Chiral Perturbation Theory for Neutron-antineutron Oscillations

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Abstract

We construct the Chiral Perturbation Theory operators for neutron-antineutron oscillations and use these to estimate chiral and finite volume corrections at one-loop order.

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1 Introduction

The baryon asymmetry of the universe is one of the open problems in particle physics. One possible solution is to have $B - L$ violation as exemplified in $\Delta B = 2$ transitions and in particular neutron-antineutron oscillations. This has been suggested long ago, see e.g. [1, 2, 3, 4, 5]. Recent reviews are [6, 7]. $\Delta B = 2$ transitions require a six-quark operator. These were classified in [8, 9, 10]. To obtain predictions of a particular model the coefficients of these operators need to be evolved to a low scale and then the matrix elements computed. This running is known to two-loop order [11]. We will also use the notation of the operators used in that reference. In the past these matrix elements were estimated using models but now the first lattice calculations have appeared [12, 13]. These can be done at different quark masses from the physical ones and are necessarily at finite volume. Chiral Perturbation Theory (ChPT) allows to do estimate both of these effects.

The bounds on the mean oscillation time τ are $\tau > 8.6\ 10^7$ s from free neutrons [14] and $\tau > 2.7\ 10^8$ s from bound neutrons [15]. The reason why the bound from bound neutrons is much lower than those for proton decay is that the antineutron inside nuclei is far off-shell, see e.g. [16] for a clear explanation. For the same reason, strong magnetic shielding is needed for the free neutron experiments. A new free neutron experiment is proposed for ESS in Lund [17] so a better estimate of the matrix elements will be very useful to put limits on $\Delta B = 2$ effects in theories beyond the Standard Model.

In this paper we construct the ChPT equivalents of the six-quark operators of [11] and use these then to calculate the chiral and finite volume corrections in the isospin limit. The finite volume corrections are found to be small for $m_\pi L > 4$ for the physical pion mass but chiral extrapolations can be substantial already for pion masses of order 200 MeV.

In Sect. 2 we discuss shortly the quark operators of [11] and their chiral representation. Sect. 3 discusses the ChPT aspects. The main new result is the construction of the ChPT operators for neutron-antineutron transitions. This is done using the spurion technique. In Sect. 4 we calculate the one-loop corrections in ChPT to the matrix elements and in Sect. 5 we give some numerical results. Our main conclusions are given in Sect. 6. App. A recalls some $SU(2)$ identities used heavily in deriving the ChPT operators and the needed integrals are discussed in App. B.

Preliminary results of this work were presented in the master thesis [18] and at Lattice 2017 [19].

2 Quark operators and chiral properties

The operator structure needed for $n\bar{n}$-transitions contains six quark fields $dddduu$ where under the chiral symmetry group $SU(2)_L \times SU(2)_r$ each quark field can be in a left- or right-handed doublet. The operators were classified in [8, 9, 10] and rewritten in a basis that shows the chiral properties in [11]. It was found that there are 14 operators that have six types of representations under the chiral group. There are three $(1_L, 3_R)$, one $(1_L, 7_R)$ and three $(5_L, 3_R)$ operators, as well as their parity conjugates. The chiral loop corrections

Chiral	#operators	Chiral		Spurion	#operators
$(3_L, 1_R)$	3: P_1, P_2, P_3	$\theta_i^{i_L j_L}$ $(i = 1, 2, 3)$		$(1_L, 3_R)$	3: Q_1, Q_2, Q_3
$(3_L, 5_R)$	3: P_5, P_6, P_7	$\theta_i^{i_L j_L k_R l_R m_R n_R}$ $(i = 4, 5, 6)$		$(3_R, 5_L)$	3: Q_5, Q_6, Q_7
$(7_L, 1_R)$	1: P_4	$\theta_4^{i_L j_L k_L l_L m_L n_L}$		$(1_L, 7_R)$	1: Q_4

Table 1: The chiral representations of the dimension-9 six-quark operators as listed in [11] as well as the corresponding spurions. The indices on the spurions are $SU(2)_L \times SU(2)_R$ upper doublet, fully symmetrized in the indices of the same type.

for the parity-conjugates are the same since the strong interactions are invariant under parity.

If we assume isospin conservation, only an $I = 1$ operator can contribute to $n\bar{n}$-transitions. So only the $I = 1$ projection of the different $(5_L, 3_R)$ and $(3_L, 5_R)$ operators contributes, this explains why the loop contributions for all those operators are the same, in fact one can show that the operators P_5, P_6, P_7 (and similarly Q_5, Q_6, Q_7) are related by isospin. The $(1_L, 7_R)$ and $(7_L, 1_R)$ operators do not contribute in the isospin limit. The operators are summarized in Tab. 1.

We can add spurion fields transforming under $G_\chi = SU(2)_L \times SU(2)_R$ such that the combination of quark-operators with chiral flavour indices and the spurions is invariant under G_χ. These will be used to construct the operators in ChPT. There is a corresponding set for the opposite parity operators Q_i.

3   Chiral perturbation theory

We work in two-flavour ChPT and we use the heavy-baryon formalism [20] (HBCHPT), a review and introduction is [21]. The notation we use can be found in [21] or [22]. The lowest order meson Lagrangian is

$$\mathcal{L}_2 = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle \,, \qquad u_\mu = i \left[u^\dagger (\partial_\mu - i r_\mu) u - u (\partial_\mu - i l_\mu) u^\dagger \right] \,,$$

$$\chi = 2B \left(s + ip \right) \,, \qquad \Gamma_\mu = \frac{1}{2} \left[u^\dagger (\partial_\mu - i r_\mu) u - u (\partial_\mu - i l_\mu) u^\dagger \right] \,,$$

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u \,, \qquad \langle A \rangle \equiv \mathrm{tr}(A) \,. \tag{1}$$

u is a 2×2 unitary matrix that contains the pion fields π^a via $u = \exp(\pi^a \tau^a / (2F))$, with τ^a the Pauli matrices. B, F are the two lowest-order (LO) low-energy constants (LECs). The 2×2 matrices s, p, l_μ, r_μ are the usual ChPT external fields. The last one defines U.

Under a chiral transformation g_L, g_R the objects above transform as

$$u \to g_L u h^\dagger \equiv h u g_L^\dagger \,, \qquad u_\mu \to h u_\mu h^\dagger \,, \qquad \chi \to g_R \chi g_L^\dagger \,,$$

$$\chi_\pm \to h \chi_\pm h^\dagger \,, \qquad U = u^2 \to g_R U g_L^\dagger \,. \tag{2}$$

The first equation is the definition of the compensator transformation h which depends on u, g_L, g_R.
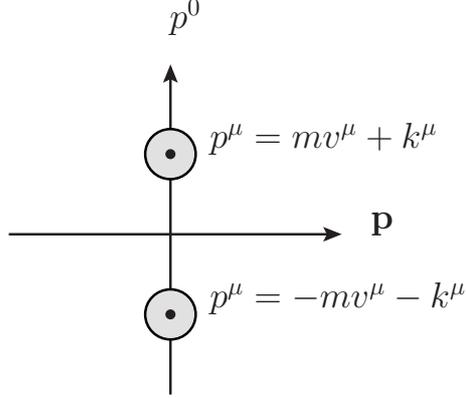
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Figure 1: A pictorial representation of the velocity regions relevant for projection on the nucleon and antinucleon HBCHPT fields.

Nucleons in a relativistic normalization can be included via a doublet field Ψ at LO as [23]

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad\qquad \psi \to h\Psi,$$

$$\mathcal{L}_R = \overline{\Psi} \left(iD_\mu\gamma^\mu - m + \frac{g_A}{2}u_\mu\gamma^\mu\gamma_5 \right)\Psi, \qquad\qquad D_\mu \equiv \partial_\mu + \Gamma_\mu. \qquad (3)$$

In HBCHPT we project on velocity-dependent fields $\mathcal{N}$ via

$$\mathcal{N} = (1/2)(1 + v_\mu\gamma^\mu)\exp(imv\cdot x)\Psi, \qquad\qquad (4)$$

with v a four-velocity with $v^2 = 1$. However, in this paper we need to introduce also an antinucleon field with the same velocity v. The charge conjugate fermion spinor is $\psi^c \equiv -i\gamma^2\psi^*$. We then define

$$\Psi^c \equiv i\tau^2 \begin{pmatrix} p^c \\ n^c \end{pmatrix} = \begin{pmatrix} n^c \\ -p^c \end{pmatrix}, \qquad\qquad \Psi^c \to h\Psi^c. \qquad (5)$$

The transformation under the chiral group follows from the properties of $SU(2)$ using the identities in App. A. We then define a HBCHPT field for the antineutron as

$$\mathcal{N}^c = (1/2)(1 + v_\mu\gamma^\mu)\exp(imv\cdot x)\Psi^c. \qquad\qquad (6)$$

Compared to the first projection (4), this is at $-v$ if formulated in terms of Ψ. $\mathcal{N}$ and $\mathcal{N}^c$ are in HBCHPT independent fields, since they are from expansions around different widely-separated velocities as depicted in Fig. 1. The lowest order Lagrangian for the HBCHPT fields is

$$\mathcal{L}_N = \overline{\mathcal{N}}\left(iv^\mu D_\mu + g_A u^\mu S_\mu\right)\mathcal{N} + \overline{\mathcal{N}^c}\left(iv^\mu D_\mu - g_A u^\mu S_\mu\right)\mathcal{N}^c. \qquad (7)$$

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The signs can be derived using charge conjugation. The spin vector S_μ has the properties

$$S_\mu = -\frac{1}{4}\gamma_5 \left[\gamma_\mu, \gamma_\nu\right] v^\nu\,, \quad S^2 = \frac{1-d}{4}\,, \quad \{S_\mu, S_\nu\} = \frac{1}{2}\left(v_\mu v_\nu - g_{\mu\nu}\right)\,, \quad v \cdot S = 0\,. \quad (8)$$

These properties are sufficient for our calculation. Higher order Lagrangians can be constructed in the same way as usual.

The operators that give neutron-antineutron transitions have to be written with doublet indices and must create the antineutron. For this we introduce[1]

$$\widetilde{\mathcal{N}^c} = \left(\begin{array}{c} \overline{p^c} \\ \overline{n^c} \end{array} \right) = -i\tau^2 \overline{\mathcal{N}^c}^T\,, \qquad\qquad \widetilde{\mathcal{N}^c} \to h\widetilde{\mathcal{N}^c}\,. \quad (9)$$

We need to construct operators that transform with left- or right-handed doublet indices under $SU(2)_L \times SU(2)_R$. These can then be contracted with the spurion operators given in Tab. 1 to make invariant quantities.

To be precise, a lower index on an object x_{i_L} leads to the transformation $x_{i_L} \to \sum_{j_L}(g_L)_{i_L}^{j_L} x_{j_L}$ and equivalently for a right-handed lower index. Some examples of objects with the corresponding indices are:

$$\left(Ui\tau^2\right)_{i_R j_L}\,, \quad \left(u\mathcal{N}\right)_{i_R}\,, \quad \left(u^\dagger\mathcal{N}\right)_{i_L}\,, \quad \left(u\widetilde{\mathcal{N}^c}\right)_{i_R}\,, \quad \left(u^\dagger\widetilde{\mathcal{N}^c}\right)_{i_L}\,, \quad \left(u^\dagger u_\mu u i\tau_2\right)_{i_L j_L}\,. \quad (10)$$

To get a neutron to antineutron transition we need an $\widetilde{\mathcal{N}^c}$ and a $\mathcal{N}$ field. Dirac (or fermion) indices are contracted between these.

The lowest order, p^0, operators are

$$(3_L, 1_R) : R_{i_L j_L} = \left(u^\dagger\widetilde{\mathcal{N}^c}\right)_{i_L} \left(u^\dagger\mathcal{N}\right)_{j_L}$$

$$(3_L, 5_R) : R_{i_L j_L k_R l_R m_R n_R} = \left(u\widetilde{\mathcal{N}^c}\right)_{k_R} \left(u\mathcal{N}\right)_{l_R} \left(Ui\tau^2\right)_{m_R i_L} \left(Ui\tau^2\right)_{n_R j_L}$$

$$(7_L, 1_R) : \text{---} \quad (11)$$

and the parity-conjugates. There is no lowest order operator for $(7_L, 1_R)$. The first operator that appears for $(7_L, 1_R)$ is at order p^2:

$$(7_L, 1_R), p^2 : \left(u^\dagger\widetilde{\mathcal{N}^c}\right)_{i_L} \left(u^\dagger\mathcal{N}\right)_{j_L} \left(u^\dagger u_\mu u i\tau_2\right)_{k_L l_L} \left(u^\dagger u_\mu u i\tau_2\right)_{m_L n_L} \quad (12)$$

At higher orders there are very many operators. A partial list can be found in [18]. We will restrict ourselves to comments sufficient for the application to neutron-antineutron transitions. The relevant independent combinations we refer to as δ_i below.

At order p, the operators must contain a derivative D_μ or u_μ. As such, they will contain either dependence on the neutron or antineutron four momentum, or contain an extra pion. For a neutron-antineutron transition at rest the HBCHPT momentum k_μ vanishes. There is

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[1]In this equation, the fields p^c, n^c are the HBCHPT ones for the antinucleons, not the relativistic fields.

thus no tree level contribution to neutron-antineutron transitions. Loop level contributions from these operators will start at p^3, which is beyond what is considered in this paper.

At order p^2 there are very many operators that contribute, a rather extensive list is in [18]. Two examples are

$$\left(u^\dagger D_\mu \widetilde{\mathcal{N}^c}\right)_{i_L} \left(u^\dagger D^\mu \mathcal{N}\right)_{j_L}, \quad \left(u^\dagger \widetilde{\mathcal{N}^c}\right)_{i_L} \left(\chi^\dagger u \mathcal{N}\right)_{j_L}. \tag{13}$$

For this paper it is sufficient to notice that there is a free parameter at order p^2 associated with each operator.

How many parameters do we need to order p^2 to describe neutron-antineutron transitions given the operators $P_1, \ldots, P_7$ with a given coefficient? The operators P_1, P_2, P_3 are all $(3_L, 1_R)$, however the quark-operators are not related by a chiral transformation. This leads to three free parameters at order p^0 and three more at order p^2. The three operators P_5, P_6, P_7 belong to same chiral multiplet, i.e. they are related via a chiral transformation. This leads to one parameter at p^0 and one more at p^2. The $(7:L, 1_R)$ operator at order p^2 does not contribute to neutron-antineutron transitions.

The values to which the spurions need to be set to reproduce the quark level operators can be derived from the expressions in [11]. They are (1 corresponds to an up-quark, 2 to a down-quark):

$$\theta_1^{ij} = \theta_2^{ij} = \theta_3^{ij} = \delta_2^i \delta_2^j,$$
$$\theta_5^{ijklmn} = \delta_1^i \delta_1^j \delta_2^k \delta_2^l \delta_2^m \delta_2^n,$$
$$\theta_6^{ijklmn} = \frac{1}{2\sqrt{2}} \left(\delta_1^i \delta_2^j + \delta_2^i \delta_1^j\right) \left(\delta_1^k \delta_2^l \delta_2^m \delta_2^n + \delta_2^k \delta_1^l \delta_2^m \delta_2^n + \delta_2^k \delta_2^l \delta_1^m \delta_2^n + \delta_2^k \delta_2^l \delta_2^m \delta_1^n\right),$$
$$\theta_7^{ijklmn} = \frac{1}{\sqrt{6}} \delta_2^i \delta_2^j \left(\delta_1^k \delta_1^l \delta_2^m \delta_2^n + \delta_1^k \delta_2^l \delta_1^m \delta_2^n + \delta_1^k \delta_2^l \delta_2^m \delta_1^n + \delta_2^k \delta_1^l \delta_1^m \delta_2^n + \delta_2^k \delta_1^l \delta_2^m \delta_1^n + \delta_2^k \delta_2^l \delta_1^m \delta_1^n\right).$$
$$\tag{14}$$

Note that these are normalized to 1, slightly different from [11].

To summarize the neutron-antineutron part. If the Lagrangian at the quark-level is of the form

$$\sum_{i=1,7} \alpha_i P_i \tag{15}$$

then the LO ChPT Lagrangian has the form

$$\mathcal{L}_{n\bar{n}} = \left(\beta_1 \alpha_1 + \beta_2 \alpha_2 + \beta_3 \alpha_3\right) \theta_1^{i_L j_L} R_{i_L j_L}$$
$$+ \beta_5 \left(\alpha_5 \theta_5^{i_L j_L k_R l_R m_R n_R} + \alpha_6 \theta_6^{i_L j_L k_R l_R m_R n_R} + \alpha_7 \theta_7^{i_L j_L k_R l_R m_R n_R}\right) R_{i_L j_L k_R l_R m_R n_R} \tag{16}$$

with the spurions as defined in (14) and the operators in (11). The α_i are short-distance parameters while the β_i are long-distance parameters. The parity-conjugate operators can be included similarly.
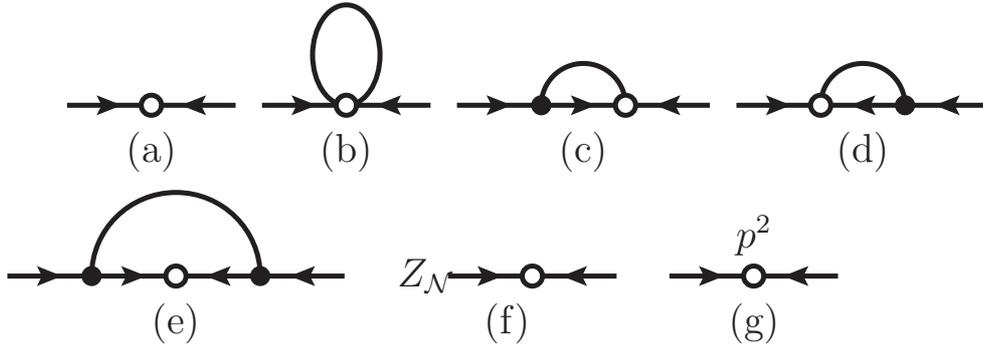
Figure 2: The diagrams for $n\bar{n}$ transitions to order p^2. An open dot indicates a vertex from the $n\bar{n}$ Lagrangian (16), a dot from the LO normal Lagrangian (7). The contributions from wave-function renormalization are indicated schematically in(f) and from the p^2 $n\bar{n}$-Lagrangian in (g). A right-pointing line is a neutron, a left-pointing line an antineutron.

4 Analytical results

The diagrams needed for $n\bar{n}$ transition to order p^2 are shown in Fig. 2.

The LO, p^0, result from Fig. 2(a) is

$$A(n \to \bar{n})_{LO} = \beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3 + \beta_5\left(\alpha_5 - \frac{\alpha_6}{\sqrt{2}} + \frac{\alpha_7}{\sqrt{6}}\right). \qquad (17)$$

The integrals we use are defined in App. B. The tadpole diagram of Fig. 2(b) contributes

$$A(n \to \bar{n})_{(b)} = \frac{1}{F^2}A(m_\pi^2)\left[(\beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3) + 7\beta_5\left(\alpha_5 - \frac{\alpha_6}{\sqrt{2}} + \frac{\alpha_7}{\sqrt{6}}\right)\right]. \qquad (18)$$

The diagrams (c) and (d) contain the integral

$$\frac{1}{i}\int \frac{d^d r}{(2\pi)^d} \frac{S \cdot r}{(r^2 - m_\pi^2)(v \cdot (r + k))}. \qquad (19)$$

We work in the frame where the external momentum k vanishes. In infinite volume the integral is proportional to $v \cdot S = 0$. In finite volume for a neutron and antineutron at rest, S is purely spatial, and the integral/sum is odd under $\vec{r} \to -\vec{r}$ and vanishes for periodic boundary conditions. So (c) and (d) give no contribution.

Diagram (e) can be rewritten in terms of the integral

$$I(m_\pi^2) = \frac{1}{i}\int \frac{d^d r}{(2\pi)^2} \frac{(S \cdot r)^2}{(r^2 - m_\pi^2)(v \cdot r)^2}. \qquad (20)$$

The central vertex is directly the LO contribution so (f) contributes

$$A(n \to \bar{n})_{(f)} = -\frac{g_A^2}{F^2}I(m_\pi^2)A(n \to \bar{n})_{LO}. \qquad (21)$$

6

Wave-function renormalization can be computed from the derivative of the nucleon (and antinucleon) selfenergy. This leads again to the occurrence of the integral $I(m_\pi)^2$ in this contribution. We get

$$A(n \to \bar{n})_{(f)} = \frac{3g_A^2}{F^2} I(m_\pi^2) A(n \to \bar{n})_{LO} \,. \tag{22}$$

Depending on the form of p^3 Lagrangian in the pion nucleon sector chosen, we have a contribution proportional to m_π^2 and a possible p^3 pion-nucleon LEC. This is nonzero if choosing the Lagrangian in [21] and vanishes if the version of [22] is chosen. The two choices are related by a field redefinition. The effect is that the p^2 $n\bar{n}$ LECs (referred to as δ_i below) have different values in the two cases but such that the total result remains the same.

The final result is

$$\begin{aligned}
A(n \to \bar{n}) = {}& (\beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3)\left[1 + \frac{1}{F^2}\left(A(m_\pi^2) + 2g_A^2 I(m_\pi^2)\right)\right] \\
& + \beta_5\left(\alpha_5 - \frac{\alpha_6}{\sqrt{2}} + \frac{\alpha_7}{\sqrt{6}}\right)\left[1 + \frac{1}{F^2}\left(7A(m_\pi^2) + 2g_A^2 I(m_\pi^2)\right)\right] \\
& + m_\pi^2(\delta_1\alpha_1 + \delta_2\alpha_2 + \delta_3\alpha_3) + m_\pi^2\delta_5\left(\alpha_5 - \frac{\alpha_6}{\sqrt{2}} + \frac{\alpha_7}{\sqrt{6}}\right) \,.
\end{aligned} \tag{23}$$

In order to get the infinite volume finite result, replace the δ_i by their finite parts δ_i^r and the integrals I, A by $\overline{I}, \overline{A}$. The finite volume correction is obtained by dropping terms not involving an integral and replacing I, V by I^V, A^V. Expressions for these integrals are in App. B.

5 Numerical results

We set in this section all p^2 LECs, δ_i^r, to zero.

The relative chiral correction from the loops to $(3_L, 1_R)$ $(D_1))$ and $(3_L, 5_R)$ (D_5) operators is given by keeping the I, A terms in (23) and replacing them by $\overline{I}, \overline{A}$. The result is

$$\begin{aligned}
D_1 &= \frac{m_\pi^2}{16\pi^2 F^2}\left[\left(-1 - \frac{3g_A^2}{2}\right)\log\frac{m_\pi^2}{\mu^2} - g_A^2\right] \,, \\
D_5 &= \frac{m_\pi^2}{16\pi^2 F^2}\left[\left(-7 - \frac{3g_A^2}{2}\right)\log\frac{m_\pi^2}{\mu^2} - g_A^2\right] \,.
\end{aligned} \tag{24}$$

These are plotted in Fig. 3(a) for a range of m_π^2 with $F = 92.2$ MeV fixed and $g_A = 1.25$. Note that they are large for the $(3_L, 5_R)$ operators alread at $m_\pi \approx 200$ MeV.
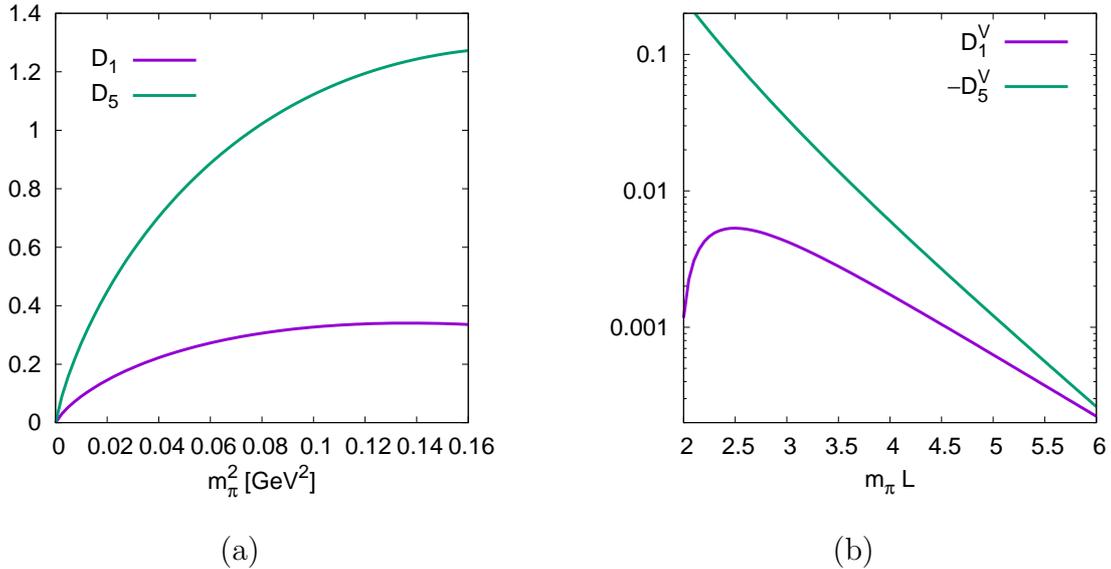
Figure 3: The numerical results of the pure loop contributions. (a) The infinite volume correction of (24) (b) The finite volume correction of (25).

The correction due to finite volume is obtained by replacing I, A by I^V, A^V in (23):

$$
\begin{aligned}
D_1^V &= \frac{1}{F^2}\left[\left(1+\frac{g_A^2}{2}\right)A^V(m_\pi^2,1)+m_\pi^2 g_A^2 A^V(m_\pi^2,2)\right], \\
D_5^V &= \frac{1}{F^2}\left[\left(7+\frac{g_A^2}{2}\right)A^V(m_\pi^2,1)+m_\pi^2 g_A^2 A^V(m_\pi^2,2)\right].
\end{aligned} \tag{25}
$$

These are plotted in Fig. 3(b) for $m_\pi = 135$ MeV and $F = 92.2$ MeV as a function of $m_\pi L$. D_5^V is negative over the whole region while D_1^V is positive. D_1^V goes through zero just below the region plotted. The finite volume corrections are small for $m_\pi L > 4$.

6 Conclusions

In this paper we have constructed ChPT operators for the dimension 9 six-quark operators that contribute to neutron-antineutron oscillations. At order p^0 there is one term each transforming as $(3_L, 1_R)$ and $(3_L, 5_R)$. The $(7_L, 1_R)$ operators only contribute at order p^3 by power-counting but do require isospin violation. We showed that the order p operators only contribute from order p^3. There is a large number of operators contributing at order p^2, a partially complete list can be found in [18]. The same is true for the parity-conjugate operators.

Our main results are the one-loop corrections in (23), (24) and (25). We have shown numerical results. The finite volume corrections are small for $m_\pi L > 4$. We found that chiral corrections are reasonable for the $(3_L, 1_R)$ operators but can be sizable for the $(3_L, 5_R)$ operators.

8

Acknowledgements

A Group theory

$SU(2)$ is a pseudoreal group with as generators $T^a = (1/2)\tau^a$. The Pauli matrices τ^a are Hermitian and satisfy

$$\tau^{aT} = \tau^{a*} = -\tau^2 \tau^a \tau^2 \,. \tag{26}$$

As a consequence the special unitary matrices $x = g_L, g_R, u, h$ all satisfy

$$\tau^2 x \tau^2 = x^* \,, \qquad\qquad \tau^2 x^T \tau^2 = x^\dagger \,. \tag{27}$$

These identities are used a lot in the construction of the transformations and operators in the main text.

B Integrals

The integrals we need to calculate both at infinite and finite volume. In finite volume we replace the integral over spatial momenta by a sum. The techniques are well known both at finite and infinite volume, we use here [21] and [24].

The mesonic integral/sum needed is

$$A(m^2) = \frac{1}{i} \int \frac{d^d r}{(2\pi)^d} \frac{1}{r^2 - m_\pi^2} = \frac{\lambda_0}{16\pi^2} + \overline{A}(m^2) + A^V(m^2, 1) \,, \tag{28}$$

with

$$\lambda_0 = \frac{1}{\epsilon} + 1 + \log(4\pi) - \gamma_E \,, \qquad\qquad d = 4 - 2\epsilon \,. \tag{29}$$

The terms with λ_0 are removed by the renormalization procedure and the logarithms of m_π^2 obtain the subtraction scale μ^2 via the renormalization as well. We therefore quote the integrals including μ^2. The finite volume part depends on the spatial length scale L. The results are, see e.g. [24],

$$\overline{A}(m_\pi^2) = -\frac{m_\pi^2}{16\pi^2} \log \frac{m_\pi^2}{\mu^2} \,,$$

$$A^V(m^2, n) = \frac{(-1)^n}{16\pi^2} \left(\frac{L^2}{4}\right)^{n-2} \int \frac{d\lambda}{\Gamma(n)} \lambda^{n-3} e^{-\lambda m^2 L^2/4} \left[\theta_3 \left(e^{-1/\lambda}\right)^3 - 1\right] \,,$$

$$\theta_3 = \sum_{n=-\infty,\infty} q^{(n^2)} \,. \tag{30}$$

The other integral/sum needed is

$$I(m_\pi^2) = \frac{1}{i} \int \frac{d^d r}{(2\pi)^d} \frac{(S \cdot r)^2}{(r^2 - m_\pi^2)(v \cdot r)^2} \,. \tag{31}$$

The numerator can be rewritten via

$$(S \cdot r)^2 = \frac{1}{2} r^\mu r_\nu \{S_\mu, S_\nu\} = \frac{1}{4} \left[(v \cdot r)^2 - r^2 \right] = \frac{1}{4} \left[(v \cdot r)^2 - (r^2 - m_\pi^2) - m_\pi^2 \right] \,. \tag{32}$$

The second term leads to an integral with only $v \cdot r$ in the denominator. These vanish both at infinite and finite volume. We thus get

$$I(m_\pi^2) = \frac{1}{4} A(m_\pi^2) - \frac{m_\pi^2}{4} \frac{1}{i} \int \frac{d^d r}{(2\pi)^d} \frac{1}{(r^2 - m_\pi^2)(v \cdot r)^2} \tag{33}$$

We combine the propagators in the second term with $1/(ab^2) = \int_0^\infty d\lambda \, 8\lambda/(a + 2b\lambda)^3$ and shift the momentum to^2 $\tilde{r} = r + v\lambda$ and obtain the integral

$$\frac{1}{i} \int \frac{d^d \tilde{r}}{(2\pi)^d} \int_0^\infty d\lambda \, 8\lambda \frac{1}{\left(r^2 - (m_\pi^2 + \lambda^2)\right)^3} = -\frac{1}{i} \int \frac{d^d \tilde{r}}{(2\pi)^d} \frac{2}{(\tilde{r}^2 - m_\pi^2)^2} \,. \tag{34}$$

In the second step we have done the λ-integral. The integral/sum appearing now is known and we obtain

$$I(m_\pi^2) = \frac{\lambda_0}{16\pi^2} \frac{3m_\pi^2}{4} + \overline{I}(m_\pi^2) + I^V(m_\pi^2) \,,$$
$$\overline{I}(m_\pi^2) = \frac{m_\pi^2}{16\pi^2} \left(-\frac{3}{4} \log \frac{m_\pi^2}{\mu^2} - \frac{1}{2} \right) \,,$$
$$I^V(m_\pi^2) = \frac{1}{4} A^V(m_\pi^2, 1) + \frac{m_\pi^2}{2} A^V(m_\pi^2, 2) \,. \tag{35}$$

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^{2}This also works at finite volume since v is in the temporal direction there.

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