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Bethe Lattices in Hyperbolic Space

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Abstract

A recently suggested geometrical embedding of Bethe type lattices (branched polymers) in the hyperbolic plane is shown to be only a special case of a whole continuum of possible realizations, preserving some of the symmetries of the Bethe lattice. The properties of such embeddings are investigated, and relations to Farey trees, devil's staircases and Apollonian tiling are pointed out.

1 Introduction

The term "Bethe lattice" is used for a family of regular tree-graphs, where every node is connected to a fixed number of neighbouring nodes, and no loops are present. It has been used as an artificial backbone in several areas of physics, providing for solvable models due to the high degree of symmetry and the absence of loops.

The symmetry of a Bethe lattice is considerable – all pairs of nodes with a fixed topological separation (the number of links in the connecting path) are equivalent. This implies in particular *homogeneity* (all nodes are equivalent) and *isotropy* (for every node, all its neighbours are equivalent). A Bethe lattice cannot be embedded in a finite-dimensional Euclidean space without much of the symmetry being broken.

A more suitable manifold for this purpose is offered by hyperbolic spaces of constant negative curvature, where indeed Bethe lattices can be embedded while preserving homogeneity and isotropy. A recipy for the embedding of a Bethe lattice in the hyperbolic (Lobachevsky) plane has been suggested in several papers, see, e.g., [1, 2]. As will be shown in this paper, that particular embedding is a limiting case of a continuum of inequivalent embeddings, differing in the induced metric properties of the tree.

2 Regular graphs

Regular (homogeneous and isotropic) graphs in two dimensions are labeled by the Schläfli symbol $\{k, l\}$, where the integers $k, l > 2$ give the size of the elementary loops and the coordination number of a node, respectively. Only for $(k-2)(l-2) = 4$ can the graph be embedded in the Euclidean plane. The only possibilities are $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$, corresponding to the triangular lattice, the square lattice, and the hexagonal honeycomb, respectively.

For $(k-2)(l-2) < 4$, the graph can be embedded on the sphere – this gives the familiar five Platonic polyhedra. Thus, e.g., $\{4, 3\}$ denotes a cube, and $\{3, 4\}$ an octahedron, etc.

For $(k-2)(l-2) > 4$, one must utilize the hyperbolic plane. Assuming a unit negative curvature, the geodesic link length d is given by

$$\cosh(d/2) = \frac{cos(\pi/k)}{\sin(\pi/l)} \tag{1}$$

3 Regular trees

Increasing the topological loop length k for a fixed coordination number l leads to the graph gradually opening up: $\cos(\pi/k)$ increases which leads to an increase in d. In the limit as k becomes infinite, the loops become infinite and fail to remain loops. The graph is on the verge of turning into a tree – this case will be referred to as a *critical* tree, and denoted by $\{\infty, l\}$. It has a critical link length, d_c, given by

$$\cosh d_c = \frac{3 + cos(2\pi/l)}{1 - cos(2\pi/l)} \tag{2}$$

This limiting case, $\{\infty, l\}$, was suggested in refs. [1, 2] as a geometric realization of the Bethe lattice.

It is however possible to further open up the graph in a continuous way beyond $k = \infty$. This demands a formally imaginary k, such that $\cos(\pi/k) > 1$. This leads to a *super-critical* tree with a link length above the critical link length d_c. Thus, there is a whole family of possible embeddings with identical topology and symmetry; they will be denoted by the symbol $\{*, l\}$, where the '$*$' is a place-holder for an imaginary (or infinite) k.

In order to depict such trees, a compact representation of the hyperbolic plane is needed. This is acquired by a conformal mapping to the *Poincaré disk*: i.e. the interior of the unit circle, equipped with the metric

$$ds^2 = 4\frac{dr^2 + r^2 d\phi^2}{(1 - r^2)^2}, \ r < 1. \tag{3}$$

where r, ϕ are polar coordinates.

In Fig. 1, a few (critical and super-critical) $\{*, 3\}$ trees are shown, using the Poincaré disk representation. Notice how an increase in link-length is accompanied by an opening up of the tree.

In the case of the critical $\{*, 3\}$ tree, every point on the boundary of the Poincaré disk will be approached by some part of the tree. With a suitable mapping of the boundary to the real line, the tree organizes the real numbers according to their continued fraction expansions, corresponding to a Farey tree organization of the rational numbers [3], associated with the nodes of the dual graph $\{3, \infty\}$. In contrast, for a super-critical tree, the boundary points that are approached by the tree will define a fractal, Cantor-like, subset of the boundary, with a finite gap for every rational number, reminiscent of the devil's staircase for the phenomenon of rational mode-locking in circle-maps [3]. The connection between the $\{*, 3\}$ trees and the Farey organization of the rationals is due to a common symmetry-group – the modular group SL(2,$\mathcal{Z}$).

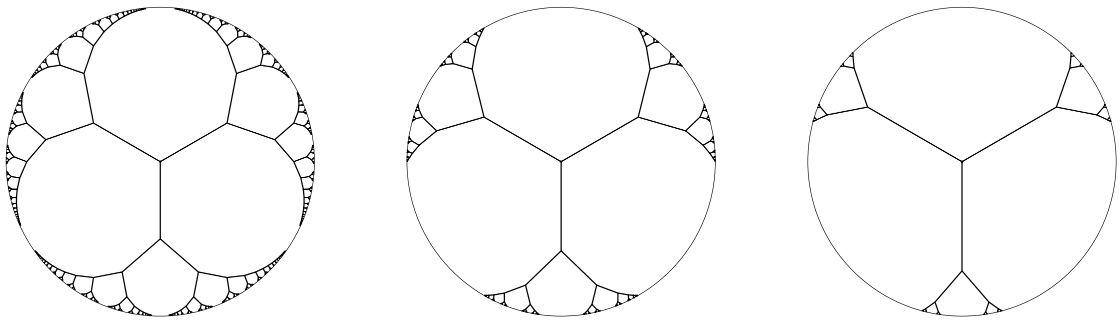

Figure 1: A critical and two super-critical $\{*, 3\}$ trees. The repective values of $\cosh d$ are (a) 5/3, (b) 2, and (c) 3, respectively.

4 Geometry

The computation of inter-node distances can be done recursively. For the hyperbolic cosine c_i of the distance from node i to a fixed node, there are simple linear relations. Thus, for the $\{*, 3\}$ tree, we have for a sequence of four consecutive nearest neighbours:

$$c_1 - Ac_2 + Ac_3 - c_4 = 0, \quad A = \frac{3\cosh d + 1}{2} \qquad \text{[cis]} \tag{4}$$

$$c_1 - Bc_2 - Bc_3 + c_4 = 0, \quad B = \frac{3\cosh d - 1}{2} \qquad \text{[trans]} \tag{5}$$

where 'cis' and 'trans' refers to whether the two consecutive turns are in the same or opposite direction, respectively (see Fig. 2). The relations (4,5) derive from the fact that to every node can be associated a time-like unit vector in the Minkowski space $\mathcal{M}_3$,[1] such that the scalar product between two vectors equals the hyperbolic cosine of the corresponding node distance. The vectors themselves satisfy the linear relations above; hence, so do their scalar products with a fixed vector [4]. The geometric distances d_D for topological distance $D = 0, 1, 2, 3$ are given by

$$\cosh d_0 = 1 \tag{6}$$

$$\cosh d_1 = C \equiv \cosh d \tag{7}$$

$$\cosh d_2 = \frac{1}{2}(3C^2 - 1) \tag{8}$$

$$\cosh d_3 = \begin{cases} \frac{1}{4}(9C^3 - 3C^2 - 5C + 3) & \text{[cis]} \\ \frac{1}{4}(9C^3 + 3C^2 - 5C - 3) & \text{[trans]} \end{cases} \tag{9}$$

Any other distance is obtained by recursively applying eqs. (4,5).

[1]This defines an embedding in Minkowski space, restricted to the unit mass-shell, analogous to the embedding of regular polyhedra in Euclidean space, restricted to the unit sphere.

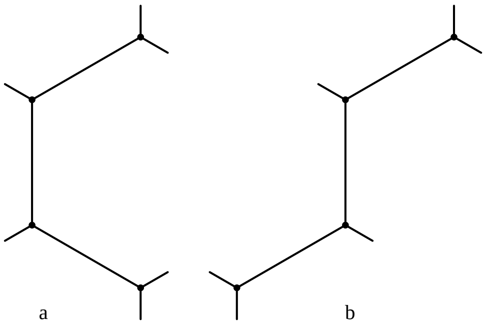

Figure 2: A "cis" (a) and a "trans" (b) sequence of nodes.

By similar means, the geometry of other two-dimensional trees (l=4, 5, etc.) can be derived. A generic feature of this kind of embedding is that, although homogeneity and isotropy are preserved, the inter-node distance is not a function of topological distance only. Thus the geometric embedding of the Bethe lattice is accompanied by the breaking of the full (topological) symmetry group down to a subgroup, defining the geometrical symmetry group.

However, the symmetry is restored in the limit of a large link-length (or, equivalently, a large curvature), where the interlink distance becomes proportional to the topological distance.

5 Higher dimensions

Analogous constructions exist in higher dimensions. Thus in three dimensions, a regular structure is denoted by a Schläfli symbol with three integers $\{k, l, m\}$, where $\{k, l\}$ determines the hyper-faces, and $\{l, m\}$ the arrangement of neighbours around a node (see e.g. ref. [4] for details). As in two dimensions, a critical $\{*, l, m\}$ tree is obtained as $k \to \infty$, and super-critical trees are obtained by further increasing the nearest-neighbour distance.

The simplest three-dimensional tree is given by $\{*, 3, 3\}$; then every node has four neighbours, tetrahedrically arranged. This is the obvious generalization of the two-dimensional $\{*, 3\}$ tree, and it gives the most symmetric embedding of a Bethe lattice of coordination number four in a finite-dimensional manifold. This embedding is preferable to the two-dimensional embedding $\{*, 4\}$, where nearest neighbours are arranged in a square, and the symmetry is lower.

When depicted in the Poincaré 'ball' (which is the 3-d analogue of the Poincaré disk) the critical $\{*, 3, 3\}$ tree approaches a fractal subset of the boundary. The complement of this subset consists of circular discs that tile the surface of the sphere (Apollonian tiling [4]). As in two dimensions, there exist linear recursion relations for the computation of inter-node distances, although they relate a sequence of five, rather than four, consecutive nodes.

The process can be generalized to Bethe lattices of arbitrary coordination number N, for which a highly symmetric embedding can be done in $(N-1)$-dimensional hyperbolic space. The neighbours of a node then will be arranged to form a regular N-simplex, $\{3,3,\ldots,3\}$, and the corresponding family of trees is $\{*,3,3,\ldots,3\}$.

6 Conclusions

In conclusion, I have shown that the embedding of a Bethe lattice as a regular tree in hyperbolic space is not unique. Even when demanding maximal symmetry, there is (at least) a one-parameter family of possible inequivalent embeddings, of which the critical one is a limiting case. Thus, any conclusion based on the critical embedding alone is bound to be non-universal.

The induced geometry of an embedding of the discussed type can be obtained by linear recursive methods.

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