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# An analytic representation of $F_K/F_\pi$

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We present an analytic representation of $F_K/F_\pi$ as calculated in three-flavoured two-loop chiral perturbation theory, and use it to extract values of the low energy constants $L_5^r$, $C_{14}^r + C_{15}^r$ and $C_{15}^r + 2C_{17}^r$ by means of fitting with data from lattice simulations. Although for the calculation of the two-loop multi-scale integrals involved we have derived exact results using Mellin-Barnes theory, the representation presented in this letter is, for practical purposes, an approximation whose accuracy may be improved to any desired level without a significant increase in its complexity.

**Introduction-** The spectrum of QCD contains as lightest particles the pseudo-scalar octet, and their properties provide a delicate test of its non-perturbative features, including that of chiral symmetry breaking in the sector involving the three lightest quarks. Of these, a special place is accorded to the decay constants of the kaon and pion, namely $F_K$ and $F_\pi$. Their ratio has been investigated on the lattice now, including at realistic quark masses [1]. On the other hand, in chiral perturbation theory (ChPT) [2] at two-loops, expressions have been available for nearly two decades, but involving certain integrals (sunsets) that are evaluated numerically [3]. In this work, we provide an analytic expression for $F_K/F_\pi$, which among other things incorporates double series derived using Mellin-Barnes (MB) representations of the sunsets. This allows us to produce a template for easy fitting to lattice simulations. We also present a first such fit.

**Methodology-** Three-flavoured ChPT expressions for the decay constants of the pseudoscalar mesons at two-loops are given in [3]. These may be decomposed as:

$$\frac{F_P}{F_0} = 1 + F_P^{(4)} + (F_P)^{(6)}_{CT} + (F_P)^{(6)}_{loop} + \mathcal{O}(p^8), \quad (1)$$

where $P$ is the particle in question. The $\mathcal{O}(p^6)$ contribution can be subdivided as:

$$F_\pi^4\,(F_P)^{(6)}_{loop} = d^P_{sunset} + d^P_{log\times log} + d^P_{log} + d^P_{log\times L_i} + d^P_{L_i} + d^P_{L_i\times L_j}. \quad (2)$$

$d^P_{L_i\times log}$ collects the terms linear in the $\mathcal{O}(p^4)$ LECs $L_i$ and containing chiral logs, $d^P_{log}$, $d^P_{log\times log}$ collect the terms linear respectively quadratic in chiral logarithms without $L_i$, $d_{L_i}$ and $d^P_{L_i\times L_j}$ the terms linear respectively quadratic in the LECs $L_i$. The term $(F_P)^{(6)}_{CT}$ is composed of the $\mathcal{O}(p^6)$ counterterms, i.e. the LECs $C_i^r$, while $d^P_{sunset}$ are the pure sunset terms.

One determines the ratio $F_K/F_\pi$ using:

$$\frac{F_K}{F_\pi} = 1 + \left(\frac{F_K}{F_0}\bigg|_{p^4} - \frac{F_\pi}{F_0}\bigg|_{p^4}\right)_{\rm NLO} + \left(\frac{F_K}{F_0}\bigg|_{p^6} - \frac{F_\pi}{F_0}\bigg|_{p^6} - \frac{F_K}{F_0}\bigg|_{p^4}\frac{F_\pi}{F_0}\bigg|_{p^4} + \frac{F_\pi}{F_0}\bigg|^2_{p^4}\right)_{\rm NNLO}. \quad (3)$$

The terms $d^P_{sunset}$ are not available fully analytically. Their determination is the goal of this work. The sunset integral, shown in Fig. 1, is defined as:

$$H^d_{\{\alpha,\beta,\gamma\}}(m_1,m_2,m_3;s) = \frac{(1/i)^2}{(2\pi)^{2d}}\int\frac{d^dq\;d^dr}{[q^2-m_1^2]^\alpha[r^2-m_2^2]^\beta[(q+r-p)^2-m_3^2]^\gamma}. \quad (4)$$

Aside from the basic scalar integral defined above, tensor integrals in which the momenta $q_\mu$ and $q_\mu q_\nu$ appear in the numerator, and derivatives with respect to the external momentum of both the scalar and tensor integrals, appear in $d^P_{sunset}$ [3]. The tensor integrals, as well as all the derivatives, may be reduced into a linear combination of scalar integrals using the methods given in [4]. Thus only a smaller set of master integrals (MI) is needed.

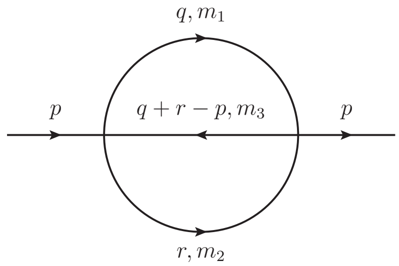


FIG. 1. The two-loop self energy "sunset" diagram

The full list of sunset integrals appearing in $d^P_{sunset}$ can thus all be expressed in terms of a set of four MIs ($H^d_{\{1,1,1\}}$, $H^d_{\{2,1,1\}}$, $H^d_{\{1,2,1\}}$ and $H^d_{\{1,1,2\}}$) and the one-loop tadpole integral. The problem reduces to solving these analytically in the required mass configurations. For the evaluation of $F_K/F_\pi$, seven distinct three mass scale MIs need evaluation.

MB theory leads to representations of these MI where each integral consists of at least one double complex plane integral. These double MB integrals are evaluated using the method proposed in [5] and fully systematized in [6] to obtain results in the form of sums of single and double infinite series [7]-[9].

**The analytic representation-** Using Eq.(3), we obtain the following representation of $F_K/F_\pi$:

$$
\begin{aligned}
\frac{F_K}{F_\pi} &= 1 + 4(4\pi)^2 L_5^r\,(\xi_K - \xi_\pi) + \frac{5}{8}\xi_\pi\lambda_\pi - \frac{1}{4}\xi_K\lambda_K \\
&+ \left(\frac{1}{8}\xi_\pi - \frac{1}{2}\xi_K\right)\lambda_\eta + \xi_K^2 F_F\left(\frac{m_\pi^2}{m_K^2}\right) + \hat{K}_1^r\lambda_\pi^2 \\
&+ \hat{K}_2^r\lambda_\pi\lambda_K + \hat{K}_3^r\lambda_\pi\lambda_\eta + \hat{K}_4^r\lambda_K^2 + \hat{K}_5^r\lambda_K\lambda_\eta \\
&+ \hat{K}_6^r\lambda_\eta^2\xi_K^2 + \hat{C}_1\lambda_\pi + \hat{C}_2\lambda_K + \hat{C}_3\lambda_\eta + \hat{C}_4, \qquad (5)
\end{aligned}
$$

where $\xi_\pi = m_\pi^2/(16\pi^2 F_\pi^2)$, $\xi_K = m_K^2/(16\pi^2 F_\pi^2)$, $\lambda_i = \log(m_i^2/\mu^2)$, and:

$$
\begin{aligned}
&\hat{K}_1^r = \frac{11}{24}\xi_\pi\xi_K - \frac{131}{192}\xi_\pi^2, \quad \hat{K}_2^r = -\frac{41}{96}\xi_\pi\xi_K - \frac{3}{32}\xi_\pi^2, \\
&\hat{K}_3^r = \frac{13}{24}\xi_\pi\xi_K + \frac{59}{96}\xi_\pi^2, \quad \hat{K}_4^r = \frac{17}{36}\xi_K^2 + \frac{7}{144}\xi_\pi\xi_K, \\
&\hat{K}_5^r = -\frac{163}{144}\xi_K^2 - \frac{67}{288}\xi_\pi\xi_K + \frac{3}{32}\xi_\pi^2, \\
&\hat{K}_6^r = \frac{241}{288}\xi_K^2 - \frac{13}{72}\xi_\pi\xi_K - \frac{61}{192}\xi_\pi^2. \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}_1^r &= -\left(\frac{7}{9} + \frac{11}{2}(4\pi)^2 L_5^r\right)\xi_\pi\xi_K \\
&- \left(\frac{113}{72} + (4\pi)^2(4L_1^r + 10L_2^r + \frac{13}{2}L_3^r - \frac{21}{2}L_5^r)\right)\xi_\pi^2, \\
\hat{C}_2^r &= \left(\frac{209}{144} + 3(4\pi)^2 L_5^r\right)\xi_\pi\xi_K \\
&+ \left(\frac{53}{96} + (4\pi)^2(4L_1^r + 10L_2^r + 5L_3^r - 5L_5^r)\right)\xi_K^2, \\
\hat{C}_3^r &= \left(\frac{13}{18} + (4\pi)^2\left(\frac{8}{3}L_3^r - \frac{2}{3}L_5^r - 16L_7^r - 8L_8^r\right)\right)\xi_K^2 \\
&- \left(\frac{4}{9} + (4\pi)^2\left(\frac{4}{3}L_3^r + \frac{25}{6}L_5^r - 32L_7^r - 16L_8^r\right)\right)\xi_\pi\xi_K \\
&+ \left(\frac{19}{288} + (4\pi)^2\left(\frac{1}{6}L_3^r + \frac{11}{6}L_5^r - 16L_7^r - 8L_8^r\right)\right)\xi_\pi^2, \\
\hat{C}_4^r &= (4\pi)^2(\xi_K - \xi_\pi) \\
&\times\Bigg\{8(4\pi)^2\Big(2(C_{14}^r + C_{15}^r)\xi_K + (C_{15}^r + 2C_{17}^r)\xi_\pi\Big) \\
&+ \Big(8(4\pi)^2 L_5^r(8L_4^r + 3L_5^r - 16L_6^r - 8L_8^r) - 2L_1^r \\
&\quad - L_2^r - \frac{1}{18}L_3^r + \frac{4}{3}L_5^r - 16L_7^r - 8L_8^r\Big)\xi_K \\
&+ \Big(8(4\pi)^2 L_5^r(4L_4^r + 5L_5^r - 8L_6^r - 8L_8^r) - 2L_1^r \\
&\quad - L_2^r - \frac{5}{18}L_3^r - \frac{4}{3}L_5^r + 16L_7^r + 8L_8^r\Big)\xi_\pi\Bigg\}. \qquad (7)
\end{aligned}
$$

$F_F$ consists of the terms arising from the pure sunset contributions. The split between the $\hat{K}_i$ terms and $F_F$ is not unique. To evaluate $F_F$, we truncate the series results of each sunset integral so that the error between the exact and truncated value is $< 1\%$ for most of the sets of masses used in the lattice study of [1]. After substituting the truncated sunset results into $d^P_{sunset}$, we take its series expansion in $\rho = m_\pi^2/m_K^2$ up to $\mathcal{O}(\rho^4)$, and express $F_F$ as:

$$
\begin{aligned}
F_F(\rho) &= a_1 + \left(a_2 + a_3\log[\rho] + a_4\log^2[\rho]\right)\rho \\
&+ \left(a_5 + a_6\log[\rho] + a_7\log^2[\rho]\right)\rho^2 \\
&+ \left(a_8 + a_9\log[\rho] + a_{10}\log^2[\rho]\right)\rho^3 \\
&+ \left(a_{11} + a_{12}\log[\rho] + a_{13}\log^2[\rho]\right)\rho^4 + \mathcal{O}\left(\rho^5\right) \qquad (8)
\end{aligned}
$$

The numerical values of the $a_i$ are:

$$
\begin{aligned}
&a_1 = 0.8740, \quad a_2 = -2.172, \quad a_3 = 0.8294, \\
&a_4 = -0.4583, \quad a_5 = 3.716, \quad a_6 = -0.1113, \\
&a_7 = 0.8776, \quad a_8 = -1.635, \quad a_9 = 1.4697, \\
&a_{10} = -0.1406, \quad a_{11} = -1.343, \quad a_{12} = 0.2731,
\end{aligned}
$$

$$a_{13} = -0.2109, \tag{9}$$

and their explicit analytic form is given by:

$$
\begin{aligned}
a_1 &= -\frac{6337}{5184}\left(\mathrm{Li}_2\left[\frac{3}{4}\right] + \log(4)\log\left[\frac{4}{3}\right]\right) + \frac{41\pi^2}{192} \\
&\quad - \frac{11\sqrt{2}\pi}{27} + \frac{85957107031}{27662342400} - \frac{119\pi}{216\sqrt{2}} \\
&\quad + \frac{62591}{612360}\log[3] + \frac{43006343}{13471920}\log\left[\frac{4}{3}\right] \\
&\quad + \left(\frac{8\sqrt{2}}{9} - \frac{41\pi}{48} - \frac{5\log[3]}{24\sqrt{2}}\right)\csc^{-1}\left[\sqrt{3}\right] \\
&\quad + \frac{41}{48}\csc^{-1}\left[\sqrt{3}\right]^2 + \frac{5}{1152}\log^2\left[\frac{4}{3}\right], \\
a_2 &= \frac{5821}{2592}\left(\mathrm{Li}_2\left[\frac{3}{4}\right] + \log[4]\log\left[\frac{4}{3}\right]\right) - \frac{25\pi^2}{96} \\
&\quad - \frac{7269419973251}{1120324867200} + \frac{145\pi}{72\sqrt{2}} + \frac{38693\pi}{25920\sqrt{3}} + \frac{82\gamma}{405} \\
&\quad - \frac{121}{576}\log^2\left[\frac{4}{3}\right] - \left(\frac{6035437}{9797760} + \frac{13\pi}{864\sqrt{3}}\right)\log[3] \\
&\quad - \left(\frac{468002719}{161663040} + \frac{13\pi}{576\sqrt{3}}\right)\log\left[\frac{4}{3}\right] - \frac{29}{324}\psi\left[\frac{5}{2}\right] \\
&\quad + \left(\frac{463\log[3]}{384\sqrt{2}} + \frac{\log\left[\frac{4}{3}\right]}{2\sqrt{2}} - \frac{11\pi}{48} - \frac{13\gamma}{18\sqrt{2}} - \frac{15875}{3456\sqrt{2}}\right) \\
&\quad\quad \times \csc^{-1}\left[\sqrt{3}\right] + \frac{11}{48}\csc^{-1}\left[\sqrt{3}\right]^2, \\
a_3 &= \frac{803}{810} + \frac{13\pi}{1728\sqrt{3}} + \frac{7}{48}\log\left[\frac{4}{3}\right] - \frac{1}{2\sqrt{2}}\csc^{-1}\left[\sqrt{3}\right], \\
a_4 &= -\frac{11}{24}, \quad a_7 = \frac{337}{384}, \quad a_{10} = -\frac{9}{64}, \quad a_{13} = -\frac{27}{128} \\
a_5 &= \frac{47}{128}\log^2\left[\frac{4}{3}\right] - \frac{845}{648}\left(\mathrm{Li}_2\left[\frac{3}{4}\right] + \log[4]\log\left[\frac{4}{3}\right]\right) \\
&\quad - \frac{1301\sqrt{3}\pi}{512} - \frac{66191\gamma}{12960} + \frac{1576413731881}{3585039575040} + \frac{5\pi^2}{18} \\
&\quad - \frac{145\pi}{144\sqrt{2}} + \frac{3572063\pi}{663552\sqrt{3}} + \frac{59}{48}\csc^{-1}\left[\sqrt{3}\right]^2 \\
&\quad + \left(\frac{744674317}{313528320} + \frac{176189\pi}{55296\sqrt{3}}\right)\log[3] + \frac{35}{144}\psi\left[\frac{5}{2}\right] \\
&\quad + \left(\frac{97621}{55296\sqrt{2}} - \frac{59\pi}{48} + \frac{3167\gamma}{288\sqrt{2}} - \frac{19589\log[3]}{4096\sqrt{2}}\right. \\
&\quad\quad \left. - \frac{115}{48\sqrt{2}}\right)\log\left[\frac{4}{3}\right]\csc^{-1}\left[\sqrt{3}\right] \\
&\quad + \left(\frac{4312709021}{1293304320} + \frac{176189\pi}{36864\sqrt{3}}\right)\log\left[\frac{4}{3}\right], \\
a_6 &= \frac{17003}{8640} - \frac{176189\pi}{110592\sqrt{3}} - \frac{155}{192}\log\left[\frac{4}{3}\right] \\
&\quad + \frac{115}{48\sqrt{2}}\csc^{-1}\left[\sqrt{3}\right], \\
a_8 &= \frac{265}{864}\left(\mathrm{Li}_2\left[\frac{3}{4}\right] + \log[4]\log\left[\frac{4}{3}\right]\right) + \frac{199393\gamma}{138240} \\
&\quad + \frac{25001310633017}{9481096396800} + \frac{4753\pi}{13824\sqrt{2}} + \frac{20910563\pi}{26542080\sqrt{3}} \\
&\quad - \frac{29\pi^2}{288} - \left(\frac{101313035}{143327232} + \frac{804611\pi}{442368\sqrt{3}}\right)\log[3] \\
&\quad - \left(\frac{129118553}{117573120} + \frac{804611\pi}{294912\sqrt{3}}\right)\log\left[\frac{4}{3}\right] - \frac{119}{288}\psi\left[\frac{5}{2}\right] \\
&\quad - \frac{5}{16}\csc^{-1}\left[\sqrt{3}\right]^2 + \csc^{-1}\left[\sqrt{3}\right]\left(\frac{823}{3072\sqrt{2}}\log\left[\frac{4}{3}\right]\right. \\
&\quad \left. + \frac{5\pi}{16} - \frac{19319\gamma}{9216\sqrt{2}} - \frac{5341499}{3538944\sqrt{2}} + \frac{104075\log[3]}{196608\sqrt{2}}\right), \\
a_9 &= -\frac{8327}{138240} + \frac{804611\pi}{884736\sqrt{3}} - \frac{1}{96}\log\left[\frac{4}{3}\right] \\
&\quad - \frac{823}{3072\sqrt{2}}\csc^{-1}\left[\sqrt{3}\right], \\
a_{11} &= -\frac{5}{192}\left(\mathrm{Li}_2\left[\frac{3}{4}\right] + \log[4]\log\left[\frac{4}{3}\right]\right) - \frac{25\pi^2}{192} \\
&\quad - \frac{1310311\gamma}{6635520} - \frac{10567863311827}{10113169489920} + \frac{4453\sqrt{3}\pi}{65536} \\
&\quad + \left(\frac{12616533707}{45864714240} + \frac{1674775\pi}{7077888\sqrt{3}}\right)\log[3] \\
&\quad + \left(\frac{17720699}{46448640} + \frac{1674775\pi}{4718592\sqrt{3}}\right)\log\left[\frac{4}{3}\right] \\
&\quad - \frac{13905571\pi}{84934656\sqrt{3}} - \frac{2135\pi}{73728\sqrt{2}} + \frac{97}{648}\psi\left[\frac{5}{2}\right] \\
&\quad + \frac{1}{\sqrt{2}}\left(\frac{605645}{18874368} - \frac{391\gamma}{49152} - \frac{121093\log[3]}{4194304}\right. \\
&\quad\quad \left. - \frac{59}{4096}\log\left[\frac{4}{3}\right]\right)\csc^{-1}\left[\sqrt{3}\right], \\
a_{12} &= \frac{5538437}{11612160} - \frac{1674775\pi}{14155776\sqrt{3}} + \frac{1}{64}\log\left[\frac{4}{3}\right] \\
&\quad + \frac{59}{4096\sqrt{2}}\csc^{-1}\left[\sqrt{3}\right],
\end{aligned} \tag{10}
$$

where $\gamma$ is the Euler–Mascheroni constant, $\csc^{-1}$ is the

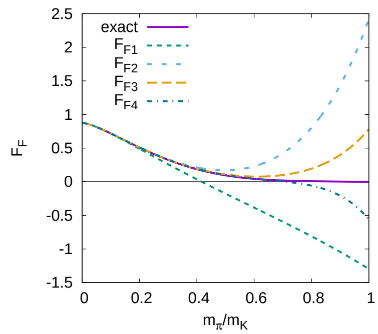


FIG. 2. Comparison of the exact and approximate $F_F$

arccosec function, $\mathrm{Li}_2$ is the dilogarithm function, and $\psi$ is the digamma function.

The range of validity of Eq.(10) is shown in Fig. 2, in which the exact value of $F_F$, calculated numerically, is plotted against $x = \sqrt{\rho}$, as are the approximate $F_F$ retained upto various orders of $\rho$. The expansion up to $\mathcal{O}(\rho^4)$ approximates the exact value of $F_F$ to 1% for $m_\pi/m_K < 3$ and to 6% for $m_\pi/m_K < 0.5$. If greater accuracy is required, the sunset MB series may be truncated with a larger number of terms. This leads to the same general analytic form of the $a_i$, but with different coefficients.

**Lattice Fittings-** We fit Eq.(5) with the data of the lattice study [1] to determine best-fit values of the NLO LEC $L_5^r$ and the NNLO LEC combinations $C_{14}^r + C_{15}^r$ and $C_{15}^r + 2C_{17}^r$. We perform the fit (using [10]) on the mass sets for which $m_\pi < 0.40$GeV. We do the fit with the exact $F_F$, the approximate version gives compatible results. We fix the renormalization scale $\mu$ at $m_\rho = 0.77$GeV, and use the values of the BE14 fit [11] for the other $L_i^r$, to obtain:

$$\begin{aligned} L_5^r &= (3.92 \pm 0.55)\ 10^{-4} \\ C_{14}^r + C_{15}^r &= (2.59 \pm 0.63)\ 10^{-6} \\ C_{15}^r + 2C_{17}^r &= (6.10 \pm 1.41)\ 10^{-6}. \end{aligned} \tag{11}$$

The uncertainties on the values derive from the errors of the $F_K/F_\pi$ data of the lattice study, but do not take into account other uncertainties. In addition we fixed $F_\pi$ in the determination of $\xi_\pi$ and $\xi_K$ to 92.2 MeV.

With these LEC values and the physical meson masses as inputs, we get for the value of $F_K/F_\pi$:

$$F_K/F_\pi = 1.194, \tag{12}$$

| | $L_5$ | $C_{14}+C_{15}$ |
|---|---|---|
| $C_{14}+C_{15}$ | $-0.93$ | 1.00 |
| $C_{15}+2C_{17}$ | 0.35 | $-0.66$ |

TABLE I. Correlation values of the fit in (11).

which agrees well with the literature value of [11].

The values of Eq.(11) differ from those of the BE14 exact fit ($L_5 = 10.1 \times 10^{-4}, C_{14} + C_{15} = -4.00 \times 10^{-6}, C_{15}+2C_{17} = -5.00\times10^{-6}$), as well as from the fit of [12] ($L_5 = 7.60\times10^{-3}, C_{14}+C_{15} = 0.37\times10^{-6}, C_{15}+2C_{17} = 1.29\times10^{-6}$) by a significant amount. However, the strong correlation among these values values must be taken into account. The correlation parameters are given in Table I.

The quality of the fit is shown in Fig. 3(Left). The correlation is shown graphically in Fig. 3(Middle, Right) by plotting a number of random points in a distribution given by the correlation matrix of the fit projected on the two different planes.

A similar fit, but now with $F_\pi$ also varied in $\xi_\pi, \xi_K$ requires the use of lattices common to [1] and [13] to obtain the values of $F_\pi$ for each lattice. This fit leads to:

$$\begin{aligned} L_5^r &= (0.49 \pm 1.08)\ 10^{-4} \\ C_{14}^r + C_{15}^r &= (5.59 \pm 1.08)\ 10^{-6} \\ C_{15}^r + 2C_{17}^r &= (39.7 \pm 2.10)\ 10^{-6}. \end{aligned} \tag{13}$$

The lattice data clearly have a significant impact on fitting the LECs.

**Conclusions-** The ratio $F_K/F_\pi$ is a quantity at the heart of chiral symmetry breaking, a fundamental property of the strong interactions that is measured in ab initio calculations on the lattice. Tuning of the quark masses to physical values is now possible. Thus an analytic expansion for this in masses of the quarks or the mesons is the order of the day. By suitably adapting two-loop expansions for this in ChPT [14], and using modern loop calculation techniques, we have achieved this goal, and thereafter fit lattice data to the expressions obtained.

This work is a product of combining techniques developed independently in various branches of elementary particle physics and field theory, and represents an important advance on the results that appeared nearly two decades ago, when many sunsets were evaluated numerically. We hope this work will pave the way for detailed comparisons of other similar quantities with lattice simulations, and help improve our understand-

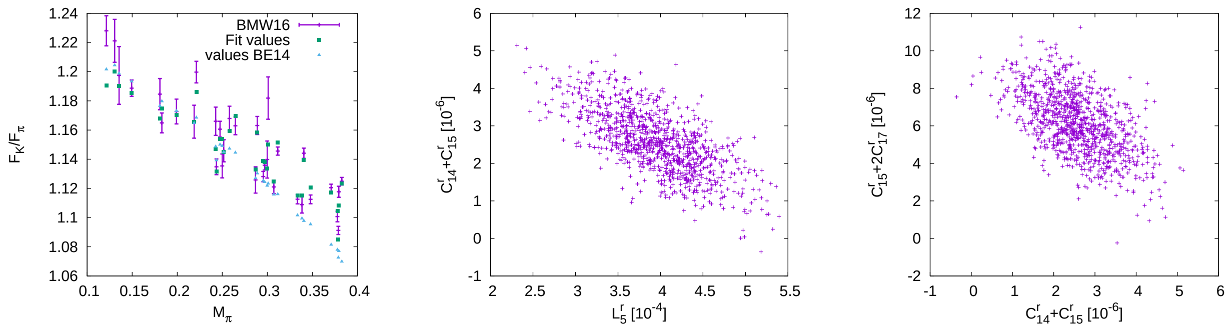


FIG. 3. Left: Quality of the fit. 'Values BE14' plots use the BE14 numbers for $L_5^r$, $C_{14}^r$, $C_{15}^r$ and $C_{17}^r$. Middle: Correlation of $L_5^r$ and $C_{14}^r + C_{15}^r$. Right: Correlation of $C_{14}^r + C_{15}^r$ and $C_{15}^r + 2C_{17}^r$.

ing of both ChPT and lattice studies.

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