cusp divergence is a function of the angle $\gamma_{12}$ between the vectors. This angle is defined in Minkowski space as $\cosh \gamma=\frac{\nu_{12}}{\sqrt{\nu_{1}^{2} \nu_{2}^{2}}}$, where $\nu_{12}=\left(\nu_{1} \cdot \nu_{2}\right)$. If both vectors are off lightcone, the cusp divergence can be multiplicatively renormalized by a factor $Z_{\gamma}$ [13]. For loops with a finite amount of cusps off lightcone, multiply by the renormalization factor of each cusp.

In the RGE for cusped Wilson lines, the cusp renormalization factors give rise to the cusp anomalous dimension. The RGE comes from the independece of the original loop $\Phi(C)=Z_{\gamma} \Phi_{R}(C)$ on $\mu$. Differentiating with respect to $\mu$ we have

$$
\begin{equation*}
\mu \frac{d}{d \mu} Z_{\gamma} \Phi_{R}\left(\mu^{2} ; C\right)=\left(\mu \frac{\partial}{\partial \mu}+\beta \frac{\partial}{\partial g}+\Gamma_{\operatorname{cusp}(\gamma, g)}\right) \Phi_{R}\left(\mu^{2} ; C\right)=0 \tag{2.6}
\end{equation*}
$$

where $\Gamma_{\text {cusp }}(\gamma, g)=\frac{\mu}{Z_{\gamma}} \frac{d Z_{\gamma}}{d \mu}$ is the cusp anomalous dimension and $\beta=\frac{\mu}{Z_{g}} \frac{\partial Z_{g}}{\partial \mu}$.
The cusp anomalous dimension has been known to two-loop order for a long time [15]. To three-loop order it was calculated in [16].

When one or both of the vectors are on light cone, this is not valid anymore. This follows from the fact that when $\nu$ approaches lightcone, $\gamma\left(\nu_{1}, \nu_{2}\right)$ blows up. For large $\gamma$, it



Figure 2: Renormalization of loops with self-intersection mixes all possible ways of connecting incoming with outgoing lines.
is proven in [24] that $\Gamma_{\text {cusp }}$ is linear in $\gamma$ to all orders in perturbation theory. Then we see in [15] that by explicit calculation $\Phi_{R}$ is finite. The RGE equation cannot be satisfied.

Nevertheless, it is possible to find a similar equation that does hold [15]. The trick is to differentiate $\Gamma_{\text {cusp }}(\gamma, g)$ with respect to $\nu_{12}$. This will remove all dependence on $\nu^{2}$. This removes the problem of $\Gamma_{\text {cusp }}(\gamma, g)$ blowing up. Integrate back over $\nu_{12}$ to find

$$
\begin{equation*}
\left(\mu \frac{\partial}{\partial \mu}+\beta \frac{\partial}{\partial g}+\right) \Phi_{R}\left(\mu^{2} ; C\right)=-\Gamma_{c u s p}(g) \log \nu_{12}-\Gamma(g) \tag{2.7}
\end{equation*}
$$

where $\Gamma(g)$ is some integration constant. This let's us find $\Gamma_{\text {cusp }}(g)$ from knowing $\Phi_{R}$.
For Wilson lines with endpoints, they will also contribute with renormalization factors $[12$ and a corresponding endpoint anomalous dimension. In this thesis, we study semiinfinite lines with endpoints at infinity. We will introduce a regulator that exponentially suppresses these contributions.

For completeness, we also mention the case where loops self-intersect. This case reduces to the case with cusps [14]. There is mixing between all ways of connecting the incoming lines with outgoing lines, see figure 2.1.1.

### 2.2 Feynman diagrams

We are looking to calculate the vacuum averages of Wilson lines. Here is where the machinery of QFT comes in. The path integral approach takes us quickly to what we need. As any introductory textbook on QFT explains, the vacuum average of an operator can be calculated as

$$
\begin{equation*}
\langle\mathcal{O}\rangle=\frac{1}{Z_{0}} \int \mathcal{D} A \mathcal{O} e^{i S_{0}[A]+i S_{\mathrm{int}}} \tag{2.8}
\end{equation*}
$$

where $Z_{0}$ is the same integral without the operator $\mathcal{O}$ and the integration is over all possible configurations of the fields $A$. This is very similar to calculating expectation values in statistical mechanics using the partition function. With functional derivatives we can write it as

$$
\begin{equation*}
\langle T \mathcal{O}[A]\rangle=\left.Z_{0}^{-1} e^{\int \frac{d x d x^{\prime}}{2} \frac{\delta}{\delta A_{x}} \Delta\left(x, x^{\prime}\right) \frac{\delta}{\delta A_{x^{\prime}}}} \mathcal{O}[A] e^{i S_{\text {int }}[A]}\right|_{A=0} \tag{2.9}
\end{equation*}
$$

A derivation is in appendix C. For perturbative calculations, expand the exponentials in a power series. The functional derivatives will act according to the ordinary product rule


Figure 3: The Feynman graph for $\int \frac{\delta \mathcal{O}\left(x_{1}\right)}{\delta A\left(x_{1}\right)} \Delta\left(x_{1}, x_{2}\right) \frac{\delta \mathcal{O}^{\prime}\left(x_{2}\right)}{\delta A\left(x_{2}\right)} \frac{\delta \mathcal{O}^{\prime}\left(x_{3}\right)}{\delta A\left(x_{3}\right)} \Delta\left(x_{3}, x_{4}\right) \frac{\delta \mathcal{O}^{\prime}\left(x_{4}\right)}{\delta A\left(x_{4}\right)}$
for derivatives, generating a large amount of terms. These can be kept track of using Feynman diagrams. When $n$ functional derivatives act on operator, such as $F$ or those in $S_{\text {int }}$, denote it by a vertex with $n$ external legs. These vertices are joined up by the propagator $\Delta\left(x, x^{\prime}\right)$. As an example, with two operators $\mathcal{O}$ and $\mathcal{O}^{\prime}$, in figure 3 the graph corresponding to the term $\left.\left.\int \frac{\delta}{\delta A_{1}} \mathcal{O}\right|_{A=0} \Delta\left(x_{1}, x_{2}\right) \Delta\left(x_{3}, x_{4}\right) \frac{\delta}{\delta A_{2}} \frac{\delta}{\delta A_{3}} \frac{\delta}{\delta A_{4}} \mathcal{O}^{\prime}\right|_{A=0}$ is pictured. We have completely glossed over the combinatorial factors.

In appendix we present the Feynman rules for QCD and a derivation of the Feynman rules for the semi-infinite Wilson lines we study.

## 3 Wilson lines

### 3.1 Definitions, elementary properties

Consider we have a contour consisting of the points $z(\tau)$ with $\tau \in[0,1]$ and $z(0)=a, z(1)=$ b. At each point $z$ on the contour we have the tangent vector $\gamma^{\mu}(z(\tau))$. Define the Wilson line to satisfy

$$
\begin{align*}
\gamma^{\mu}(z(\tau)) \vec{D}_{\mu} \Phi(z(\tau), b) & =0 \\
\Phi(z(\tau), z(\tau)) & =1  \tag{3.1}\\
\Phi(a, z(\tau)) \Phi(z(\tau), b) & =\Phi(a, b)
\end{align*}
$$

for each $z$. Here, the covariant derivative is $D_{\mu}=\partial_{\mu}-i g A_{\mu}$. The reason for introducing the parameter $\tau$, rather than just using $z$, is to avoid ambiguities in the path taken. If the contour loops back on itself, $z$ does not specify the path. The equation $\Phi(z, z)=1$ only holds when it refers to the trivial path of a single point. Being aware of this, we will drop $\tau$ for simplicity.

Basing on these equations let us prove some basic properties of wilson lines. From the first two equations one can derive how the covariant derivative acts on the second argument.

$$
\begin{equation*}
\gamma^{\mu}(z) \Phi(a, z) \overleftarrow{D}_{\mu}=0 \tag{3.2}
\end{equation*}
$$

The transformation properties of the gauge field $A$ determines the transformation properties of the Wilson line $\Phi$. The gauge field $A$ transforms as

$$
\begin{equation*}
A_{\mu}(z) \rightarrow U(z) A_{\mu}(z) U^{\dagger}(z)-\frac{i}{g}\left(\partial_{\mu} U(z)\right) U^{\dagger}(z) \tag{3.3}
\end{equation*}
$$

The covariant derivative transforms as $D_{\mu}(z) \rightarrow U(z) D_{\mu}(z) U^{\dagger}(z)$. Then if $\Phi \rightarrow \Phi^{\prime}$, one has

$$
\begin{gather*}
\gamma^{\mu}(z) U \vec{D}_{\mu} U^{\dagger} \Phi^{\prime}(z, b)=0 \\
\gamma^{\mu}(z) \Phi^{\prime}(a, z) U \overleftarrow{D^{\dagger}}{ }_{\mu} U^{\dagger}=0  \tag{3.4}\\
\Phi^{\prime}(a, z) \Phi^{\prime}(z, b)=\Phi^{\prime}(a, b)
\end{gather*}
$$

Clearly, the transformation $\Phi(a, b) \rightarrow U(a) \Phi(a, b) U^{\dagger}(b)$ satisfies all of these equations. Under hermitian conjugate, the first equation in (3.1) becomes

$$
\begin{equation*}
\gamma^{\mu}(z) \Phi^{\dagger}(z, a) \overleftarrow{D}_{\mu}=0 \tag{3.5}
\end{equation*}
$$

Comparing with equation (3.2), we find that

$$
\begin{equation*}
\Phi(a, b)^{\dagger}=\Phi(b, a) . \tag{3.6}
\end{equation*}
$$

Finally, $\Phi$ is unitary: $\Phi(a, b) \Phi(a, b)^{\dagger}=\Phi(a, b) \Phi(b, a)=\Phi(a, a)=1$.
It is clear from the definition that $\Phi$ is invariant under rescaling of the tangent vector.

We will look at semi-infinite straight Wilson lines $\Phi_{\nu}(0, \infty)$. These are specified by the constant tangent vector $\nu$. The unbounded integration gives rise to an infrared divergence. To regularize it, we introduce a regulator that exponentially suppress the integrand away from the origin. This amounts to including a factor $e^{-\delta \tau}$ for each $A$. The regularized Wilson line takes the form

$$
\begin{align*}
& \Phi_{\nu}(0, \infty)=1-i g \int_{0}^{\infty} A_{\mu} e^{-\delta \tau_{1}} v^{\mu} d \tau_{1}  \tag{3.7}\\
&-g^{2} \int_{0}^{\infty} d \tau_{1} \int_{\tau_{1}}^{\infty} d \tau_{2} A_{\mu}\left(\tau_{1}\right) A_{\nu}\left(\tau_{2}\right) e^{-\delta\left(\tau_{1}+\tau_{2}\right)} v^{\mu} v^{\nu}+\ldots
\end{align*}
$$

Using the path ordering operator, which orders the fields later along the path to the right of earlier fields, we write the solution compactly as

$$
\begin{equation*}
\Phi_{\nu}(0, \infty)=\mathcal{P} \exp \left(-i g \int_{0}^{\infty} A_{\mu} e^{-\delta \tau} v^{\mu} d \tau\right) \tag{3.8}
\end{equation*}
$$

The path ordering acts termwise in the expansion of the exponential.
Such a regulator is used in for example [25], but with $\delta \sqrt{\nu^{2}}$ instead of just $\delta$. This factor is there to make sure the integral is invariant under rescaling of the tangent vector. However, we consider Wilson lines on lightcone where $\nu^{2}=0$. Therefore, this factor is not available to us. We will come back to this when we study the cusp with three lines.

WRITE THIS Infrared divergences.

### 3.2 Exponentiation

The exponentiation of disconnected diagrams is standard material in textbooks like [10. We present similar arguments here, tailored to our purpose. Exponentiating means that a sum of diagrams can be represented as the exponential of another sum of diagrams. Connected diagrams play a crucial role here. If a diagram is made out of disconnected pieces, the diagram is equal to the product of those pieces.

Consider an operator of the form $\mathcal{O}=e^{\mathcal{F}}$. For a moment, we assume $\mathcal{F}$ to be a scalar operator, so that there is no trouble with reordering. Expanding $\mathcal{O}$ in a power series, we find terms like $\frac{\mathcal{F}^{n}}{n!}$. This operator gives rise to diagrams with $n$ insertions of $\mathcal{F}$-vertices. We can factor such a diagram into connected pieces $C_{i}$. Denote by $k_{i}$ the number of $\mathcal{F}$-vertices in the piece and by $m_{i}$ the number of such pieces in the diagram. Taking into account the combinatorics of how many ways one can partition the $\mathcal{F}$-vertices into the connected pieces, the diagram is

$$
\begin{equation*}
\prod_{i} \frac{C_{i}^{m_{i}}}{k_{i}!m_{i}!} \tag{3.9}
\end{equation*}
$$

The series for $\langle\mathcal{O}\rangle$ is the sum of all such diagrams. It's easy to see that all terms are generated by

$$
\begin{equation*}
\exp \left(\sum_{i} \frac{C_{i}}{k!}\right) . \tag{3.10}
\end{equation*}
$$



Figure 4: The diagrammatic relation that allows for exponentiation at two-loop order. The crossed diagram should be taken with a modified color factor.
where the sum runs over all connected diagrams. We conclude that operators of the form $\mathcal{O}=e^{\mathcal{F}}$ exponentiate. The original series which includes disconnected diagrams is generated by the exponential of a series of only connected diagrams. Abelian Wilson lines can be exponentiated in this manner [19].

If $\mathcal{F}$ is a matrix, the order of $\mathcal{F}$-vertices is important. Then different partitions of $\mathcal{F}$-vertices are not equivalent. Nonetheless, non-abelian Wilson lines also exponentiate [26, ?, 27, 28], as we will see in the next section.

### 3.2.1 Non-abelian exponentiation

The non-abelian exponentiation theorem was proven in [27]. That renormalization may be performed in the exponent was shown in [28]. The fact that the perturbative series can be presented by the exponential of another series is trivial. One only needs to solve the following equation for $w_{n}$.

$$
\begin{equation*}
\Phi=1+\sum_{n=1}^{\infty}\left(\frac{\alpha_{s}}{\pi}\right)^{n} W_{n}=\exp \sum_{n=1}^{\infty}\left(\frac{\alpha_{s}}{\pi}\right)^{n} w_{n} \tag{3.11}
\end{equation*}
$$

The non-abelian exponentiation theorem [27] [28] specifies the form of $w_{n}$. It states that $w_{n}$ consists of the same diagrams as $W_{n}$, but with modified color factors. The advantage of the approach is that only a subset of diagrams, called webs, have non-zero modified color factors. Webs are those diagrams which cannot be separated into two lower order diagrams by two cuts of Wilson lines.

Considering cusp anomalous dimension at two-loop order with the help of webs, we can eliminate one diagram. On the diagrammatic level, it works like figure 4 shows. The ladder diagram can be eliminated in the exponent, while the crossed diagram must be taken with a modified color factor.

### 3.2.2 Generating Function approach

This approach is explained in detail in [19, 20]. Here, we extract only the general idea and some expressions we need.

In the previous section, the Wilson line $\Phi$ has been expressed as a path ordered exponent, but it can be represented by an ordinary exponential. The price to pay is that the
exponent is more complicated.

$$
\begin{equation*}
\Phi(a, b)=\exp \left(\sum_{k=0}^{\infty} \Omega_{k}(a, b)\right) \tag{3.12}
\end{equation*}
$$

where $\Omega_{k} \sim g^{k}$. The series is known as the Magnus series. It resembles the Baker-CampbellHausdorff formula, but we don't need the exact form of it. It is important to us that each term consists of completely nested commutators of the gauge field $A$. Using algebra of generators

$$
\begin{equation*}
\left[t^{a}, t^{b}\right]=f^{a b c} t^{d}, \tag{3.13}
\end{equation*}
$$

where $f^{a b c}$ is the structure constant, one can extract single generator out of the commutators and each $\Omega_{k}$ can be written as $\Omega_{k}=t^{a} V_{k}^{a}$, where $V_{k}^{a} \sim g^{k}$. Only non-comutativity of the generators prevent us from the ordinary exponentiation with connected diagrams. The trick is to replace generators by scalars $t^{a} \rightarrow M^{a}$, one for each generator. We can at any stage go back to matrices by means of the matrix shift operator. Action of the matrix shift operator on a scalar function $f(x)$ is defined by

$$
\begin{equation*}
\widetilde{f}(t)=\left.\exp \left(t^{a} \frac{\partial}{\partial x^{a}}\right) f(x)\right|_{x=0}, \tag{3.14}
\end{equation*}
$$

where $\sim$ denotes the matrix function .
Following [?], we introduce the scalar Wilson line

$$
\begin{equation*}
\phi=e^{M^{a} V_{a}} . \tag{3.15}
\end{equation*}
$$

Its vacuum expectation value can be exponentiated in the usual manner.

$$
\begin{equation*}
\langle\phi\rangle=e^{W[M]} \tag{3.16}
\end{equation*}
$$

where $W[M]$ is a sum over connected diagrams with insertions of $V_{a}$-vertices. Shifting back to matrices we have

$$
\begin{equation*}
\langle\Phi\rangle=\left.\exp \left(t^{a} \frac{\partial}{\partial M^{a}}\right) e^{W[M]}\right|_{M=0} \tag{3.17}
\end{equation*}
$$

The matrix shift and the exponential do not commute, hence $\langle\Phi\rangle \neq \exp (\widetilde{W[T]})$. By defining the defect of exponentiation as

$$
\begin{equation*}
\widetilde{\delta W}[T]=\left.\left[\log , \exp \left(t^{a} \frac{\partial}{\partial M^{a}}\right)\right] e^{W[M]}\right|_{M=0} \tag{3.18}
\end{equation*}
$$

we obtain the expectation value of the Wilson line

$$
\begin{equation*}
\langle\Phi\rangle=e^{\widetilde{W[T]}+\widetilde{\delta W}} \tag{3.19}
\end{equation*}
$$

The defect is a function of $\widetilde{W[T]}$, which is called the kernel of matrix exponentiation (MEK). For perturbative calculations, it is convenient to decompose it into orders of $g$ : $\widetilde{\delta W}=\sum_{n=1}^{\infty} \widetilde{\delta_{n} W}$, where $\widetilde{\delta_{n} W} \sim g^{n}$. A recursive formula for the defect is

$$
\begin{equation*}
\widetilde{\delta_{n} W}[T]=\frac{1}{n!}\left\{\widetilde{W^{n}}\right\}-\sum_{k=2}^{n} \frac{1}{k!} \sum_{i>1, \sum i=n}\left(\widetilde{\delta_{i_{1}} W} \ldots \widetilde{\delta_{i_{k}} W}\right), \tag{3.20}
\end{equation*}
$$

where $\widetilde{\delta_{1} W}=\widetilde{W}$. We write down the explicit formula at second order as we will use it later,

$$
\begin{equation*}
\widetilde{\delta_{2} W}[T]=\frac{1}{2}\left(\left\{\widetilde{W^{2}}\right\}-(\widetilde{W})^{2}\right) . \tag{3.21}
\end{equation*}
$$

It is useful to represent the MEK in the following form

$$
\begin{align*}
\widetilde{W}= & \sum_{k=1}^{N} t_{k}^{a}\left\langle V_{\gamma_{k}}^{a}\right\rangle+\sum_{\substack{k_{l} l=1 \\
k<l}}^{N} t_{k}^{a} t_{l}^{b}\left\langle V_{\gamma_{k}}^{a} V_{\gamma_{l}}^{b}\right\rangle+\sum_{k}^{N} \frac{\left(t_{k}^{\{a b\}}\right.}{2!}\left\langle V_{\gamma_{k}}^{a} V_{\gamma_{l}}^{b}\right\rangle  \tag{3.22}\\
& +\sum_{\substack{k, l, m=1 \\
k<l<m}}^{N} t_{k}^{a} t_{l}^{b} t_{m}^{c}\left\langle V_{\gamma_{k}}^{a} V_{\gamma_{l}}^{b} V_{\gamma_{m}}^{c}\right\rangle+\sum_{\substack{k_{2} l=1 \\
k<l}}^{N} \frac{t_{k}^{\{a b\}}}{2!} t_{l}^{c}\left\langle V_{\gamma_{k}}^{a} V_{\gamma_{k}}^{b} V_{\gamma_{l}}^{c}\right\rangle+\sum_{\substack{k, l=1 \\
k<l}}^{N} \frac{t_{k}^{a} t_{l}^{\{b c\}}}{2!}\left\langle V_{\gamma_{l}}^{a} V_{\gamma_{k}}^{b} V_{\gamma_{k}}^{c}\right\rangle \\
& +\sum_{k=1}^{N} \frac{t_{k}^{\{a b c\}}}{3!}\left\langle V_{\gamma_{k}}^{a} V_{\gamma_{k}}^{b} V_{\gamma_{k}}^{c}\right\rangle+\ldots,
\end{align*}
$$

where $t^{\left\{a_{1} \ldots a_{n}\right\}}$ is the symmetric sum of the generators $t^{a_{i}}$ weighted by $\frac{1}{n!}$, the dots denote the correlators with the higher number of operators $V$, that are not neccesary in this work.


Figure 5: All diagrams contributing to the cusp on lightcone up to two loops. Diagram $e$ is the one-loop contributions to the gluon propagator including counterterms.

## 4 Cusp for two lines on light cone

In this section we consider a configuration of two lightlike Wilson lines forming a cusp. Their directions will be denoted $\nu_{1}$ and $\nu_{2}$. For evaluation we will use dimensional regularization in the $M S$-scheme, defined in ??, for regulation of ultraviolet divergences; $\delta$-regularization defined in ?? for regularization of collinear divergences. Soft divergences should not appear in our calculation, however to prevent possible problems we used a finite shift of Feynman propagator $\Delta$.

We will calculate the cusp on lightcone up to two loops. There are many diagrams which contribute to this quantity. However, for our configuration all diagrams with one or more propagators that start and end at the same Wilson line are zero, since they are proportional to $\nu^{2}=0$. Since we are working in exponentiated form, the ladder diagram is absent and the crossed diagram should be taken with a modified color factor $C_{F}\left(C_{F}-C_{A} / 2\right) \rightarrow-C_{F} C_{A} / 2$. The remaining non-zero diagrams contributing to the exponent are shown in figure ??. Diagram $b$ comes with a modified color factor.

In the following we present the results of the calculation and comment on the course of evaluation. In appendix A, we present the calculation of diagram $c$ in momentum space.

### 4.1 One-loop

Only diagram $a$ in figure 5 contributes at the one-loop level. Let us perform the one-loop calculation in the coordinate space. The Feynman rules can be used in calculation can be found in the appendix B.

The diagram a is the one-gluon-exchange diagram. Its color factor is $t_{i k}^{a} t_{k j}^{a}=C_{F} \delta_{i j}$. Its kinematic part is

$$
\begin{equation*}
-g^{2} \nu_{12} \int d x_{1,2} \int_{\infty}^{0} d \tau_{1} \int_{0}^{\infty} d \tau_{2} \delta\left(\nu_{1} \tau_{1}-x_{1}\right) \delta\left(\nu_{2} \tau_{2}-x_{2}\right) \Delta\left(x_{1}-x_{2}\right) e^{-\delta\left(\tau_{1}+\tau_{2}\right)} \tag{4.1}
\end{equation*}
$$

where we introduce the shorthand notation $\nu_{12}=\nu_{1} \nu_{2}$. The expression for the Feynman propagator in the coordinate space reads

$$
\begin{equation*}
\Delta(x-y)=\frac{\Gamma(1-\epsilon)}{4 \pi^{2-\epsilon}} \frac{g^{\mu \nu} \delta_{a b}}{\left(-(x-y)^{2}+i 0\right)^{1-\epsilon}} . \tag{4.2}
\end{equation*}
$$

Integrating expresion (4.1) over $x$ we find

$$
\begin{equation*}
\frac{-g^{2} \nu_{12} \Gamma(1-\epsilon)}{4 \pi^{2-\epsilon}} \int_{0}^{\infty} \int_{0}^{\infty} d \tau_{1} d \tau_{2} \frac{e^{-\left(\tau_{1}+\tau_{2}\right) \delta}}{\left(2 \nu_{12} \tau_{1} \tau_{2}+i \Delta\right)^{1-\epsilon}}=-2 g^{2}\left(\frac{\nu_{12}}{2 \delta^{2}}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{(4 \pi)^{2-\epsilon}} \Gamma(\epsilon)^{2} \tag{4.3}
\end{equation*}
$$

The $\Delta$ regulator can be removed, as the dimension regularization and $\delta$-regularization is enough to regularize the divergences of the integral. Then the evaluation of 4.3) consists of only $\Gamma$ function integrals.

From equation (2.7), we find the cusp anomalous dimension at one loop as

$$
\begin{equation*}
\Gamma_{\text {cusp }}(\alpha)=\frac{\alpha_{s}}{\pi} C_{F} . \tag{4.4}
\end{equation*}
$$

This expression coincides with the well known value calculated, see for example [4].

### 4.2 Two-loop

At two-loop order, we have three diagrams (but note that diagram $c$ also contributes with permuted lines).

The blob in diagram $d$ is the renormalized gluon propagator at one-loop. It adds the contribution of quark-, gluon- and ghost-loops, with corresponding counterterms, within the propagator. In $\overline{M S}$-scheme the glion-polarization operator is

$$
\begin{align*}
\Pi_{a b}^{\mu \nu}(k)=i \delta_{a b} \frac{\alpha_{s}}{4 \pi}\left(g^{\mu \nu} k^{2}-k^{\mu} k^{\nu}\right)[ & \left(C_{A}(5-\epsilon)-4(1-\epsilon) T_{f} n_{f}\right)\left(\frac{-\mu^{2}}{k^{2}+i \Delta}\right)^{\epsilon} \\
& \left.\times \frac{\Gamma^{2}(1-\epsilon)}{\epsilon(3-2 \epsilon) \Gamma(1-2 \epsilon)}-\frac{1}{\epsilon}\left(\frac{5}{3} C_{A}-\frac{4}{3} T_{f} n_{f}\right)\right] \tag{4.5}
\end{align*}
$$

where the last term in the parenthesis represents the counterterms.
In our regularization, the $k^{\mu} k^{\nu}$ part of this propagator creates problems. In fact, that term should be removed from the calculation for the following reasons. Recall that $\delta$ regulates collinear divergences and $\Delta$ regulates soft divergences. The soft divergences cancel in the sum of diagrams while the collinear may remain. With $\left(k_{1}^{\mu}\right)\left(k_{2}^{\nu}\right)$ in the numerator, the $\delta$ is not needed to regularize the integral. But, we do find a $\delta$ in the result. We conclude that it has forcibly taken the job of $\Delta$, regularizing a soft divergence. The soft divergences of this diagram should cancel with those in the $k^{\mu} k^{\nu}$ part of the self-interaction diagrams. But since we are one lightcone, those diagrams disappear. We conclude that we must set this part to zero as well, in order to support gauge invariance violated by $\delta$-regulation. In all expressions below, the $k^{\mu} k^{\nu}$ contribution has been removed.

The expression for individual diagrams and results of their evaluation are

$$
\begin{align*}
& w_{\text {1-loop }}=i C_{F} g^{2} \nu_{12} \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{1}{\left(k^{2}+i \Delta\right)\left(k \nu_{1}-i \delta\right)\left(k \nu_{2}+i \delta\right)}  \tag{4.6}\\
& =-2 \frac{\alpha_{s}}{4 \pi} C_{F} e^{L \epsilon} \frac{\Gamma(1-\epsilon) \Gamma(\epsilon+1)}{\epsilon^{2}}  \tag{4.7}\\
& w_{\text {cross }}=\frac{C_{F} C_{A}}{2} g^{4} \nu_{12}^{2} \int \frac{d^{d} k d^{d} l}{(2 \pi)^{2 d}} \frac{1}{\left(k^{2}+i \Delta\right)\left(l^{2}+i \Delta\right)} \\
& \times \frac{1}{\left(k \nu_{1}-i \delta\right)\left((k+l) \nu_{1}-2 i \delta\right)\left(l \nu_{2}+i \delta\right)\left((k+l) \nu_{2}+2 i \delta\right)}  \tag{4.8}\\
& =-\left(\frac{\alpha_{s}}{4 \pi}\right)^{2} \frac{C_{A} C_{F}}{2} e^{2 L \epsilon} \frac{\Gamma^{2}(1-\epsilon) \Gamma^{2}(\epsilon+1)}{\epsilon^{4}}  \tag{4.9}\\
& w_{3 \mathrm{~g}}=\frac{C_{F} C_{A}}{2} g^{4} \nu_{12} \int \frac{d^{d} k_{1,2,3}}{(2 \pi)^{3 d}} \frac{\delta\left(k_{1}+k_{2}+k_{3}\right)}{\left(k_{1}^{2}+i \Delta\right)\left(k_{2}^{2}+i \Delta\right)\left(k_{3}^{2}+i \Delta\right)} \\
& \times \frac{\nu_{1}\left(k_{1}-k_{2}\right)}{\left(k_{1} \nu_{1}-i \delta\right)\left(\left(k_{1}+k_{2}\right) \nu_{1}-2 i \delta\right)\left(-k_{3} \nu_{2}-i \delta\right)}  \tag{4.10}\\
& =\left(\frac{\alpha_{s}}{4 \pi}\right)^{2} C_{A} C_{F} e^{2 L \epsilon} \frac{\Gamma(1+2 \epsilon)^{2} \Gamma(1-2 \epsilon)}{4 \epsilon^{3} \Gamma^{2}(1+\epsilon)}\left(\frac{\Gamma(1-\epsilon) \Gamma(1+\epsilon)}{\epsilon}\right. \\
& \left.-\frac{2^{1-2 \epsilon} \Gamma^{2}(1-\epsilon)}{\Gamma(2-2 \epsilon)}\right)  \tag{4.11}\\
& w_{\mathrm{se}}+w_{\mathrm{se}, \mathrm{ct}}=-C_{F} g^{2} \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{\nu_{1}^{\mu} \nu_{2}^{\nu} \Pi_{\mu \nu}(k)}{\left(k^{2}+i \Delta\right)^{2}\left(k \nu_{1}-i \delta\right)\left(k \nu_{2}+i \delta\right)}  \tag{4.12}\\
& w_{\mathrm{se}}=-\left(\frac{\alpha_{s}}{4 \pi}\right)^{2} C_{F} e^{2 L \epsilon} \frac{\Gamma^{2}(1-\epsilon) \Gamma^{2}(1+2 \epsilon)}{2 \epsilon^{3}(1-2 \epsilon) \Gamma^{2}(1+\epsilon)} \frac{C_{A}(5-3 \epsilon)-4 n_{f} T_{f}(1-\epsilon)}{3-2 \epsilon}(  \tag{4.13}\\
& w_{\mathrm{se}, \mathrm{ct}}=-2\left(\frac{\alpha_{s}}{4 \pi}\right)^{2} C_{F} e^{L \epsilon} \frac{\Gamma(1-\epsilon) \Gamma(1+\epsilon)}{\epsilon^{3}}\left(\frac{5 C_{A}-4 n_{f} T_{f}}{3}\right), \tag{4.14}
\end{align*}
$$

where we define for convenience

$$
L=\log \left(\frac{\nu_{12} \mu^{2}}{2 \delta^{2}}\right)
$$

We observe that the parameter $\Delta$ does not appear in our result.
At two-loop level one should also include the the renormalization constants of the gauge field and the gauge coupling. The renormalization factor $Z_{g} Z_{3}^{1 / 2}=1-C_{A} \frac{\alpha_{s}}{4 \pi \epsilon}+\mathcal{O}\left(\alpha_{s}^{2}\right)$ contributes with a term proportional to the one-loop result. Adding all pieces together we have

$$
\begin{align*}
\left(\frac{\alpha_{s}}{4 \pi}\right)^{2} w_{2} & =w_{\text {cross }}+2 w_{3 \mathrm{~g}}+w_{\mathrm{se}}+w_{\mathrm{se}, \mathrm{ct}}+\left(Z_{g}^{2} Z_{3}-1\right) w_{1-\mathrm{loop}}  \tag{4.15}\\
& =C_{F}\left(\frac{\alpha_{s}}{4 \pi}\right)^{2}\left(A L^{3}+B L^{2}+C L^{1}\right)+\text { finite terms } \tag{4.16}
\end{align*}
$$



Figure 6: The set of diagrams in the generating function approach. The ellipse means the symmetrized sum of vertex-orderings. Figure taken from [20].
where

$$
\begin{aligned}
& A=-\frac{\left(11 C_{A}-4 n_{f} T_{f}\right)}{9}, \quad B=C_{A}\left(\frac{2 \pi^{2}}{3}-\frac{67}{9}+\log (4)\right)+\frac{20 n_{f} T_{f}}{9} \\
& C=-C_{A}\left(2 \zeta(3)+\frac{11 \pi^{2}}{9}+\frac{404}{27}+4 \log ^{2}(2)-8 \log (2)\right)+n_{f} T_{f}\left(112 / 27+\left(4 \pi^{2}\right) / 9\right) .
\end{aligned}
$$

The corresponding cusp anomalous dimension is

$$
\begin{equation*}
\Gamma_{\text {cusp }}=C_{F}\left(\frac{\alpha_{s}}{\pi}\right)+\left(\frac{\alpha_{s}}{\pi}\right)^{2} \frac{C_{F}}{36}\left(C_{A}\left(67-6 \pi^{2}-36 \log (2)\right)-20 n_{f} T_{f}\right) \tag{4.17}
\end{equation*}
$$

This expression does not coincide with the well known value of $\Gamma_{\text {cusp }}$ :

$$
\Gamma_{c u s p}=\left(\frac{\alpha_{s}}{\pi}\right) C_{F}+\left(\frac{\alpha_{s}}{\pi}\right)^{2} \frac{C_{F}}{36}\left(C_{A}\left(67-3 \pi^{2}\right)-20 n_{f} T_{f}\right)
$$

for the first time calculated in [15]. One can see that we have an additional term proportional to $\pi^{2}$ and $\log 2$. For the moment, we do not have complete explanation of this discrepancy. However, we suppose that these terms are artificial and arise from the $\delta$ regulator. That observation is novel and have not been discussed in the literature to our best knowledge.

### 4.2.1 In generating function approach

In the generating function approach, a different set of diagrams arises. These are shown in figure 6. In [20], these diagrams are compared to the previous ones to find how their contributions are encoded in the MEK and the defect. Here we give present short conclusion.

Diagrams F, G, H, I are zero because of the contraction of antisymmetric three-gluon vertex with the symmetric sum of generators on one line. Diagrams A,C,D and E are equal to their corresponding diagrams in the ordinary approach.

The main difference of the approach is in the diagram B. This is not surprising, as in the ordinary exponentiation it is the ladder and crossed ladder diagrams that combines to
make exponentiation possible. It has been shown in [20], that on the lightcone, diagram $B$ is zero. Therefore, the two-gluon exchange diagrams are given entirely by the defect, which is a function of the one-loop result. The defect was calculated in [20] and found to be

$$
\begin{equation*}
\widetilde{\delta W_{2}}=-\frac{C_{A}}{C_{F}} \frac{w_{1}^{2}}{8} . \tag{4.18}
\end{equation*}
$$

This expression coincides with our expression ??. That implies that both approaches to exponentiation are equivalent.

## 5 Three cusp

We will work entirely in the generating function approach for this cusp. For three Wilson lines, we get additional diagrams. The previous diagrams with permuted Wilson lines and new diagrams in which all three lines are connected. Figure 7 shows the new diagrams. We omit the cusp as it is convenient to consider each line in a different matrix space. They can in the end be joined up in the desired order.

Compare approaches.


Figure 7: For the three-cusp in the generating function approach, these are the additional diagrams one needs to consider. Diagram 3a reduces to a product of already calculated diagrams. Diagram 3b is zero. Diagram 3c vanishes when we restore scale-invariance by modifying our $\delta$ regulator.

Diagram 2b. This diagram comes from the term $t_{1}^{a} t_{2}^{b} t_{3}^{c}\left\langle V_{\nu_{1}}^{a} V_{\nu_{2}}^{b} V_{\nu_{3}}^{c}\right\rangle$, where the subscript on the generators denote which matrix space it's in. The lowest order term from this diagram takes $V_{1}$ for two lines and $V_{2}$ for the remaining one. On lightcone, this is zero. This can be seen before doing any integrals. The diagram is proportional to the integral

$$
\begin{equation*}
\int_{0}^{\infty} d x_{1,2} d y_{1,2} \frac{\left(\theta\left(x_{1}>x_{2}\right)-\theta\left(x_{2}>x_{1}\right)\right) e^{-\delta\left(x_{1}+x_{2}+y_{1}+y_{2}\right)}}{\left(2 x_{1} y_{1} \nu_{12}+i 0\right)^{1-\epsilon}\left(2 x_{2} y_{2} \nu_{23}+i 0\right)^{1-\epsilon}} \tag{5.19}
\end{equation*}
$$

The $+i 0$ can be removed since $\epsilon$ regularizes the integral. Removing it, the integral vanishes by symmetry. The same argument is the reason why diagram B vanishes in figure 6. This argument generalizes to all multiple gluon exchange webs [20], those diagrams with no direct interaction among gluon.

Let's consider diagram 3a. This diagram is the lowest order contribution from $\frac{1}{2!} t_{1}^{a} t_{2}^{b b^{\prime}} t_{3}^{c}\left\langle V_{\nu_{1}}^{a} V_{\nu_{2}}^{b} V_{\nu_{2}}^{b^{\prime}} V_{\nu_{3}}^{c}\right\rangle$.

$$
\begin{align*}
\left\langle V_{1}^{a} V_{2}^{b} V_{2}^{b^{\prime}} V_{3}^{c}\right\rangle=-\left(\delta^{a b} \delta^{b^{\prime} c}\right. & \left.+\delta^{a b^{\prime}} \delta^{b c}\right) g^{4}\left(\frac{\Gamma(1-\epsilon)}{4 \pi^{2-\epsilon}}\right)^{2}  \tag{5.20}\\
& \times \int_{0}^{\infty} d x_{1,2} d y_{1,2} \frac{e^{-\delta\left(x_{1}+x_{2}+y_{1}+y_{2}\right)}}{\left(2 x_{1} y_{1} \nu_{12}+i 0\right)^{1-\epsilon}\left(2 x_{2} y_{2} \nu_{13}+i 0\right)^{1-\epsilon}}
\end{align*}
$$

Up to color factors, it is essentially the product of two one-gluon exchange diagrams.

$$
\begin{equation*}
w_{3 a}=2 t_{1}^{a} t_{3}^{c}\left(t^{a} t^{c}+t^{c} t^{a}\right)_{2}\left(\frac{\alpha_{s}}{4 \pi}\right)^{2} \frac{\Gamma^{2}(1-\epsilon) \Gamma^{2}(1+\epsilon)}{\epsilon^{4}}\left(\frac{\nu_{12} \mu^{2}}{2 \delta^{2}}\right)^{\epsilon}\left(\frac{\nu_{13} \mu^{2}}{2 \delta^{2}}\right)^{\epsilon} \tag{5.21}
\end{equation*}
$$

Diagram 3c arises in both the generating function and the ordinary approach. On lightcone, it has been calculated in [22]. They find that is is zero.

For diagram 3c, the details are in appendix A.2. The calculation there reveals that it vanishes when we choose a suitable $\delta$ for each line. For line 1 one should choose the regulator $\delta_{1}=\delta \sqrt{\frac{\nu_{12} \nu_{13}}{\nu_{23}}}$, and for the other lines cyclic permutations of it. We then have to go back and rescale the $\delta$ 's for the previous results as well. This is very simple; the result is that all the scalar products $\nu_{i j}$ in the logs disappear. Actually, [23] finds that is zero even keeping the scalar products. I get this numerically as well, but what symmetry-argument works?

What remains is the defect. It is a function of the one-loop which we write as

$$
\begin{equation*}
w_{1}=t_{1}^{a} t_{2}^{a} w_{12}+t_{2}^{a} t_{3}^{a} w_{23}+t_{1}^{a} t_{3}^{a} w_{13} \tag{5.22}
\end{equation*}
$$

where $w_{i j}$ is the one-gluon exchange diagram between lines $i$ and $j$. For the defect we need $w_{1}^{2}$. $w_{1}^{2}$ will contain two classes of terms, squares and cross terms. We only need to consider one representative from each.

$$
\begin{equation*}
w_{1}^{2}=t_{1}^{a} t_{1}^{b} t_{2}^{a} t_{2}^{b} w_{12}^{2}+\left(t_{1}^{a} t_{1}^{b}+t_{1}^{b} t_{1}^{a}\right) t_{2}^{a} t_{3}^{b} w_{12} w_{13}+\ldots \tag{5.23}
\end{equation*}
$$

Let's look at the square.

$$
\begin{equation*}
\frac{1}{2}\left(\frac{1}{4}\left(t_{1}^{a} t_{1}^{b}+t_{1}^{b} t_{1}^{a}\right)\left(t_{2}^{a} t_{2}^{b}+t_{2}^{b} t_{2}^{a}\right)-t_{1}^{a} t_{1}^{b} t_{2}^{a} t_{2}^{b}\right)=\frac{C_{A} t_{1}^{a} t_{2}^{a}}{8} \tag{5.24}
\end{equation*}
$$

Need to discuss sign
The defect removes the symmetric part of the color factor. The cross term is fully symmetric in color space, hence the cross terms disappear.

At two-loop order, the three-cusp is completely determined by the interactions between pairs of lines.

## 6 Conclusion

In the thesis, we have presented one and two loop calculation involving lightcone wilson lines. We have considered both in traditional and generating function approach to exponentiation. We found agreement between these approaches. Conclude which is better. The two loop calculation in the generating function approach is done for the first time, and will serve as a base for the future publication.

We have considered the problems arising from the use of the $\delta$-regulator. These problems include violation of guage invariance and scale invariance, and we present possible solutions to these problems. We conclude that while the $\delta$-regulator, which has been known as very convenient at one-loop order, it presents difficulties at two-loop order. We could solve these at two-loop order with reasonable efforts, at higher orders they can present serious problems, making the $\delta$-regulator inappropriate for use for lightlike Wilson line configurations.

We have observed that the two-loop cusp anoamlous dimension calculated within $\delta$ regulator differs from the standard value by terms proportional to $\pi^{2}$ and $\log 2$. We conclude that this terms are artificial and are a result of the usage of $\delta$-regularization for lightlike Wilson lines. This observation is novel.

The results of our presented work are to be used for three-loop anomalous dimension. Within the generating function approach, the results of the calculation can be used for calculation of defect of exponentiation at all loop order, which may have important applications in diffractive processes. Our calculation confirms the dipole formula for soft factor [?], which is widely used for description of multi-hadron processes at high energies.

## A Sample calculations

## A. $1 \quad W_{3 \mathrm{~g}}$ calculation

In this section, we show in detail one way of calculating diagram c in figure 5 .

$$
\begin{equation*}
I_{3 g}=\int \frac{d^{d} k_{1,2,3}}{(2 \pi)^{3 d}} \frac{(2 \pi)^{d} \delta\left(k_{1}+k_{2}+k_{3}\right)\left(\left(2 k_{1} \nu_{1}-i \delta\right)-\left(\left(k_{1}+k_{2}\right) \nu_{1}-2 i \delta\right)\right)}{\left(k_{1}^{2}+i \Delta\right)\left(k_{2}^{2}+i \Delta\right)\left(k_{3}^{2}+i \Delta\right)\left(k_{1} \nu_{1}-i \delta\right)\left(\left(k_{1}+k_{2}\right) \nu_{1}-2 i \delta\right)\left(-k_{3} \nu_{2}-i \delta\right)} \tag{A.1}
\end{equation*}
$$

The integral splits in two simpler ones by cancellation of numerator and denominator.

$$
\begin{equation*}
I_{3 g}=2 I_{3 g 1}-I_{3 g 2} \tag{A.2}
\end{equation*}
$$

Calculation is basically the same, so we only consider

$$
\begin{equation*}
I_{3 g 1}=\int \frac{d^{d} k_{1,2,3}}{(2 \pi)^{3 d}} \frac{(2 \pi)^{d} \delta\left(k_{1}+k_{2}+k_{3}\right)}{\left(k_{1}^{2}+i \Delta\right)\left(k_{2}^{2}+i \Delta\right)\left(k_{3}^{2}+i \Delta\right)\left(\left(k_{1}+k_{2}\right) \nu_{1}-2 i \delta\right)\left(-k_{3} \nu_{2}-i \delta\right)} . \tag{A.3}
\end{equation*}
$$

Use $\alpha$-representation (equation E.7) for each propagator, introducing 5 new integration parameters going from 0 to $\infty$. For notational simplicity, we will not explicitly write the new integration variables and intervals. In addition, represent the Dirac delta function by $\delta(a)=\int_{-\infty}^{\infty} \frac{d^{d} x}{(2 \pi)^{d}} e^{-i x a}$. With all momentum in the exponent, complete the squares to get Gaussian integrals over momentum

$$
\begin{align*}
& I_{3 g 1}=i^{-1} \int \frac{d^{d} k_{1,2,3} d^{d} x}{(2 \pi)^{3 d}} \exp ( \\
& \qquad \begin{array}{l}
i \alpha_{1}\left(k_{1}-\frac{x+\beta_{1} \nu_{1}}{2 \alpha_{1}}\right)^{2}+i \alpha_{2}\left(k_{2}-\frac{x+\beta_{1} \nu_{1}}{2 \alpha_{2}}\right)^{2}+i \alpha_{3}\left(k_{3}-\frac{x-\beta_{2} \nu_{2}}{2 \alpha_{3}}\right)^{2} \\
\\
\quad-i \frac{\alpha_{123}}{4 \alpha_{1} \alpha_{2} \alpha_{3}}\left(x+\frac{\alpha_{1} \alpha_{2} \alpha_{3}}{\alpha_{123}}\left(\frac{\beta_{1} \nu_{1}}{\alpha_{1}}+\frac{\beta_{1} \nu_{1}}{\alpha_{2}}-\frac{\beta_{2} \nu_{2}}{\alpha_{3}}\right)\right)^{2} \\
\\
\left.\quad-i \frac{\alpha_{1}+\alpha_{2}}{2 \alpha_{123}} \beta_{1} \beta_{2} \nu_{12}-\Delta\left(\alpha_{1}+\alpha_{2}+\alpha_{3}\right)-\delta\left(2 \beta_{1}+\beta_{2}\right)\right)
\end{array}
\end{align*}
$$

where $\alpha_{123}=\alpha_{1} \alpha_{2}+\alpha_{2} \alpha_{3}+\alpha_{3} \alpha_{1}$. Doing the Gaussian integrals as in equation E. 9 we find

$$
\begin{equation*}
I_{3 g 1}=\frac{i^{-1+d-2}}{(4 \pi)^{d}} \int \alpha_{123}^{-d / 2} \exp \left(-i \frac{\alpha_{1}+\alpha_{2}}{2 \alpha_{123}} \beta_{1} \beta_{2} \nu_{12}-\Delta\left(\alpha_{1}+\alpha_{2}+\alpha_{3}\right)-\delta\left(2 \beta_{1}+\beta_{2}\right)\right) . \tag{A.5}
\end{equation*}
$$

Inserting $1=\int_{0}^{\infty} \frac{d \lambda}{\lambda} \delta\left(1-\lambda \sum \alpha\right)$ and rescaling $\alpha_{n} \rightarrow \alpha_{n} / \lambda$ we find

$$
\begin{equation*}
I_{3 g 1}=\frac{i^{d-3}}{(4 \pi)^{d}} \int \lambda^{-4+d} \alpha_{123}^{-d / 2} \delta\left(1-\sum \alpha\right) \exp \left(-i \lambda \frac{\alpha_{1}+\alpha_{2}}{2 \alpha_{123}} \beta_{1} \beta_{2} \nu_{12}-\Delta / \lambda-\delta\left(2 \beta_{1}+\beta_{2}\right)\right) \tag{A.6}
\end{equation*}
$$

The regulator $\Delta$ is used only for it's sign. So we may change $\Delta / \lambda \rightarrow \Delta \lambda$. The $\lambda$ integral followed by the $\beta$ integrals is straightforward, resulting in

$$
\begin{equation*}
I_{3 g 1}=\frac{2^{2 d-7}}{(4 \pi)^{d}} \Gamma(d-3) \Gamma(4-d)^{2} \delta^{2 d-8} \nu_{12}^{3-d} \int \alpha_{123}^{d / 2-3}\left(\alpha_{1}+\alpha_{2}\right)^{3-d} \delta\left(1-\sum \alpha\right) \tag{A.7}
\end{equation*}
$$

The Cheng-Wu theorem [29] can be applied here, which states that we may choose any subset of $\alpha$ 's in the delta function. Choose $\alpha_{1}+\alpha_{2}$. Then the remaining $\alpha$ integrals present no difficulties. The result is

$$
\begin{align*}
& \frac{2^{2 d-7}}{(4 \pi)^{d}} \Gamma(d-3) \Gamma(4-d)^{2} \delta^{2 d-8} \frac{\nu_{12}^{3-d}}{2-d / 2} \frac{\Gamma(d / 2-1)^{2}}{\Gamma(d-2)}  \tag{A.8}\\
= & \nu_{12}^{-1} \frac{2^{1-2 \epsilon}}{(4 \pi)^{4-2 \epsilon}} \Gamma(1-2 \epsilon) \Gamma(2 \epsilon)^{2}\left(\frac{\nu_{12}}{2 \delta^{2}}\right)^{2 \epsilon} \frac{1}{\epsilon} \frac{\Gamma(1-\epsilon)^{2}}{\Gamma(2-2 \epsilon)} .
\end{align*}
$$

## A. 2 Calculation of 3cusp with 3 gluon vertex



Figure 8: The only diagram requiring some effort of those where all three lines interact.
V1 on each line, 3 gluon vertex:

$$
\begin{equation*}
-g^{4} \delta^{a a^{\prime}} \delta^{b b^{\prime}} \delta^{b b^{\prime}} f^{a^{\prime} b^{\prime} c^{\prime}} \nu_{1}^{\mu_{1}} \nu_{2}^{\mu_{2}} \nu_{3}^{\mu_{3}} I_{\mu_{1} \mu_{2} \mu_{3}} \tag{A.9}
\end{equation*}
$$

where

$$
\begin{align*}
I_{\mu_{1} \mu_{2} \mu_{3}} & =\int \frac{d^{d} p d^{d} k}{(2 \pi)^{2 d}} \frac{V_{\mu_{1} \mu_{2} \mu_{3}}(-k, p, k-p)}{\left(k^{2}+i \Delta\right)\left(p^{2}+i \Delta\right)\left((p-k)^{2}+i \Delta\right)\left(k \nu_{1}-i \delta\right)\left(p \nu_{2}+i \delta\right)\left((p-k) \nu_{3}-i \delta\right)} \text { (A.10) }  \tag{A.10}\\
I_{\mu_{1} \mu_{2} \mu_{3}} & =\int \frac{d^{d} k_{1} d^{d} k_{2} d^{d} k_{3}}{(2 \pi)^{3 d}} \frac{V_{\mu_{1} \mu_{2} \mu_{3}}\left(-k_{1},-k_{2},-k_{3}\right)(2 \pi)^{d} \delta\left(k_{1}+k_{2}+k_{3}\right)}{\left(k_{1}^{2}+i \Delta\right)\left(k_{2}^{2}+i \Delta\right)\left(k_{3}^{2}+i \Delta\right)\left(k_{1} \nu_{1}-i \delta\right)\left(k_{2} \nu_{2}-i \delta\right)\left(k_{3} \nu_{3}-i \delta\right)} . \tag{A.11}
\end{align*}
$$

All terms from coming from the three gluon vertex can be obtained by substitutions from the integral

$$
\begin{equation*}
I_{1}^{\mu}=\int \frac{d^{d} k_{1} d^{d} k_{2} d^{d} k_{3}}{(2 \pi)^{3 d}} \frac{k_{1}^{\mu}(2 \pi)^{d} \delta\left(k_{1}+k_{2}+k_{3}\right)}{\left(k_{1}^{2}+i \Delta\right)\left(k_{2}^{2}+i \Delta\right)\left(k_{3}^{2}+i \Delta\right)\left(k_{1} \nu_{1}-i \delta\right)\left(k_{2} \nu_{2}-i \delta\right)\left(k_{3} \nu_{3}-i \delta\right)} . \tag{A.12}
\end{equation*}
$$

The index on $I^{\mu}$ refers to the which line the $k^{\mu}$ in the numerator is related to. It is convenient to use $k_{\mu}=\left.\frac{\partial}{\partial z^{\mu}} e^{k z}\right|_{z=0}$ to write the integral as

$$
\begin{equation*}
I_{1 \mu}=\left.\frac{\partial}{\partial z^{\mu}} \int \frac{d^{d} k_{1} d^{d} k_{2} d^{d} k_{3}}{(2 \pi)^{3 d}} \frac{(2 \pi)^{d} \delta\left(k_{1}+k_{2}+k_{3}\right) e^{k_{1} z}}{\left(k_{1}^{2}+i \Delta\right)\left(k_{2}^{2}+i \Delta\right)\left(k_{3}^{2}+i \Delta\right)\left(k_{1} \nu_{1}-i \delta\right)\left(k_{2} \nu_{2}-i \delta\right)\left(k_{3} \nu_{3}-i \delta\right)}\right|_{z=0} . \tag{A.13}
\end{equation*}
$$

This can be decomposed into

$$
\begin{equation*}
I_{1}^{\mu}=I_{11} \nu_{1}^{\mu}+I_{12} \nu_{2}^{\mu}+I_{13} \nu_{3}^{\mu} . \tag{A.14}
\end{equation*}
$$

Contracting the $\nu$ 's with the three-vertex shows that the diagram is proportional to

$$
\begin{array}{r}
\nu_{12} \nu_{23}\left(I_{12}-I_{32}\right)+\nu_{23} \nu_{31}\left(I_{23}-I_{13}\right)+\nu_{31} \nu_{12}\left(I_{31}-I_{21}\right) \\
=\left(\nu_{12} \nu_{23} I_{12}-\nu_{23} \nu_{31} I_{13}\right)+\left(\nu_{23} \nu_{31} I_{23}-\nu_{31} \nu_{12} I_{21}\right)+\left(\nu_{31} \nu_{12} I_{31}-\nu_{12} \nu_{23} I_{32}\right) . \tag{A.16}
\end{array}
$$

Let's manipulate $I_{1}$ into a form where we can extract $I_{12}$. All $I_{i j}$ with $i \neq j$ can be obtained from it by substitutions. We represent the Dirac delta function by $\delta(k)=$ $\int_{-\infty}^{\infty} \frac{d^{d} x}{(2 \pi)^{d}} e^{-i x k}$. Send all propagators to $\alpha$-representation, introducing 6 new parameters with integration regions from 0 to $\infty$. Completing the squares in the exponent we get

$$
\begin{align*}
& I_{1 \mu}=\frac{\partial}{\partial z^{\mu}} \int \frac{d^{d} k_{1,2,3} d^{d} x}{(2 \pi)^{d}} d \alpha_{1,2,3} d \beta_{1,2,3} \exp \{ \\
& i \alpha_{1}\left(k_{1}-\frac{x+\beta_{1} \nu_{1}-z}{2 \alpha_{1}}\right)^{2}+i \alpha_{2}\left(k_{2}-\frac{x+\beta_{2} \nu_{2}}{2 \alpha_{2}}\right)^{2}+i \alpha_{3}\left(k_{3}-\frac{x+\beta_{3} \nu_{3}}{2 \alpha_{3}}\right)^{2} \\
& -i \frac{\alpha_{123}}{4 \alpha_{1} \alpha_{2} \alpha_{3}}\left(x+\frac{\alpha_{1} \alpha_{2} \alpha_{3}}{\alpha_{123}}\left(-\frac{z}{\alpha_{1}}+\frac{\beta_{1} \nu_{1}}{\alpha_{1}}+\frac{\beta_{2} \nu_{2}}{\alpha_{2}}+\frac{\beta_{3} \nu_{3}}{\alpha_{3}}\right)\right)^{2} \\
& +i \frac{\alpha_{1} \alpha_{2} \alpha_{3}}{2 \alpha_{123}}\left(\frac{\beta_{1} \beta_{2} \nu_{12}}{\alpha_{1} \alpha_{2}}+\frac{\beta_{2} \beta_{3} \nu_{23}}{\alpha_{2} \alpha_{3}}+\frac{\beta_{3} \beta_{1} \nu_{31}}{\alpha_{3} \alpha_{1}}\right)-i \frac{z}{\alpha_{1}} \frac{\alpha_{1} \alpha_{2} \alpha_{3}}{2 \alpha_{123}}\left(-\beta_{1} \nu_{1}\left(\frac{1}{\alpha_{2}}+\frac{1}{\alpha_{3}}\right)+\frac{\beta_{2} \nu_{2}}{\alpha_{2}}+\frac{\beta_{3} \nu_{3}}{\alpha_{3}}\right) \\
& -\Delta \Sigma \alpha-\delta \Sigma \beta\}\left.\right|_{z=0} \tag{A.17}
\end{align*}
$$

where $\alpha_{123}=\alpha_{1} \alpha_{2}+\alpha_{2} \alpha_{3}+\alpha_{3} \alpha_{1}$. Terms with $z^{2}$ have been removed as they will vanish when we take the $z$ derivative. First we do the Gaussian integrals.

$$
\begin{align*}
I_{1 \mu}= & \frac{i^{d-2}}{(4 \pi)^{d}} \frac{\partial}{\partial z^{\mu}} \int d \alpha d \beta \alpha_{123}^{-d / 2} \exp \left\{i \frac{\alpha_{1} \alpha_{2} \alpha_{3}}{2 \alpha_{123}}\left(\frac{\beta_{1} \beta_{2} \nu_{12}}{\alpha_{1} \alpha_{2}}+\frac{\beta_{2} \beta_{3} \nu_{23}}{\alpha_{2} \alpha_{3}}+\frac{\beta_{3} \beta_{1} \nu_{31}}{\alpha_{3} \alpha_{1}}\right)\right. \\
& \left.-i \frac{z}{\alpha_{1}} \frac{\alpha_{1} \alpha_{2} \alpha_{3}}{2 \alpha_{123}}\left(-\beta_{1} \nu_{1}\left(\frac{1}{\alpha_{2}}+\frac{1}{\alpha_{3}}\right)+\frac{\beta_{2} \nu_{2}}{\alpha_{2}}+\frac{\beta_{3} \nu_{3}}{\alpha_{3}}\right)-\Delta \Sigma \alpha-\delta \Sigma \beta\right\}\left.\right|_{z=0} \tag{A.18}
\end{align*}
$$

Now the $z$-derivative

$$
\begin{align*}
& I_{1 \mu}=\frac{i^{d-3}}{2(4 \pi)^{d}} \int d \alpha d \beta \alpha_{123}^{-d / 2-1}\left(-\beta_{1} \nu_{1 \mu}\left(\alpha_{2}+\alpha_{3}\right)+\alpha_{3} \beta_{2} \nu_{2 \mu}+\alpha_{2} \beta_{3} \nu_{3 \mu}\right) \\
& \quad \exp \left\{\frac{i \omega}{2 \alpha_{123}}-\Delta \Sigma \alpha-\delta \Sigma \beta\right\} \tag{A.19}
\end{align*}
$$

Where $\omega=\beta_{1} \beta_{2} \nu_{12} \alpha_{3}+\beta_{2} \beta_{3} \nu_{23} \alpha_{1}+\beta_{3} \beta_{1} \nu_{31} \alpha_{2}$. From here we can read off $I_{12}$ as

$$
\begin{equation*}
I_{12}=\frac{i^{d-3}}{2(4 \pi)^{d}} \int d \alpha d \beta \alpha_{123}^{-d / 2-1} \alpha_{3} \beta_{2} \exp \left\{\frac{i \omega}{2 \alpha_{123}}-\Delta \Sigma \alpha-\delta \Sigma \beta\right\} \tag{A.20}
\end{equation*}
$$

Rescale the $\beta$ 's by $\beta_{1} \rightarrow \sqrt{\frac{\nu_{23}}{\nu_{12} \nu_{13}}} \beta_{1}$ and similar for the other $\beta$ 's. This removes all scalar products from $\omega$, call it $\omega^{\prime}$.

$$
\begin{align*}
I_{12}=\frac{i^{d-3}}{2(4 \pi)^{d}} \frac{1}{\nu_{12} \nu_{23}} & \int d \alpha d \beta \alpha_{123}^{-d / 2-1} \alpha_{3} \beta_{2} \exp \left\{\frac{i \omega^{\prime}}{2 \alpha_{123}}-\Delta \Sigma \alpha\right. \\
& \left.-\delta\left(\beta_{1}\left(\frac{\nu_{23}}{\nu_{12} \nu_{13}}\right)^{1 / 2}+\beta_{2}\left(\frac{\nu_{13}}{\nu_{21} \nu_{23}}\right)^{1 / 2}+\beta_{3}\left(\frac{\nu_{12}}{\nu_{13} \nu_{23}}\right)^{1 / 2}\right)\right\} \tag{A.21}
\end{align*}
$$

Taking the first parenthesis from equation A.16, it is proportional to

$$
\begin{align*}
\left(\nu_{12} \nu_{23} I_{12}-\nu_{23} \nu_{31} I_{13}\right) & \propto \int d \alpha d \beta \alpha_{123}^{-d / 2-1}\left(\alpha_{3} \beta_{2}-\alpha_{2} \beta_{3}\right) \exp \left\{\frac{i \omega^{\prime}}{2 \alpha_{123}}-\Delta \Sigma \alpha\right. \\
& \left.-\delta\left(\beta_{1}\left(\frac{\nu_{23}}{\nu_{12} \nu_{13}}\right)^{1 / 2}+\beta_{2}\left(\frac{\nu_{13}}{\nu_{21} \nu_{23}}\right)^{1 / 2}+\beta_{3}\left(\frac{\nu_{12}}{\nu_{13} \nu_{23}}\right)^{1 / 2}\right)\right\} \tag{A.22}
\end{align*}
$$

This would be antisymmetric (and therefore $=0$ ) under the trivial operation of $\alpha_{1} \leftrightarrow \alpha_{2}$, $\beta_{1} \leftrightarrow \beta_{2}$ if it werent for the scalar products in the exponent. This can be fixed by using a different $\delta$ for each line or by making the scalar products equal to each other. Then the diagram vanishes.

Checking numerically, the following integral seems to vanish even without rescaling, but I haven't found the right symmetry-argument.

$$
\begin{array}{r}
\text { diagram } \propto \int d \alpha d \beta \alpha_{123}^{-d / 2-1}\left(\left(\alpha_{3} \beta_{2}+\alpha_{2} \beta_{1}+\alpha_{1} \beta_{3}\right)-\left(\alpha_{2} \beta_{3}+\alpha_{1} \beta_{2}+\alpha_{3} \beta_{1}\right)\right) \exp \left\{\frac{i \omega^{\prime}}{2 \alpha_{123}}-\Delta \Sigma \alpha\right. \\
\left.-\delta\left(\beta_{1}\left(\frac{\nu_{23}}{\nu_{12} \nu_{13}}\right)^{1 / 2}+\beta_{2}\left(\frac{\nu_{13}}{\nu_{21} \nu_{23}}\right)^{1 / 2}+\beta_{3}\left(\frac{\nu_{12}}{\nu_{13} \nu_{23}}\right)^{1 / 2}\right)\right\} . \tag{A.23}
\end{array}
$$

## B Feynman rules

## B. 1 QCD

Quark propagator

$$
\begin{equation*}
i \frac{\gamma^{\mu} p_{\mu}}{p^{2}} \tag{B.1}
\end{equation*}
$$

Gluon propagator

$$
\begin{equation*}
\frac{-i \delta_{a b}}{p^{2}} g^{\mu \nu} \tag{B.2}
\end{equation*}
$$

Renormalizing to one-loop order adds the contribution $\Pi_{a b}^{\mu \nu}(p)=$

$$
\begin{align*}
& i \delta_{a b} \alpha_{s}\left(g^{\mu \nu} p^{2}-p^{\mu} p^{\nu}\right)\left[\left(C_{A}\left(\frac{5}{3}-\epsilon\right)-\frac{4}{3}(1-\epsilon) T_{f} n_{f}\right)\left(\frac{-\mu^{2}}{p^{2}}\right)^{\epsilon} \frac{\Gamma^{2}(1-\epsilon)}{\epsilon\left(1-\frac{2 \epsilon}{3}\right) \Gamma(1-2 \epsilon)}\right. \\
&\left.-\frac{1}{\epsilon}\left(\frac{5}{3} C_{A}-\frac{4}{3} T_{f} n_{f}\right)\right] \tag{B.3}
\end{align*}
$$

Ghost propagator

$$
\begin{equation*}
\frac{i \delta_{a b}}{p^{2}} \tag{B.4}
\end{equation*}
$$

Quark-gluon vertex

$$
\begin{equation*}
t^{a} i g_{0} \gamma^{\mu} \tag{B.5}
\end{equation*}
$$

Three gluon vertex

$$
\begin{equation*}
V_{\mu_{1} \mu_{2} \mu_{3}}^{a_{1} a_{2} a_{3}}\left(k_{1}, k_{2}, k_{3}\right)=-g_{0} f^{a_{1} a_{2} a_{3}}\left(\left(k_{3}-k_{2}\right)_{\mu_{1}} g_{\mu_{2} \mu_{3}}+\left(k_{1}-k_{3}\right)_{\mu_{2}} g_{\mu_{3} \mu_{1}}+\left(k_{1}-k_{3}\right)_{\mu_{2}} g_{\mu_{3} \mu_{1}}\right) \tag{B.6}
\end{equation*}
$$

## B. 2 Semi-infinite Wilson lines

Image of incoming Wilson line

$$
\begin{align*}
& \left.\frac{\delta}{\delta A\left(x_{1}\right)} \cdots \frac{\delta}{\delta A\left(x_{n}\right)} \Phi_{\nu}^{\dagger}[0, \infty]\right|_{A=0} \theta\left(x_{n}>\cdots>x_{1}\right)= \\
& \quad(-i g)^{n} t_{i k_{1}}^{a_{1}} \ldots t_{k_{n-1} j}^{a_{n}} \nu^{\mu_{1}} \ldots \nu^{\mu_{n}} \times \\
& \int_{\infty}^{0} d \tau_{1} \int_{\tau_{1}}^{0} d \tau_{2} \ldots \int_{\tau_{n-1}}^{0} d \tau_{n} \delta\left(\tau_{1} \nu-x_{1}\right) \ldots \delta\left(\tau_{n} \nu-x_{n}\right) e^{-\delta \sum_{i} \tau_{i}} \tag{B.7}
\end{align*}
$$

$\nu$ points from 0 to $\infty$.
In momentum space:

$$
\begin{equation*}
g^{n} t_{i k_{1}}^{a_{1}} \ldots t_{k_{n-1} j}^{a_{n}} \nu^{\mu_{1}} \ldots \nu^{\mu_{n}} \frac{1}{p_{1} \nu-i \delta} \cdots \frac{1}{\left(p_{1}+\cdots+p_{n}\right) \nu-i n \delta} \tag{B.8}
\end{equation*}
$$

Image of outgoing Wilson line

$$
\begin{align*}
& \left.\frac{\delta}{\delta A\left(x_{1}\right)} \cdots \frac{\delta}{\delta A\left(x_{n}\right)} \Phi_{\nu}[0, \infty]\right|_{A=0} \theta\left(x_{1}>\cdots>x_{n}\right)= \\
& \quad(-i g)^{n} t_{i k_{n-1}}^{a_{n}} \ldots t_{k_{1} j}^{a_{1}} \nu^{\mu_{n}} \ldots \nu^{\mu_{1}} \times \\
& \int_{0}^{\infty} d \tau_{n} \int_{\tau_{n}}^{\infty} d \tau_{n-1} \ldots \int_{\tau_{2}}^{\infty} d \tau_{1} \delta\left(\tau_{1} \nu-x_{1}\right) \ldots \delta\left(\tau_{n} \nu-x_{n}\right) e^{-\delta \sum_{i}} \tag{B.9}
\end{align*}
$$

In momentum space:

$$
\begin{equation*}
(-g)^{n} t_{i k_{n-1}}^{a_{n}} \cdots t_{k_{1} j}^{a_{1}} \nu^{\mu_{n}} \cdots \nu^{\mu_{1}} \frac{1}{p_{1} \nu-i \delta} \cdots \frac{1}{\left(p_{1}+\cdots+p_{n}\right) \nu-i n \delta} \tag{B.10}
\end{equation*}
$$

In generating function approach: For semi-infinite Wilson lines, the Feynman rules for $V_{1}$ and $V_{2}$ are in position space

$$
\begin{align*}
V_{a, a_{1}}^{\mu_{1}}\left(x_{1}\right) & =-i g \delta_{a a_{1}} \nu^{\mu_{1}} \theta(x 1>0)  \tag{B.11}\\
V_{a, a_{1} a_{2}}^{\mu_{1} \mu_{2}}\left(x_{1}, x_{2}\right) & =-i g^{2} f_{a a_{1} a_{2}} \nu^{\mu_{1}} \nu^{\mu_{2}}\left(\theta\left(x_{1}>x_{2}>0\right)-\theta\left(x_{2}>x_{1}>0\right)\right)
\end{align*}
$$

and in momentum space

$$
\begin{align*}
V_{\mu_{1}}^{a, a_{1}}\left(p_{1}\right) & =\delta^{a a_{1}} \nu_{\mu_{1}} \frac{-i g}{\left(p_{1} \nu-i \delta\right)} \\
V_{\mu_{1} \mu_{2}}^{a, a_{1} a_{2}}\left(p_{1}, p_{2}\right) & =f^{a a_{1} a_{2}} \nu_{\mu_{1}} \nu_{\mu_{2}} \frac{i g^{2}}{\left(p_{1}+p_{2}\right) \nu-2 i \delta}\left(\frac{1}{p_{1} \nu_{1}-i \delta}-\frac{1}{p_{2} \nu_{1}-i \delta}\right) \tag{B.12}
\end{align*}
$$

## C Dimensional reduction formula

$$
\begin{equation*}
Z[J]=\int \mathcal{D} \Phi e^{i S[\Phi]} \tag{C.1}
\end{equation*}
$$

where $S[\Phi]$ is the action of the theory, $\mathcal{D} \Phi$ tells us to integrate over all field configurations.
The time ordered expectation value of some operator is then

$$
\begin{equation*}
\langle F\rangle=\frac{1}{Z[0]} \int \mathcal{D} A F[A] e^{i S[A]} . \tag{C.2}
\end{equation*}
$$

Dimensional reduction formula.

$$
\begin{equation*}
\langle T F[A]\rangle=\frac{1}{Z[0]} \int \mathcal{D} A F[A] e^{i S[A]} \tag{C.3}
\end{equation*}
$$

Use functional derivative to shift the argument.

$$
\begin{equation*}
\langle T F[A]\rangle=\left.\frac{1}{Z[0]} \int \mathcal{D} A^{\prime} e^{i S_{0}\left[A^{\prime}\right]} e^{\int d x A_{x}^{\prime} \frac{\delta}{\delta A_{x}}} F[A] e^{i S_{i n t}[A]}\right|_{A=0}=\left.\left\langle e^{\frac{\delta}{\delta A}}\right\rangle_{0} F[A] e^{i S[A]}\right|_{A=0} \tag{C.4}
\end{equation*}
$$

The 0 on the brackets means expectation value taken in the free theory. Expanding the exponential, we get terms proportional to factors like

$$
\begin{equation*}
\int d x_{k} \ldots d x_{1} \frac{1}{k!}\left\langle A\left(x_{k}\right) \ldots A\left(x_{1}\right)\right\rangle_{0} . \tag{C.5}
\end{equation*}
$$

By Wick's theorem, the correlator is the product of Feynman propagators summed over all pairings of fields. Odd terms vanish. The integral over $x$ 's makes all pairs identical, replacing the sum over pairs by the combinatorial factor $\frac{k!}{2^{k / 2}\left(\frac{k}{2}\right)!}$. Each term then looks like

$$
\begin{equation*}
\frac{1}{\left(\frac{k}{2}\right)!}\left(\int \frac{d x d x^{\prime}}{2} \frac{\delta}{\delta A_{x}} \Delta\left(x, x^{\prime}\right) \frac{\delta}{\delta A_{x^{\prime}}}\right)^{k / 2} \tag{C.6}
\end{equation*}
$$

This is again an exponential function which gives the result

$$
\begin{equation*}
\langle T F[A]\rangle=\left.\frac{1}{Z[0]} e^{\int \frac{d x d x^{\prime}}{2} \frac{\delta}{\delta A_{x}} \Delta\left(x, x^{\prime}\right) \frac{\delta}{\delta A_{x^{\prime}}}} F[A] e^{i S_{i n t}[A]}\right|_{A=0} \tag{C.7}
\end{equation*}
$$

## D Algebra

We present here some of the formulas used in the $\mathrm{SU}(\mathrm{n})$ algebra $S U(N)$ algebra:

$$
\begin{gather*}
{\left[t^{a}, t^{b}\right]=i f^{a b c} t^{c}}  \tag{D.1}\\
t_{i k}^{a} t_{k j}^{a}=C_{F} \delta_{i j}  \tag{D.2}\\
t_{i k}^{a} t_{k j}^{b} a^{a b c}=C_{A} \delta_{i j} t^{c}  \tag{D.3}\\
f^{a b c} f^{a b d}=C_{A} \delta^{c d}  \tag{D.4}\\
C_{F}=\frac{N^{2}-1}{2 N}  \tag{D.5}\\
C_{A}=N  \tag{D.6}\\
\operatorname{tr}\left[t_{r}^{a} t_{r}^{b}\right]=C(r) \delta^{a b} d(S U(N))=N^{2}-1 \tag{D.7}
\end{gather*}
$$

## E Various formulas

Gamma, beta function:

$$
\begin{gather*}
\Gamma(z)=\int_{0}^{\infty} t^{z-1} e^{-t} d t  \tag{E.1}\\
\Gamma(z+1)=z \Gamma(z)  \tag{E.2}\\
\ln \Gamma(1+z)=-\gamma z+\sum_{k=2}^{\infty} \frac{\zeta(k)}{k}(-z)^{k}  \tag{E.3}\\
\Gamma(z)=\frac{1}{z}-\gamma z+\frac{1}{2}\left(\gamma^{2}+\frac{\pi^{2}}{6}\right) z^{2}+\ldots  \tag{E.4}\\
B(a, b)=\int_{0}^{1} x^{a-1}(1-x)^{b-1} d x=\frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \tag{E.5}
\end{gather*}
$$

$\alpha$-representation.

$$
\begin{equation*}
\frac{i^{ \pm \lambda}}{(A \pm i 0)^{\lambda}}=\int_{0}^{\infty} d \alpha \alpha^{\lambda-1} e^{ \pm i \alpha A} \tag{E.7}
\end{equation*}
$$

Gaussian integral, $k^{2}=k_{0}^{2}-\vec{k}^{2}$ :

$$
\begin{gather*}
\int_{-\infty}^{\infty} d x e^{i a x^{2}}=\sqrt{\frac{\pi}{i a}}, \operatorname{Im}[a]>0  \tag{E.8}\\
\int_{-\infty}^{\infty} \frac{d^{d} k}{(2 \pi)^{d}} e^{i a k^{2}}=\frac{1}{(4 \pi)^{d / 2}}(i a)^{-1 / 2}(-i a)^{-(d-1) / 2}=\frac{i^{d / 2-1}}{(4 \pi)^{d / 2}} a^{-d / 2} \tag{E.9}
\end{gather*}
$$

Feynman propagator:

$$
\begin{equation*}
\Delta_{a b}^{\mu \nu}(x-y)=\int_{\infty}^{\infty} \frac{d^{d} p}{(2 \pi)^{d}} \frac{i}{p^{2}+i 0} e^{-i p(x-y)}=\frac{\Gamma(1-\epsilon)}{4 \pi^{2-\epsilon}} \frac{g^{\mu \nu} \delta_{a b}}{\left(-(x-y)^{2}+i 0\right)^{1-\epsilon}} \tag{E.10}
\end{equation*}
$$

## References

[1] S. Mandelstam, "Quantum electrodynamics without potentials," Annals of Physics, vol. 19, pp. 1-24, July 1962.
[2] R. A. Brandt, A. Gocksch, M. A. Sato, and F. Neri, "Loop space," Physical Review D, vol. 26, pp. 3611-3640, Dec. 1982.
[3] R. Brandt, A. Gocksch, M.-A. Sato, and F. Neri, "Loop space," Physical Review D, vol. 26, no. 12, pp. 3611-3640, 1982.
[4] A. M. Polyakov, "Gauge fields as rings of glue," Nuclear Physics B, vol. 164, pp. 171188, 1980.
[5] Y. M. Makeenko and A. A. Migdal, "Quantum chromodynamics as dynamics of loops," Nuclear Physics B, vol. 188, pp. 269-316, Sept. 1981.
[6] K. G. Wilson, "Confinement of quarks," Physical Review D, vol. 10, pp. 2445-2459, Oct. 1974.
[7] J. Collins, "Foundations of perturbative QCD," 2011.
[8] J. C. Collins, D. E. Soper, and G. Sterman, "Factorization of Hard Processes in QCD," arXiv:hep-ph/0409313, Sept. 2004. arXiv: hep-ph/0409313.
[9] J. C. Collins, D. E. Soper, and G. F. Sterman, "Transverse Momentum Distribution in Drell-Yan Pair and W and Z Boson Production," Nucl.Phys., vol. B250, p. 199, 1985.
[10] M. E. Peskin and D. V. Schroeder, An Introduction To Quantum Field Theory. Reading, Mass: Westview Press, first edition edition ed., Oct. 1995.
[11] M. D. Schwartz, Quantum Field Theory and the Standard Model. New York: Cambridge University Press, 1 edition ed., Dec. 2013.
[12] G. P. Korchemsky and A. V. Radyushkin, "Loop-space formalism and renormalization group for the infrared asymptotics of QCD," Physics Letters B, vol. 171, pp. 459-467, May 1986.
[13] V. S. Dotsenko and S. N. Vergeles, "Renormalizability of phase factors in non-abelian gauge theory," Nuclear Physics B, vol. 169, pp. 527-546, Aug. 1980.
[14] R. A. Brandt, F. Neri, and M.-a. Sato, "Renormalization of Loop Functions for All Loops," Phys.Rev., vol. D24, p. 879, 1981.
[15] I. A. Korchemskaya and G. P. Korchemsky, "On light-like Wilson loops," Physics Letters B, vol. 287, pp. 169-175, Aug. 1992.
[16] A. Grozin, J. M. Henn, G. P. Korchemsky, and P. Marquard, "The three-loop cusp anomalous dimension in QCD," Physical Review Letters, vol. 114, Feb. 2015. arXiv: 1409.0023.
[17] E. Gardi, "Progress on soft gluon exponentiation and long-distance singularities," arXiv:1401.0139 [hep-ph, physics:hep-th], Dec. 2013. arXiv: 1401.0139.
[18] L. Magnea, "Progress on the infrared structure of multi-particle gauge theory amplitudes," arXiv:1408.0682 [hep-ph], Aug. 2014. arXiv: 1408.0682.
[19] A. A. Vladimirov, "Generating function for web diagrams," Physical Review D, vol. 90, Sept. 2014. arXiv: 1406.6253.
[20] A. A. Vladimirov, "Exponentiation for products of Wilson lines within the generating function approach," arXiv:1501.03316 [hep-ph, physics:hep-th], Jan. 2015. arXiv: 1501.03316.
[21] T. Becher and M. Neubert, "Infrared singularities of scattering amplitudes in perturbative QCD," Physical Review Letters, vol. 102, Apr. 2009. arXiv: 0901.0722.
[22] S. M. Aybat, L. J. Dixon, and G. Sterman, "The Two-loop Anomalous Dimension Matrix for Soft Gluon Exchange," Physical Review Letters, vol. 97, Aug. 2006. arXiv: hep-ph/0606254.
[23] S. M. Aybat, L. J. Dixon, and G. Sterman, "The Two-loop Anomalous Dimension Matrix for Soft Gluon Exchange," Physical Review Letters, vol. 97, Aug. 2006. arXiv: hep-ph/0606254.
[24] G. P. Korchemsky and A. V. Radyushkin, "Renormalization of the Wilson loops beyond the leading order," Nuclear Physics B, vol. 283, pp. 342-364, 1987.
[25] E. Gardi, J. M. Smillie, and C. D. White, "On the renormalization of multiparton webs," arXiv:1108.1357 [hep-ph, physics:hep-th], Aug. 2011. arXiv: 1108.1357.
[26] G. F. Sterman, "Infrared Divergences in Perturbative \{QCD\}. (Talk)," AIP Conf.Proc., vol. 74, pp. 22-40, 1981.
[27] J. G. M. Gatheral, "Exponentiation of eikonal cross sections in nonabelian gauge theories," Physics Letters B, vol. 133, pp. 90-94, Dec. 1983.
[28] J. Frenkel and J. C. Taylor, "Non-abelian eikonal exponentiation," Nuclear Physics $B$, vol. 246, pp. 231-245, Nov. 1984.
[29] V. A. Smirnov, "Evaluating multiloop Feynman integrals by Mellin-Barnes representation," arXiv:hep-ph/0406052, June 2004. arXiv: hep-ph/0406052.

