Scalar Kinetic Mixing and the Renormalization Group

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Abstract

Quantum field theories containing scalar fields with equal quantum numbers allow for a mixed kinetic term in the Lagrangian. It has been argued that this mixing must be taken into consideration when performing renormalization group (RG) analyses of such a theory. From the fact that scalar kinetic mixing does not correspond to a physical observable, we show that no extra parameters need to be introduced. Using a toy model, we explicitly derive the 1-loop RG equations (RGEs) in three different renormalization schemes to demonstrate how the issue can be dealt with. In schemes without kinetic mixing, either the fields mix during renormalization to produce non-diagonal anomalous dimensions or the RGEs explicitly depend on the scalar masses. Finally, we show how the different schemes are related to each other by scale dependent field redefinitions.

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Contents

1 Introduction

With the discovery of the Higgs boson at the LHC [1, 2, 3], the particle content of the Standard Model (SM) is now complete. Still, as is well known, the SM is not a complete theory and there are many proposals for physics Beyond the Standard Model (BSM), such as supersymmetry, in order to solve some of its outstanding problems.

One common feature of BSM models is the inclusion of extra scalar fields. A very well studied example is the so called two Higgs Doublet Model (2HDM) which was introduced in 1974 by T. D. Lee [4]. As its name implies, it has two Higgs doublets instead of one as in the SM. Most phenomenological studies of the 2HDM assumes that the two Higgs doublets are charged differently under a discrete $\mathbb{Z}_2$ symmetry and that this symmetry is at most softly broken, *i.e.* by mass-terms, in the scalar potential. By assigning appropriate charges to the right-handed fermion fields, this ensures that there are no tree-level Flavor Changing Neutral Currents (FCNC). However, in general there is no justification to make this assumption, since the 2HDM in itself is also not a complete theory. For example having the same fine-tuning or hierarchy problem as the SM. In fact, in a supersymmetric version of the theory, the $\mathbb{Z}_2$ symmetry is broken when going beyond tree-level [5] and even though this breaking is only soft it gives rise to so called non-holomorphic corrections.

In this paper we want to clarify in a pedagogical way what happens when the $\mathbb{Z}_2$ symmetry is broken. One immediate consequence is that the two Higgs doublets will mix, giving rise to both kinetic and mass mixing. At tree level, as is well known[4], the kinetic term as well as the mass matrix can be diagonalized. The question we investigate in this paper is whether this can be done consistently also at loop-level, *i.e.* after renormalization, and whether the kinetic term can be kept diagonal also under renormalization group evolution,

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[4]See for example sec.12.5 in ref. [6].

even though the fields mix. In refs. [7, 8] it was claimed that whereas one can always make the kinetic term diagonal at a given renormalization scale, the mixing will reappear if the renormalization scale is changed, thus meaning that one gets an additional parameter.

As we will show in this paper, this claim is not correct and one does not need an additional parameter in order to renormalize the theory. Instead, one can choose to work with different renormalization schemes where this parameter is either present or not, all giving the same results. The underlying reason behind this result is that the two-point functions are in fact not observables and as such do not need to be finite. The only observables are masses and scattering amplitudes. We also show that in those schemes where there is no kinetic mixing, one instead has mixing of the fields under renormalization corresponding to a change of basis which has to be taken into account.

A similar problem occurs in the renormalization of the CKM matrix. For a recent discussion of how that is resolved in different renormalization schemes for the case of an extended scalar sector, see ref. [9].

We present three different renormalization schemes with or without kinetic mixing and show how they are related at one-loop level under renormalization through orthogonal and non-orthogonal transformations. A difference compared to tree-level is that these transformations are scale-dependent. As a by-product of our considerations we also show that in one of these schemes, the renormalization group equations in the Minimal Subtraction (MS) scheme for the quartic couplings depend on the masses of the scalar particles - in contradiction with statements often made in textbooks.

As a pedagogical example and to simplify the discussions, we will not consider the full 2HDM but instead only consider a toy model with $N \geq 2$ real pseudo scalar fields coupled to one Dirac fermion. This model has the same generic features as the 2HDM when it comes to renormalization properties of the scalar fields.

This paper is organized as follows. We start in section 2 by defining the theory we are considering and introduce the three different renormalization schemes that we are considering. In the following section we define the three schemes in more detail, perform the renormalization at one-loop level and calculate the renormalization group equations for each case. In section 4 we then show how the schemes are related to each other and that they in fact are equivalent. Finally section 5 contains our conclusions.

2 Kinetic mixing and renormalizability

$$G_{ij}(p^2) = i \; \text{---}\!\!\bigotimes\!\!\text{---} \; j = \frac{1}{\epsilon} \left(\alpha_{ij} p^2 + \beta_{ij} \right) \tag{1}$$

Figure 1: Two point function of N real scalar fields. At 1-loop order, it contains divergences proportional to p^2; α_{ij} and β_{ij} are momentum independent.

To discuss the renormalizability of theories with mixed scalar kinetic terms, we have chosen the simplest case with N real pseudo scalar fields coupled to a Dirac fermion. The reason to include a fermion is because then the 2-point Green functions in figure 1, $G_{ij}(p^2)$, exhibit divergences proportional to p^2 at 1-loop level; instead of at 2-loop level, which would be the case in a theory with only scalars.

The only requirement for a theory to be renormalizable is that all physical quantities are free from divergences. So even if some component of $G_{ij}(p^2)$ contains divergences, it in itself does not spoil renormalizability since $G_{ij}(p^2)$ is not a physical observable. In our theory, the physical observables are scattering amplitudes and masses of the particles. Thus, the minimal set of parameters and counterterms consists only of interaction and diagonal mass terms. We will show in section 3.2 how this set is sufficient to absorb all divergences. This merely corresponds to one particular renormalization scheme.

Another renormalization scheme is considered in section 3.1, where we work with the most general Lagrangian consistent with the symmetries of the theory, including kinetic mixing operators, and renormalize every parameter. Since the parameters in the Lagrangian are not directly related to any physical observables, one instead requires all Green functions to be free of divergences.

A third renormalization scheme, and the one most often used, is to work with diagonal kinetic terms and renormalized fields. To be able to absorb the divergences in all Green functions into counterterms, the scalar fields must mix during renormalization. This will induce the necessary non-diagonal kinetic counterterms to ensure renormalizability. The anomalous dimensions of the fields will then enter the RGEs for the couplings. We show how this is done in section 3.3.

Throughout all of our calculations, we use dimensional regularization in $d = 4 - 2\epsilon$ dimensions and MS for the counterterms. We will also introduce the renormalization scale μ to make all the couplings in the Lagrangian have their natural dimension. All bare quantities that are μ independent will be denoted by a b superscript. For notational convenience, we define

$$\Lambda \equiv \frac{1}{16\pi^2\epsilon}, \qquad\qquad \mathcal{D} \equiv 16\pi^2\mu\frac{\mathrm{d}}{\mathrm{d}\mu}.$$

3 Three renormalization schemes

3.1 Most general case

The most general Lagrangian with the imposed symmetries in terms of bare fields and parameters is given by

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}\partial_\mu\varphi_i^b a_{ij}^b\partial^\mu\varphi_j^b - \frac{1}{2}\varphi_i^b b_{ij}^b\varphi_j^b + c^b\overline{\psi}^b i\slashed{\partial}\psi^b - d^b\overline{\psi}^b\psi^b \\ &- e_i^b\varphi_i^b\overline{\psi}^b i\gamma_5\psi^b - \frac{1}{4!}f_{ijkl}^b\varphi_i^b\varphi_j^b\varphi_k^b\varphi_l^b\,. \end{aligned} \tag{2}$$

3

The bare parameters are divided into renormalized parameters and counter terms as

$$a_{ij}^b = a_{ij} + \delta a_{ij}, \qquad b_{ij}^b = b_{ij} + \delta b_{ij}, \qquad c^b = c + \delta c,$$
$$d^b = d + \delta d, \qquad e_i^b = \mu^\epsilon \left(e_i + \delta e_i \right), \qquad f_{ijkl}^b = \mu^{2\epsilon} \left(f_{ijkl} + \delta f_{ijkl} \right), \qquad (3)$$

where a_{ij}, b_{ij}, f_{ijkl} are fully symmetric in all their indices and appropriate powers of μ have been inserted to give all terms correct dimensions. We will also use matrix and vector notation for the coefficients a_{ij}, b_{ij}, e_i.

This Lagrangian contains a number of redundant parameters. In the case of two scalar fields, we have 15 free parameters with their associated counterterms. These do not directly correspond to physical observables. If one transforms the Lagrangian to the minimal form, as we do in section 3.2, the total number of free parameters is reduced to 10; which then directly correspond to physical observables.

One complication of working with this general Lagrangian is the non-diagonal kinetic terms; which gives rise to a matrix scalar propagator. In the calculations we treat the mass parameters as small perturbations $b_{ij} \ll p^2$ and work with an expanded propagator,

$$i \text{-------} j = \left(\frac{i}{p^2 a - b} \right)_{ij} \simeq \frac{i}{p^2} \left(a^{-1} + \frac{a^{-1} b a^{-1}}{p^2} \right)_{ij}. \qquad (4)$$

This we can do since in the end we are only interested in the RGEs and therefore we only need to keep the terms that give rise to UV divergences in the loop calculations.

Ignoring the finite pieces, the 1-loop calculation of the diagrams in figure 2 results in

$$\Sigma(\not{p}) = \left(\frac{e_i e_j a_{ij}^{-1}}{2c} \not{p} - \frac{e_i e_j a_{ij}^{-1}}{c^2} d \right) \Lambda + (\not{p} \delta c - \delta d), \qquad (5)$$

$$\Pi_{ij}(p^2) = \left[\frac{1}{2} f_{ijkl} (a^{-1} b a^{-1})_{kl} + (4d^2 - 2c^2 p^2) \frac{e_i e_j}{c^4} \right] \Lambda + (p^2 \delta a - \delta b), \qquad (6)$$

$$\Gamma_i^{\text{amp}} = -\frac{e_i e_j e_k a_{jk}^{-1}}{c^2} \Lambda + \delta e, \qquad (7)$$

$$\Gamma_{ijkl}^{\text{amp}} = \left[\frac{1}{2} \left(a^{-1} \right)_{mn} \left(a^{-1} \right)_{op} \left(f_{ijmo} f_{klnp} + f_{ikmo} f_{jlnp} + f_{ilmo} f_{jknp} \right) - \frac{24 e_i e_j e_k e_l}{c^4} \right] i\Lambda - i\delta f_{ijkl}. \qquad (8)$$

As renormalization conditions we will require all Green functions to be be finite which is accomplished by absorbing all the infinites in the 1PI and amputated diagrams into the counterterms. At one loop order this is realized by

$$\delta a_{ij} = \frac{2 e_i e_j}{c^2} \Lambda,$$
$$\delta b_{ij} = \left[\frac{4 d^2 e_i e_j}{c^4} + \frac{1}{2} f_{ijkl} \left(a^{-1} b a^{-1} \right)_{kl} \right] \Lambda,$$
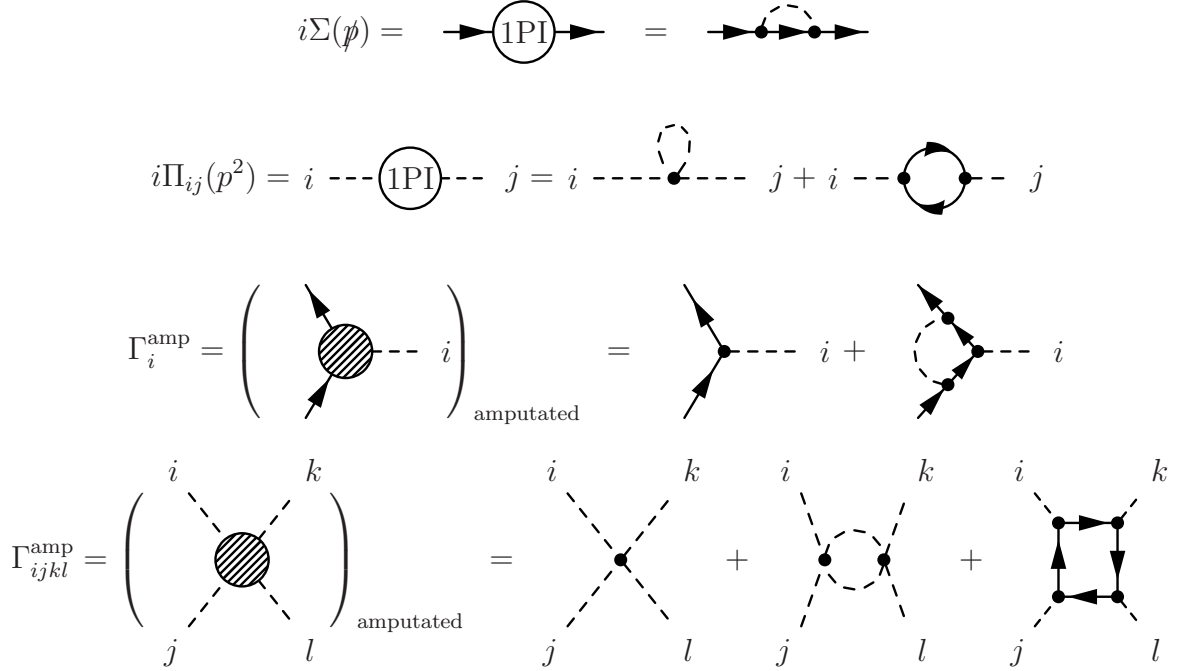$$\delta c = -\frac{e^T a^{-1} e}{2c} \Lambda,$$

4

Figure 2: The structure of 1-loop diagrams that contribute to the Green functions in all three renormalization schemes. Note that there are additional permutations of the various diagrams as well as counterterms that are not displayed.

$$\delta d = -\frac{e^T a^{-1} e \, d}{c^2}\Lambda,$$

$$\delta e_i = \frac{e_i}{c^2}\left(e^T a^{-1} e\right)\Lambda,$$

$$\delta f_{ijkl} = \left[\frac{1}{2}\left(a^{-1}\right)_{mn}\left(a^{-1}\right)_{op}\left(f_{ijmo}f_{klnp} + f_{ikmo}f_{jlnp} + f_{ilmo}f_{jknp}\right) - \frac{24 e_i e_j e_k e_l}{c^4}\right]\Lambda. \quad (9)$$

Since we work with bare fields and the bare parameters do not depend on the renormalization scale μ, the RGEs for the parameters are easily derived from their respective counterterm by requiring the expressions for the bare parameters in eq. (3) to be μ-independent. In four dimensions they are

$$\mathcal{D}a_{ij} = \frac{4 e_i e_j}{c^2},$$

$$\mathcal{D}b_{ij} = \frac{8d^2 e_i e_j}{c^4} + f_{ijkl}\left(a^{-1}ba^{-1}\right)_{kl},$$

$$\mathcal{D}c = -\frac{\left(e^T a^{-1} e\right)}{c},$$

$$\mathcal{D}d = -\frac{2\left(e^T a^{-1} e\right) d}{c^2},$$

5

$$\mathcal{D}e_i = \frac{2e_i}{c^2}\left(e^T a^{-1}e\right),$$

$$\mathcal{D}f_{ijkl} = \left(a^{-1}\right)_{mn}\left(a^{-1}\right)_{op}\left(f_{ijmo}f_{klnp} + f_{ikmo}f_{jlnp} + f_{ilmo}f_{jknp}\right) - \frac{48e_ie_je_ke_l}{c^4}. \tag{10}$$

From the first equation above we see that in this renormalization scheme there is indeed kinetic mixing if the fermion couples to more than one of the scalar fields; much like the scheme used in ref. [8]. However, not all of the parameters above are physical and as we will see later the parameter describing kinetic mixing is redundant.

3.2   Minimal case

Here we only add the minimal number of free parameters and counterterms. This is similar to what is used in effective field theory methods where one removes the so-called equations of motion terms or alternatively brings the Lagrangian into the minimal form using field redefinitions [5]. The physical observables in our theory are the masses and scattering amplitudes; therefore we only require counterterms for the masses and couplings. The Lagrangian in terms of the bare quantities is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi_i^b \partial^\mu \Phi_i^b - \frac{1}{2}m_i^{2b}\Phi_i^b\Phi_i^b + \overline{\Psi}^b i\slashed{\partial}\Psi^b - m^b\overline{\Psi}^b\Psi^b$$

$$+ Y_i^b\Phi_i^b\overline{\Psi}^b i\gamma_5\Psi^b - \frac{1}{4!}\Lambda_{ijkl}^b\Phi_i^b\Phi_j^b\Phi_k^b\Phi_l^b, \tag{11}$$

where Λ_{ijkl}^b is fully symmetric in i, j, k, l. The bare parameters are related to the renormalized ones and counterterms via

$$m_i^{2b} = m_i^2 + \delta m_i^2, \qquad\qquad m^b = m + \delta m,$$

$$Y_i^b = \mu^\epsilon\left(Y_i + \delta Y_i\right), \qquad\qquad \Lambda_{ijkl}^b = \mu^{2\epsilon}\left(\Lambda_{ijkl} + \delta\Lambda_{ijkl}\right). \tag{12}$$

Here we only have the physically relevant number of parameters and the same number of counterterms. For the case of two scalars this would be 10, although we will work with any number of scalars.

We will also assume the masses to be non-degenerate. In the case of degenerate masses one cannot tell the scalar particles apart and the theory would exhibit an additional $O(N)$ symmetry. This symmetry could then be used to rotate the fields to a basis where the 2-point Green functions are diagonal and the whole discussion of renormalizability would become simpler.

In the minimal scheme, the UV divergent pieces and counterterms of the 1PI and

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[5]See, for example, sec. 6 in ref. [10] for a pedagogical introduction.

amputated Green functions in figure 2 are

$$\Sigma(p^2) = \sum_i Y_i^2 \left(\frac{\not{p}}{2} - m\right) \Lambda - \delta m, \tag{13}$$

$$\Pi_{ij}(p^2) = \left[\frac{1}{2}\Lambda_{ijkk}m_k^2 + \frac{1}{2}Y_iY_j(8m^2 - 4p^2)\right]\Lambda - \delta_{ij}\delta m_i^2, \tag{14}$$

$$\Gamma_i^{\text{amp}} = -Y_i \sum_j Y_j^2 \Lambda + \delta Y_i, \tag{15}$$

$$\Gamma_{ijkl}^{\text{amp}} = \left[\frac{1}{2}\left(\Lambda_{ijmm}\Lambda_{klmm} + \Lambda_{ikmm}\Lambda_{jlmm} + \Lambda_{ilmm}\Lambda_{jkmm}\right) - 24Y_iY_jY_kY_l\right]i\Lambda - i\delta\Lambda_{ijkl}. \tag{16}$$

The full 2-point Green function in terms of the 1PI functions, $i\Pi_{ij}(p^2)$, is

$$G_{ij}(p^2) \equiv \int d^4x e^{ip\cdot x} \langle\Omega| T\{\Phi_i^b(x)\Phi_j^b(0)\} |\Omega\rangle = \left(\frac{i}{p^2 - M + \Pi(p^2)}\right)_{ij}, \quad M_{ij} = m_i^2\delta_{ij}, \tag{17}$$

where Ω as usual denotes the physical ground state. To 1-loop order it reduces to

$$G_{ij} = \begin{cases} \dfrac{i}{p^2 - m_i^2}\left(1 - \dfrac{\Pi_{ii}}{p^2 - m_i^2}\right), & \text{for } i = j, \\[3mm] \dfrac{-i\Pi_{ij}}{(p^2 - m_i^2)(p^2 - m_j^2)}, & \text{for } i \neq j. \end{cases} \tag{18}$$
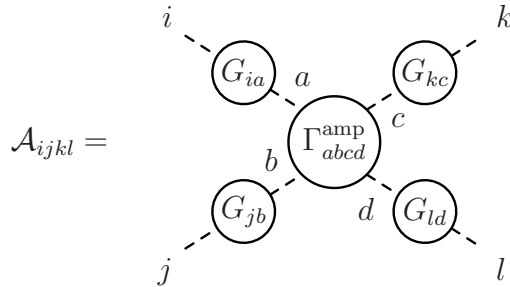

Figure 3: The physical amplitude for scattering of scalar mass eigenstates ij to kl can be expressed in terms of 2-point Green functions, G_{ij} and amputated 4-point functions, $\Gamma_{abcd}^{\text{amp}}$, with the LSZ theorem.

A crucial observation is that one cannot make all components of G_{ij} to be finite since we cannot absorb the off-diagonal p^2 divergent piece in Π_{ij} in any of the counterterms. However, this is not a problem since 2-point Green functions are not observables in quantum field theory and can thus contain divergences. The relevant physical observables are the poles of the 2-point functions, i.e. the masses of the particles, as well as the S-matrix elements, related to correlation functions through the LSZ theorem [11]. The masses are

fixed by $\det G^{-1} = 0$ on-shell and to make them finite, the δm_i^2 will absorb the infinities in $\Pi_{ii}(p^2 = m_i^2)$.

In a theory where external particles have the same quantum numbers, the conventional LSZ theorem needs to be modified to include mixing on the legs [12]. The S-matrix element describing scattering of scalar mass eigenstates ij going into kl, see figure 3, is then given by[6]

$$\mathcal{A}_{ijkl} = \frac{1}{\sqrt{Z_{ii}^{(i)} Z_{jj}^{(j)} Z_{kk}^{(k)} Z_{ll}^{(l)}}} \times \lim_{p_1^2 \to m_i^2} (p_1^2 - m_i^2) G_{ia}(p_1^2) \times \lim_{p_2^2 \to m_j^2} (p_2^2 - m_j^2) G_{jb}(p_2^2)$$
$$\times \lim_{p_3^2 \to m_k^2} (p_3^2 - m_k^2) G_{kc}(p_3^2) \times \lim_{p_4^2 \to m_l^2} (p_4^2 - m_l^2) G_{ld}(p_4^2)$$
$$\times \Gamma_{abcd}^{\mathrm{amp}}(p_1, p_2, p_3, p_4), \tag{19}$$

where the $Z_{ij}^{(k)}$ factors are defined as the residues of the corresponding 2-point correlation functions,

$$G_{ij}(p^2) \to \frac{i Z_{ij}^{(k)}}{p^2 - m_k^2 + i\epsilon} + \mathcal{O}(1), \tag{20}$$

as $p^2 \to m_k^2$, i.e.

$$Z_{ij}^{(k)} = -i \lim_{p^2 \to m_k^2} (p^2 - m_k^2) G_{ij}(p^2). \tag{21}$$

Taking the external momenta on-shell, the amplitude becomes

$$\mathcal{A}_{ijkl} = \left(1 - \frac{1}{2} \sum_{a=i,j,k,l} \frac{d}{dp^2} \Pi_{aa} \Big|_{p^2 = m_a^2} \right) \Gamma_{ijkl}^{\mathrm{amp}}$$
$$+ \sum_{a \neq i} \frac{\Pi_{ia}(p^2 = m_i^2)}{m_a^2 - m_i^2} \Gamma_{ajkl}^{\mathrm{amp}} + \sum_{a \neq j} \frac{\Pi_{ja}(p^2 = m_j^2)}{m_a^2 - m_j^2} \Gamma_{iakl}^{\mathrm{amp}}$$
$$+ \sum_{a \neq k} \frac{\Pi_{ka}(p^2 = m_k^2)}{m_a^2 - m_k^2} \Gamma_{ijal}^{\mathrm{amp}} + \sum_{a \neq l} \frac{\Pi_{la}(p^2 = m_l^2)}{m_a^2 - m_l^2} \Gamma_{ijka}^{\mathrm{amp}}. \tag{22}$$

A similar discussion applies when considering $\Phi_i - \bar{\Psi}\Psi$ scattering and here we only write the final expression in terms of 1PI and amputated diagrams,

$$\mathcal{A}_i = \left(1 - \frac{d}{d\not{p}} \Sigma(\not{p}) \Big|_{\not{p}=m} - \frac{1}{2} \frac{d}{dp^2} \Pi_{ii} \Big|_{p^2 = m_i^2} \right) \Gamma_i^{\mathrm{amp}} + \sum_{a \neq i} \frac{\Pi_{ia}(p^2 = m_i^2)}{m_a^2 - m_i^2} \Gamma_a^{\mathrm{amp}}. \tag{23}$$

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[6]The masses in the LSZ theorem are the physical pole masses, which are equivalent to the $\overline{\mathrm{MS}}$ masses up to finite pieces. Since we are only discussing UV divergences, we will not make any distinction between pole masses and $\overline{\mathrm{MS}}$ masses. The difference would need to be taken into consideration at higher orders in perturbation theory.

As renormalization conditions we now require all observables to be finite, *i.e.* $\mathcal{A}_i$, $\mathcal{A}_{ijkl}$ and the masses m_i^2, m. The infinities in G_{ij} that can not be absorbed into counterterms from Π_{ij} will then be canceled in the observables by the counterterms δY_i and $\delta \Lambda_{ijkl}$. These conditions lead to the following counterterms [7]:

$$\delta m_i^2 = \left(\frac{1}{2} \sum_j \Lambda_{iijj} m_j^2 + 4 Y_i^2 m^2 - 2 Y_i^2 m_i^2 \right) \Lambda,$$

$$\delta m = -\frac{m}{2} \sum_i Y_i^2 \Lambda,$$

$$\delta Y_i = \left(-Y_i^3 + \frac{3}{2} Y_i \sum_k Y_k^2 + \sum_{l \neq i} Y_l C_{il} \right) \Lambda,$$

$$\delta \Lambda_{ijkl} = \Bigg[\left(-Y_i^2 - Y_j^2 - Y_k^2 - Y_l^2 \right) \Lambda_{ijkl} + \frac{1}{2} \sum_m \left(\Lambda_{ijmm} \Lambda_{klmm} + \Lambda_{ikmm} \Lambda_{jlmm} + \Lambda_{ilmm} \Lambda_{jkmm} \right)$$

$$- 24 Y_i Y_j Y_k Y_l + \sum_{m \neq i} \Lambda_{mjkl} C_{im} + \sum_{m \neq j} \Lambda_{imkl} C_{jm} + \sum_{m \neq k} \Lambda_{ijml} C_{km} + \sum_{m \neq l} \Lambda_{ijkm} C_{lm} \Bigg] \Lambda, \tag{24}$$

where we have defined

$$C_{il} \equiv \frac{1}{m_i^2 - m_l^2} \left(\sum_k \frac{\Lambda_{ilkk}}{2} m_k^2 + 4 Y_i Y_l m^2 - 2 Y_i Y_l m_i^2 \right). \tag{25}$$

It should be noted again that the scalar masses are non-degenerate by the symmetry argument. A check on these counterterms is to verify that the on-shell amplitudes for $\Phi_i \Psi \to \Phi_j \Psi$ are finite.

Since all the fields are bare and do not depend on the renormalization scale, the RGEs for the couplings and masses follow straightforwardly from the counterterms:

$$\mathcal{D} m_i^2 = \sum_j \Lambda_{iijj} m_j^2 + 8 Y_i^2 m^2 - 4 Y_i^2 m_i^2,$$

$$\mathcal{D} m = - m \sum_i Y_i^2,$$

$$\mathcal{D} Y_i = - 2 Y_i^3 + 3 Y_i \sum_k Y_k^2 + 2 \sum_{l \neq i} Y_l C_{il},$$

$$\mathcal{D} \Lambda_{ijkl} = - 2 \left(Y_i^2 + Y_j^2 + Y_k^2 + Y_l^2 \right) \Lambda_{ijkl} + \sum_m \left(\Lambda_{ijmm} \Lambda_{klmm} + \Lambda_{ikmm} \Lambda_{jlmm} + \Lambda_{ilmm} \Lambda_{jkmm} \right)$$

$$- 48 Y_i Y_j Y_k Y_l + 2 \sum_{m \neq i} \Lambda_{mjkl} C_{im} + 2 \sum_{m \neq j} \Lambda_{imkl} C_{jm} + 2 \sum_{m \neq k} \Lambda_{ijml} C_{km}$$

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[7]There is no implicit sum over repeated indices in eq. (24)-(26).

$$+ 2 \sum_{m \neq l} \Lambda_{ijkm} C_{lm}. \tag{26}$$

It is interesting and unusual that the RGEs in this MS renormalization scheme depend on the masses through the C_{ij} terms. However, one could expect this; since this scheme is related to an on-shell renormalization one. It should also be noted that the definition of the fields in this case is unchanged during the RG evolution; they are always given by the mass eigenstates. In other words, the basis is the same throughout the evolution. Another point is that the contributions from the off-diagonal C_{ij}-terms vanish for two scalars if a $\mathbb{Z}_2$ symmetry is imposed. In other words these terms contain the same information as the kinetic mixing terms in section 3.1.

3.3   Standard way with Z_{ij}

The standard renormalization scheme most often used is very similar to the case of working with a completely general Lagrangian as in section 3.1. But one obvious simplification is to transform the fields to arrive at canonical kinetic terms through a non-orthogonal[8] transformation. For this scheme we will also renormalize the fields, which gives rise to anomalous dimensions that will enter the RGEs for all the parameters.

We denote bare fields by ϕ_i^b, $\overline{\Psi}^b$, Ψ^b and renormalized fields by $\phi_i, \overline{\Psi}, \Psi$. The relations between the two are given by

$$\phi_i^b = Z_{ij} \phi_j \qquad \text{and} \qquad \Psi^b = Z_\Psi \Psi. \tag{27}$$

With the renormalization factors

$$Z_{ij} = \delta_{ij} + \delta Z_{ij} \qquad \text{and} \qquad Z_\Psi = 1 + \delta \Psi. \tag{28}$$

The Lagrangian in terms of renormalized quantities and counterterms is then given by

$$\mathcal{L} = \frac{1}{2} Z_{ik} \partial_\mu \phi_k Z_{il} \partial^\mu \phi_l - \frac{1}{2} Z_{ik} \phi_k \left(m_{ij}^2 + \delta m_{ij}^2 \right) Z_{jl} \phi_l + Z_\Psi^2 \overline{\Psi} i \slashed{\partial} \Psi - Z_\Psi^2 \left(m + \delta m \right) \overline{\Psi} \Psi$$
$$+ \mu^\epsilon \left(y_i + \delta y_i \right) \phi_i \overline{\Psi} i \gamma_5 \Psi - \frac{\mu^{2\epsilon}}{4!} \left(\lambda_{ijkl} + \delta \lambda_{ijkl} \right) \phi_i \phi_k \phi_k \phi_l, \tag{29}$$

where m_{ij}^2 and λ_{ijkl} are fully symmetric in their indices. Note that we have a different number of parameters compared to counterterms here. For two scalars we have 15 counterterms and 11 parameters, but of course only 10 physical parameters as in the other schemes. The relations of bare and renormalized parameters are somewhat more complicated than in the previous renormalization schemes,

$$m_{ij}^{b\,2} = m_{ij}^2 + \delta m_{ij}^2, \tag{30}$$
$$y_i^b = Z_{ji}^{-1} Z_\Psi^{-2} \mu^\epsilon (y_j + \delta y_j), \tag{31}$$
$$\lambda_{ijkl}^b = Z_{ai}^{-1} Z_{bj}^{-1} Z_{ck}^{-1} Z_{dl}^{-1} \mu^{2\epsilon} (\lambda_{abcd} + \delta \lambda_{abcd}). \tag{32}$$

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[8]In more general cases with complex fields it would require a non-unitary transformation.

Since the renormalization factors are μ dependent we now obtain RGEs for the fields,

$$\mathcal{D}\phi_i = - \left(Z^{-1}\mathcal{D}Z\right)_{ij}\phi_j \equiv \gamma_{ij}\phi_j,$$
$$\mathcal{D}\Psi = - Z_\Psi^{-1}\mathcal{D}Z_\Psi\Psi \equiv \gamma_\Psi\Psi. \tag{33}$$

In case of γ_{ij} not being diagonal this means that the fields will mix during renormalization and thereby the basis will also change. These anomalous dimensions, γ_{ij} and γ_Ψ, also enter the RGEs for the parameters

$$\mathcal{D}y_i = - (4\pi)^2\epsilon(y_i + \delta y_i) - \gamma_{ij}y_j - 2\gamma_\Psi g_i - \mathcal{D}\delta y_i, \tag{34}$$
$$\mathcal{D}\lambda_{ijkl} = - 2(4\pi)^2\epsilon(\lambda_{ijkl} + \delta\lambda_{ijkl})$$
$$- \sum_m (\gamma_{im}\lambda_{mjkl} + \gamma_{jm}\lambda_{imkl} + \gamma_{km}\lambda_{ijml} + \gamma_{lm}\lambda_{ijkm}) - \mathcal{D}\delta\lambda_{ijkl}. \tag{35}$$

Just as in section 3.1, the counterterms are determined by requiring that all Green functions of the fields $\phi_i, \overline{\Psi}, \Psi$ are finite. This is accomplished by making the 1PI and amputated diagrams in figure 2 finite, which we compute to be

$$\Sigma(p^2) = \sum_i y_i^2 \left(\frac{\not{p}}{2} - m\right)\Lambda + (2\not{p}\delta\psi - 2m\delta\psi - \delta m), \tag{36}$$

$$\Pi_{ij}(p^2) = \left[\frac{1}{2}\lambda_{ijkl}m_{kl}^2 + y_iy_j(4m^2 - 2p^2)\right]\Lambda + (2p^2\delta Z_{ij} - m_{ja}^2\delta Z_{ia} - m_{ia}^2\delta Z_{ja} - \delta m_{ij}^2), \tag{37}$$

$$\Gamma_i^{\text{amp}} = - y_i\sum_j y_j^2\Lambda + \delta y_i, \tag{38}$$

$$\Gamma_{ijkl}^{\text{amp}} = \left[\frac{1}{2}(\lambda_{ijmm}\lambda_{klmm} + \lambda_{ikmm}\lambda_{jlmm} + \lambda_{ilmm}\lambda_{jkmm}) - 24y_iy_jy_ky_l\right]i\Lambda - i\delta\lambda_{ijkl}. \tag{39}$$

The counterterms thus become

$$\delta Z_{ij} = y_iy_j\Lambda,$$
$$\delta Z_\Psi = -\frac{1}{4}\sum_i y_i^2\Lambda,$$
$$\delta m_{ij}^2 = \left(\frac{1}{2}\lambda_{ijkl}m_{kl}^2 + 4my_iy_j - y_iy_km_{kj}^2 - m_{ik}^2y_ky_j\right)\Lambda,$$
$$\delta m = -\frac{1}{2}m\sum_i y_i^2\Lambda,$$
$$\delta y_i = y_i\sum_k y_k^2\Lambda,$$
$$\delta\lambda_{ijkl} = \left[\frac{1}{2}(\lambda_{ijmn}\lambda_{klmn} + \lambda_{ikmn}\lambda_{jlmn} + \lambda_{ilmn}\lambda_{jkmn}) - 24y_iy_jy_ky_l\right]\Lambda. \tag{40}$$

In the end, we arrive at the following anomalous dimensions and RGEs for the parameters

$$\gamma_{ij} = 2y_i y_j,$$
$$\gamma_\Psi = -\frac{1}{2}\sum_i y_i^2,$$
$$\mathcal{D}m_{ij}^2 = \lambda_{ijkl}m_{kl}^2 + 8m^2 y_i y_j - 2y_i y_k m_{kj}^2 - 2m_{ik}^2 y_k y_j,$$
$$\mathcal{D}m = -m\sum_i y_i^2,$$
$$\mathcal{D}y_i = y_i \sum_k y_k^2,$$
$$\mathcal{D}\lambda_{ijkl} = \lambda_{ijmn}\lambda_{klmn} + \lambda_{ikmn}\lambda_{jlmn} + \lambda_{ikmn}\lambda_{jlmn} - 48y_i y_j y_k y_l$$
$$- 2y_i y_m \lambda_{mjkl} - 2y_j y_m \lambda_{imkl} - 2y_k y_m \lambda_{ijml} - 2y_l y_m \lambda_{ijkm}. \tag{41}$$

Here we again see that if one fermion couples to more than one of the scalars, then the scalars will mix under RG evolution from $y_i y_j \neq 0$ for $i \neq j$. The effect appears both in the anomalous dimensions of the scalar fields as well as in the masses and quartic couplings. This then corresponds to the kinetic mixing in section 3.1. An example of how to actually perform the RG evolution with these effects taken into consideration can be found in ref. [13].

As a side note we briefly compare these results to the renormalization scheme used in refs. [14, 15, 16, 17], where they derive the 1- and 2-loop RGEs for a general quantum field theory. There is a subtlety in their notation in that they present the RGEs for a theory with an irreducible representation of the scalar fields and the anomalous dimensions are therefore taken to be diagonal [9]. Their formulas can however be modified to include theories containing multiple scalar fields by generalizing the anomalous dimensions in a relatively straightforward way. This is discussed in great detail in ref. [19] and we will not discuss it further here.

4 Relation between the various schemes

All the renormalization schemes are built on bare Lagrangians, which of course do not depend on the renormalization scale μ. The different bare Lagrangians are then related to each other by field redefinitions and therefore the renormalized fields are as well. In this section we will show that the renormalized parameters in each of the three renormalization schemes are related to each other by orthogonal and non-orthogonal transformations. Though the transformations are somewhat trivial at bare level, the transformation matrices of renormalized quantities do obey a non-trivial μ dependence.

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[9]This has independently been pointed out by ref. [18].

4.1 Most general versus minimal

To relate the general Lagrangian in section 3.1 to the minimal Lagrangian in section 3.2 we need to first diagonalize and normalize the kinetic terms, *i.e.* a_{ij}. Following that, we need another rotation to diagonalize the new mass matrix. We write the full non-orthogonal transformation matrix as

$$R^b = O_1^{bT} A^{b-1} O_2^{bT}, \tag{42}$$

where the diagonal matrix A^b and orthogonal matrices O_i^b are defined by

$$O_1^b a^b O_1^{bT} = \text{diag}(\alpha_i^b), \qquad\qquad A^b = \text{diag}\left(\sqrt{\alpha_i^b}\right),$$

$$O_2^b A^{b-1} O_1^b b^b O_1^{bT} A^{b-1} O_2^{bT} = \text{diag}\left(m_i^{2b}\right), \tag{43}$$

such that

$$R^{bT} a R^b = 1, \qquad\qquad R^{bT} b^b R^b = \text{diag}\left(m_i^{2b}\right) \equiv M^b. \tag{44}$$

The relations of the bare fields are then

$$\varphi_i^b = R_{ij}^b \Phi_j^b \qquad\qquad \text{and} \qquad\qquad \psi^b = \frac{1}{\sqrt{c^b}} \Psi^b. \tag{45}$$

Note that since the transformation matrix R^b contains divergences, it is clear that Green functions of $\Phi_i^b, \overline{\Psi}^b, \Psi^b$ can be divergent while those of $\phi_i^b, \overline{\psi}^b, \psi^b$ are finite.

While the relation between m_i^{2b} and a^b, b^b is implicit in eqs. (42–44), the other bare parameters are related via

$$m^b = \frac{d^b}{c^b}, \qquad\qquad Y_i^b = \frac{1}{c^b} e_j^b R_{ji}^b, \qquad\qquad \Lambda_{ijkl}^b = f_{abcd}^b R_{ai}^b R_{bj}^b R_{ck}^b R_{dl}^b. \tag{46}$$

All quantities involved in eqs. (42–46) are bare quantities and are μ-independent; as a consequence, the counterterms and RGEs must also be compatible. We can then relate the renormalized quantities with a μ-dependent transformation matrix $R_{ij}(\mu)$. The relations between the parameters in the two schemes are the same as in the bare case, but without the superscript b everywhere. Using the notation $M = \text{diag}(m_i^2)$ these are

$$R^T a R = 1, \qquad\qquad R^T b R = M, \qquad\qquad \frac{d}{c} = m,$$

$$\frac{1}{c} e^T R = Y^T, \qquad\qquad f_{abcd} R_{ai} R_{bj} R_{bk} R_{dl} = \Lambda_{ijkl}. \tag{47}$$

To figure out how R_{ij} depends on the renormalization scale, one can determine its explicit form in terms of the renormalized Lagrangian parameters; which of course fully fixes the μ-dependence. Another way is using the relations in eq. (47) to transform the RGEs in the general scheme in eq. (10) to the RGEs in the minimal scheme in eq. (26) from which the required μ-dependence for R_{ij} follows. These two methods are equivalent and one finds the following non-trivial relation

$$(R^{-1}\mathcal{D}R)_{ij} = \delta_{ij} 2Y_i^2 - 2C_{ji}, \tag{48}$$

where C_{ij} is defined in eq. (25).

4.2 Standard versus minimal

To go from the standard scheme in section 3.3 to the minimal scheme in section 3.2, all one has to do is to diagonalize the mass matrix with an orthogonal transformation. At the bare level we have

$$O_{ij}^{bT} m_{jk}^{b2} O_{kl}^b = \delta_{il} m_i^{b2} \tag{49}$$

and the bare fields are then related by

$$\phi_i^b = O_{ij}^b \Phi_j^b. \tag{50}$$

The fermion field and mass are actually the same in both the schemes. However, note that the field is renormalized, $\Psi^b = Z_\Psi \Psi$, in the standard scheme.

Just as discussed in section 4.1, the schemes are related by field redefinitions and hence must be equivalent. We can then relate the renormalized quantities like the bare ones, but with a "renormalized" rotation matrix $O_{ij}(\mu)$,

$$\begin{aligned}
\delta_{ij} m_i^2 &= O_{ik}^T m_{kl}^2 O_{lj}, \\
Y_i &= y_a O_{ai}, \\
\Lambda_{ijkl} &= \lambda_{abcd} O_{ai} O_{bj} O_{ck} O_{dl}.
\end{aligned} \tag{51}$$

The rotation matrix O obeys a similar non-trivial μ-dependence as the transformation in section 4.1. Using eq. (51) to get from the RGEs in the standard scheme in eq. (41) to the RGEs in the minimal scheme in eq. (26) leads to the relation

$$(O^T \mathcal{D} O)_{ij} = \begin{cases} 0, & \text{for } i = j, \\ 2C_{ji} + 2Y_i Y_j, & \text{for } i \neq j. \end{cases} \tag{52}$$

5 Conclusions

We have shown that even in the general case, having several scalar fields with the same quantum numbers, it is possible to choose a renormalization scheme such that there is no kinetic mixing of these fields. We have shown explicitly how this can be done at 1-loop order by defining three different schemes: the most general one allowing for all kinetic mixings; the minimal one where the only parameters and counterterms are the physically relevant ones; and the standard method where the fields and masses mix during renormalization which produces general counterterms. We showed that the three cases are related by scale dependent field redefinitions and also explicitly that the renormalization group equations are equivalent.

While the general scheme exhibits kinetic mixing terms in the Lagrangian, these are not present in the other two schemes. The equivalent effect is instead encoded in other ways. In the minimal scheme the effect enters in the off-diagonal C_{ij} terms that depend on the scalar masses. In the standard scheme the fields mix under renormalization; which gives rise

to non-diagonal anomalous dimensions. Also the masses mix under renormalization and diagonalizing the mass-matrix after the RG evolution, to get the masses of the physical fields, redefines the fields in a way corresponding to the off-diagonal C_{ij} terms in the minimal scheme.

Another effect from the mixing of the fields under RG evolution in the standard scheme is that the basis changes during RG running. This is, for example, important for the case of 2HDMs with a broken $\mathbb{Z}_2$ symmetry. To circumvent this, one either has to keep track of how the basis changes by also taking into account the anomalous dimensions of the fields and the evolution of the mass terms or alternatively only use basis-invariant quantities.

Acknowledgements

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