RANDOM SURFACES WITH ISING SPINS

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INTRODUCTION

String theories have been the subject of intense studies during the last decade, motivated by the hope that they will provide a new fundamental theory of "everything". In its Euclidean version, a string theory corresponds to a theory of random surfaces. It has become evident, though, that a purely bosonic theory of random surfaces has unavoidable intrinsic diseases, which however might be cured by adding fermionic degrees of freedom (superstrings) to the Polyakov type string model.[1]

In order to allow numerical investigations, the theory must be suitably discretized. The currently most promising attempt involves using dynamically triangulated random surfaces.[2,3,4] This idea has been exploited in Monte-Carlo investigations for the purely bosonic case with encouraging results.[5,6]

It is then natural to try to generalize this line of investigation to include also fermionic degrees of freedom, in a form that can be used in Monte-Carlo experiments. An appealing way of doing this is to represent the fermions by Ising spins, which on a two-dimensional manifold are known to correspond to Majorana fermions.[7]

This approach is the subject of my contribution to this meeting, and I will, apart from briefly reviewing some analytical results, discuss the results from Monte-Carlo experiments carried out by the author in collaboration with J. Jurkiewicz, A. Krzywicki and B. Petersson.[8]

THE MODEL

The basic ingredience is a *triangulation*: A graph built out of triangles, glued together along the edges (links). It is embedded in a d-dimensional space by defining a d-dimensional variable x on every site (= vertex). The action for the spatial variables is then defined as

$$S_x = \sum_{<i,j>} (x_i - x_j)^2 \tag{1}$$

with the sum running over all pairs of sites connected by a link. Note that no coupling parameter is needed here, since it can anyway be scaled away. The fermions are implemented as Ising spins σ, also defined on the sites, with a corresponding action

$$S_\sigma = -\kappa \sum_{<i,j>} \sigma_i \sigma_j \tag{2}$$

For simplicity, only the case of one spin per site is described, but it is straight-forward to generalize to several spins. Of particular interest is the case of d spins, corresponding to the supersymmetric case.

The partition function for a surface with n sites is now defined as

$$Z_n = \sum_T W(T) \prod_i (q_i/\pi)^\alpha \sum_\sigma e^{-S_\sigma} \int d^d\!x\, e^{-S_x} \tag{3}$$

where the first sum is over all inequivalent non-degenerate triangulations T with the topology of a sphere, $W(T)$ is a symmetry factor due to the "reparametrization" invariance related to the labelling of the sites, and q_i is the coordination number of site i. The parameter α controls the fluctuations of the geometry, and should be equal to $d/2$ for consistency with Polyakov's string theory.

The system is believed (for large n) to have a second order phase transition, where the continuum limit is to be taken. Thus a primary concern is to find the phase transition point κ_c, and the corresponding critical indices. The fractal dimension d_f of the surface is also of interest in order to analyze which kind of surfaces dominate the partition function.

EXACTLY SOLVABLE CASE

An unusual feature of the above model is that it can be solved exactly in the case of $d = 0$, $\alpha = 0$ for one spin per site.[9] One method to solve the model in this case is based on the following crucial steps:

- The model is equivalent to the corresponding dual model, i.e. a model of Ising spins on the sites of the dual graph, with a dual coupling κ^*, $\sinh 2\kappa^* \sinh 2\kappa = 1$. The dual graph is obtained by placing a dual site in the center of each triangle, and connecting the dual sites with dual links crossing the original links.

- The dual graphs correspond to non-degenerate planar vacuum bubbles of a coupled two-component zero-dimensional ϕ^3 theory.

- The subset of planar bubbles is typical of large-N theories, in this case in the form of Hermitian $(N \times N)$-matrices with $N \to \infty$.

- The partition function for such a matrix theory can be rewritten in terms of the eigenvalues of the matrices, and rescaled in such a way, that the N-dependence factorizes out as a pure factor multiplying the action. That means that the partition function will be completely dominated by the saddle-point contribution from the classical minimum-action configuration.

From an analysis along the lines sketched above, the critical coupling κ_c can be derived,

$$\kappa_c = \frac{1}{2} \log \frac{131}{85} \approx 0.216, \tag{4}$$

along with a set of critical indices, all consistent with an independent continuum calculation,[10] indicating that the summation over triangulations indeed correctly represents the integration over metrics.

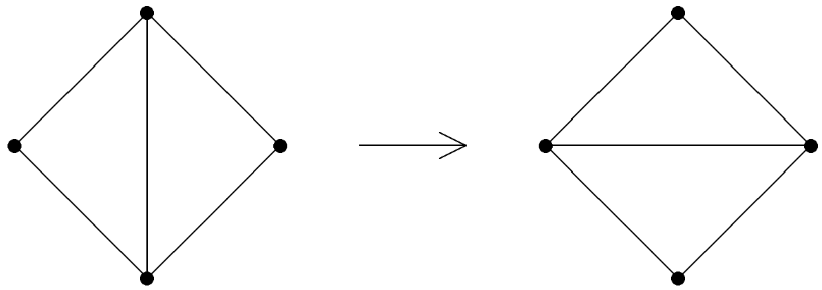

Figure 1: A flip of a link.

MONTE-CARLO METHOD

For $d \neq 0$ the analytical methods mentioned above do not work. We have investigated the model numerically[8] for $d = 0$ (as a check), 3, 5 and 10. For each value of d, we have carried out Monte-Carlo experiments for a set of different lattice sizes n and couplings κ. For a given lattice-size n, a configuration is defined by a triangulation T, and a set of positions x_i and spins σ_i. The triangulation is updated in the following way: One of the totally $n_l = 3(n-2)$ links is selected at random and flipped[4] (see fig. 1), if accepted by a Metropolis test. For the positions x_i and spins σ_i, the heatbath method is used.

A sweep consists of n_l (= the number of links) attempted flips, and an update of each x (if $d > 0$) and σ. For every value of d and n, the system was first thermalized with 2000-4000 sweeps, with nothing measured. Then, after every sweep, the internal spin energy $E = -S_\sigma$, the magnetization $M = 1/n \sum_i \sigma_i$, and the gyration radius r were measured. The radius r, defined by

$$r^2 = n_l^{-2} \sum_{ij} q_i q_j (x_i - x_j)^2, \tag{5}$$

is a natural measure of the spatial extent of the surface, when the triangles (rather than the sites) are given equal weight.

After 4096 such production sweeps (an "experiment"), estimates of the observables were computed. Finally, after 16-64 experiments, the observables were computed as averages over the experiments, using the variations between the experiments to estimate the errors.

In this way we computed $< |M| >$, $< M^2 >$, $< M^4 >$, $< E >$, $< E^2 >$ and $< r^2 >$, from which we then obtained the specific heat C,

$$C = \kappa^2 n (< E^2 > - < E >^2), \tag{6}$$

the susceptibility χ,

$$\chi = \kappa n < M^2 >, \tag{7}$$

as well as the renormalized coupling g_R for the magnetization,

$$g_R = \frac{< M^4 >}{< M^2 >^2}. \tag{8}$$

The Monte-Carlo experiments were carried out, using partially vectorized code, and analyzed on the CYBER 205 of the Bochum Computer Centre, the Cray-XMP of the HLRZ at KFA in Juelich, and the Siemens VP200 of the CNRS computer centre CIRCE in Orsay.

	d			
	0	3	5	10
κ_c	0.215(2) 0.218(3) [0.216]	0.231(4) 0.236(7)	0.292(29) 0.327(16)	0.334(4)
νD	3.3(6) 3.2(3) [3]	4.5(3) 5.0(7)	8.5(22) 2.2(10)	1.9(3)
$\beta/\nu D$	0.17(1) 0.16(3) [1/6]	0.18(3) 0.12(4)	0.02(1) 0.14(6)	0.13(10)
$\gamma/\nu D$	0.68(2) 0.71(4) [2/3]	0.67(5) 0.76(5)	0.98(5) 0.80(8)	0.77(14)
d_f		7.3(2) 8.3(4)	6.4(1) 6.2(1)	4.4(1)

Table 1: Numerical results for the critical parameters of the model with one spin per site. The results from both estimates of κ_c are given (upper: 45×500, lower: 100×500), with the error in the last digit given in parentheses. For $d = 0$, the exact results are given in brackets.

ANALYSIS AND RESULTS

For every value of d, we investigated the lattice sizes n = 20, 45, 100, 225 and 500, at typically eight different values of the coupling κ.

A finite size scaling[11] method was used to find the critical coupling κ_c, as the cross-over point where g_R as a function of κ is independent of n. One expects, close to the critical coupling, the following behaviour for g_R:

$$g_R = f\left((\kappa - \kappa_c)n^{1/\nu D}\right), \tag{9}$$

where ν is the mass-gap exponent, and D is an intrinsic fractal dimension, while f is an unknown function. In practice, κ_c was estimated from the cross-over (45×500) and (100×500), using curves obtained with a least squares fit. These combinations gave the best estimates for the case $d = 0$, where we know the exact result. Finding the cross-over was fairly straightforward for small d, but turned out to be increasingly difficult for increasing d. From the ratio of slopes, we could then estimate the combination νD.

Once we had the two estimates of κ_c, we made complementary runs at these couplings, for six different sizes ranging from $n = 100$ to $n = 1500$. We then computed estimates of the critical indices $\beta/\nu D$ and $\gamma/\nu D$ for the absolute magnetization $< |M| >$ and the magnetic susceptibility χ, based on a least squares fit to the expected finite size scaling behaviour at the critical coupling:

$$< |M| > \propto n^{-\beta/\nu D} \tag{10}$$

and

$$\chi \propto n^{\gamma/\nu D}. \tag{11}$$

The critical indices for the energy and the specific heat are more tricky, and could not be satisfactorily evaluated from our data. We also computed estimates for the fractal dimension d_f from the size-dependence of r.

The final results (for both estimates, which coincide for $d = 10$) are collected in table 1, where also an estimate of the fractal dimension d_f, obtained from the dependence of r on n at κ_c, is given. The errors quoted come from the least square fits to the data.

CONCLUSIONS

The critical indices seem to vary slowly with d. It seems $\beta/\nu D$ decreases towards zero with d, while $\gamma/\nu D$ increases towards the value one, in a way consistent with the expectation $2\beta + \gamma = \nu D$.

The fractal dimension d_f seems to decrease towards the value four, characteristic of the branched polymer type surfaces, just like in previous investigations of the purely bosonic case.[5]

The determination of κ_c (and of νD) turned out to be difficult for the high-dimensional cases, which makes the results somewhat uncertain for $d = 5$, and in particular $d = 10$. (The small apparent errors for $d = 10$ are probably too optimistic.) The cross-over for the higher d-values appears at a value of g_R very close to the asymptotic value 1, which may indicate, that the cross-over is a mere artefact of the finite lattice, and that there is no phase transition at all for 5 and 10 dimensions. Obviously, larger experiments are needed to draw a firm conclusion on this point.

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